THE FORCED VIBRATION OF A LUMPED MASS SYSTEM

by

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ABSTRACT

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The purpose of this thesis is to determine a general closed-form solution of a discrete linear dynamic system having n degrees of freedom. The solution includes the effect of axial force as well as the effect of both damped and undamped motion. Viscous-type damping is considered for the assumed mathematical model.

The solution is given in a compact matrix form which eliminates the necessity of a series-type solution. The matrix solution is given in Duhamel's integral form which allows for the application of any type of time-varying external forcing function. Two numerical problems are solved to illustrate the results.

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LIST OF SYMBOLS

SYMBOL		DEFINITION
$f_j(t)$		External time varying forces
j	=	$1, 2, \ldots, n$
p		Scalar parameter
q_j		Eigenvalues
$x_j(t)$		Generalized displacements
$\dot{x}_j(t)$		Generalized velocity
D		Dissipation function
P_{ij}		External conservative time varying forces
T		Total Kinetic energy
V_e		Total external potential energy
V_i		Total internal potential energy
ω		Natural frequency of free vibration neglecting axial forces
Ω		Natural frequency of free vibration including axial forces
$\{f(t)\}$		Column matrix of external forces
$\{x(t)\}$		Column matrix of generalized displacements
$\{\overset{*}{\ }\}$	=	2n x 1 Partitioned column matrix
$[C]$		Viscous damping matrix
$[K]$		Stiffness matrix
$[M]$		Mass matrix
$[P]$		Stability matrix
$[U]$		Model matrix of eigenvectors
$[\overset{*}{\ }]$		2n x 2n Partitioned square matrix

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CHAPTER I

INTRODUCTION

The forced vibration of lumped mass systems, both including and excluding the effect of viscous damping, is considered by Crandall and McCalley[3]. A matrix formulation is used throughout the entire analysis and a closed-form solution is given in series-type form. In addition, Foss[4] considered coordinate uncoupling of the equations of motion for linear damped systems. Also, Caughey and O'Kelly[2] considered the classical normal mode form of the linear damped systems.

The static stability problem of lumped-mass system including the effect of axial force is considered by Timoshenko,[10] where the inertial terms are neglected and general solution is given in algebraic form. The matrix formulation of static stability problem is considered by Rubinstein.[9]

The purpose of this thesis is to combine the effect of inertial forces, axial forces, damping forces, linear restoring forces and arbitrary external time varying forces. The effect of rigid body motions is neglected for the mathematical models considered. The general equations of motion for this special case are formulated in matrix form by Newmark and Rosenblueth.[8] The later reference does not present a solution to the formulated equations.

Herein, a formal closed-form matrix-type solution is presented for arbitrary external time varying forces. This solution is formulated for the special case where the mass, stiffness, and axial force matrices are simultaneously diagonalized by a non-singular matrix.

CHAPTER II

GENERAL FORMULATION OF PROBLEM

Consider a linearly dynamic system with n degrees of freedom where the motion of the system is described by n generalized displacements $x_j(t)$, $j = 1, 2, \cdots n$. Also, $f_j(t)$, $j = 1, 2, \cdots n$ represents arbitrary external time varying forces and the term P_{ij} $i = 1, 2, \cdots n$, $j = 1, 2, \cdots n$ are the scalar components defining the external conservative axial-type forces.

The total kinetic energy T and dissipation function D is then expressed in a quadratic form as a function of generalized velocity $\dot{x}_j(t)$, $j = 1, 2, \cdots n$. In matrix form it follows that,

$$T = \frac{1}{2}\{\dot{X}\}^T[M]\{\dot{X}\} \quad , \tag{1a}$$

$$\text{and} \quad D = \frac{1}{2}\{\dot{X}\}^T[C]\{\dot{X}\} \quad , \tag{1b}$$

where [M] designates the mass matrix and [C] the viscous damping matrix.

The total internal potential energy V_i is expressed in terms of the generalized displacements in the following form:

$$V_i = \frac{1}{2}\{X\}^T[K]\{X\} \tag{1c}$$

where [K] defines the stiffness matrix.

The total external potential energy is comprised of two parts: the part due to axial conservative forces and the part due to non-conservative time-varying forces. The total external potential energy

V_e is then written in the following form:

$$V_e = \frac{1}{2}\{X\}^T[P]\{X\} + \{f(t)\}^T\{X\} , \tag{1d}$$

where [P] is defined as stability matrix.

The Lagrangian of the system L is written as L = T − V, where V is defined as the total potential of the system, or $V = V_e + V_i$. Using Hamilton's principle[7], it follows that the equations of motion must satisfy the following differential equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) + \frac{\partial L}{\partial x_j} = 0 , \quad j = 1, 2, \cdots n.$$

Using the matrix quadratic forms given by equations (1a) through (1d), one obtains the following set of differential equations in matrix form:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + \Big[[K] - [P]\Big]\{X\} = \{f(t)\}. \tag{2}$$

The above equation is given by Newmark[8].

CHAPTER III

GENERAL SOLUTION OF THE EQUATIONS OF MOTION NEGLECTING THE EFFECT OF DAMPING

The undamped equations of motion are obtained by setting the matrix [C] = [0] in equation (2), and take the following matrix form:

$$[M]\{\ddot{X}\} + \Big[[K]-[P]\Big]\{X\} = \{f(t)\} \ . \tag{3}$$

Before the general closed-form solution of equation (3) is determined, one must first obtain the solutions to the following three equations:

1. the free vibration problem given by the matrix equation

$$[M]\{\ddot{X}\} + [K]\{X\} = \{0\} \ , \tag{4a}$$

2. the static stability problem given by the matrix equation

$$\Big[[K]-[P]\Big]\{X\} = \{0\} \ , \quad \text{and} \tag{4b}$$

3. the free vibration problem including the effect of axial force given by the matrix equation

$$[M]\{\ddot{X}\} + \Big[[K]-[P]\Big]\{X\} = \{0\}. \tag{4c}$$

3.1 SOLUTION OF THE FREE VIBRATION PROBLEM

Referring to equation (4a), the general solution is assumed to take the form

$$\{X\} = e^{i\omega t}\{u_f\} \ , \tag{5}$$

where ω is defined as natural frequency of free vibration and the vector $\{\mathcal{U}_f\}$ is defined as the associated eigenvector. Substituting equation (5) into equation (4a) yields

$$\left[-\omega^2[M]+[K]\right]\{\mathcal{U}_f\} = \{0\}\,, \tag{6}$$

which for non-trivial solutions of the vector $\{\mathcal{U}_f\}$ requires that,

$$\det\left[[K]-\omega^2[M]\right] = 0. \tag{7}$$

Equations (6) and (7) define the generalized eigenvalue-eigenvector problem as given by Hildebrand[5]. Equation (7) yields j values of the parameter ω_j^2, j = 1, 2, ····n. Corresponding to each value of ω_j^2, equation (6) yields a single eigenvector $\{\mathcal{U}_f\}_j$, j = 1, 2, ····n. Defining the matrix $[U_f]$ whose columns contain the eigenvectors j = 1, 2, ····n, it follows that

$$[U_f]^T[M][U_f] = [\Lambda_m]\,, \quad \text{and} \tag{8a}$$

$$[U_f]^T[K][U_f] = [\Lambda_{Kf}]\,, \tag{8b}$$

where $[\Lambda_m]$, and $[\Lambda_{Kf}]$ are diagonal matrices[1]. Referring to equation (6), it follows that,

$$[K][U_f] = [M][U_f][\Lambda_\omega]\,, \tag{9}$$

where $[\Lambda_\omega]$ is a diagonal matrix with terms ω_j^2, j = 1, 2, ····n. Premultiplying equation (9) by $[U_f]^T$ and noting equations (8a) and (8b), it follows that

$$[\Lambda_{Kf}] = [\Lambda_m][\Lambda_\omega] \tag{10}$$

3.2 SOLUTION OF THE STATIC STABILITY PROBLEM

Referring to equation (4b), the following identity is defined:

$$[P] \equiv p[\hat{P}], \tag{11}$$

where p is an arbitrary scalar constant. Using equation (11), equation (4b) is rewritten

$$\left[[K] - p[\hat{P}]\right]\{X\} = \{0\}. \tag{12}$$

For non-trivial solutions of the vector $\{X\}$ in equation (12), it follows that

$$\det\left[[K] - p[\hat{P}]\right] = 0. \tag{13}$$

Equations (12) and (13) again describe the generalized eigenvalue-eigenvector problem. Equation (13) yields $(p_{cr})_j$ values of the parameter $(p_{cr})_j$, j = 1, 2, ···n, where $(p_{cr})_j$ is defined as the critical buckling load in the jth mode shape. Corresponding to each value of the term $(p_{cr})_j$, j = 1, 2, ···n, equation (12) yields a single eigenvector $\{u_s\}_j$, j = 1, 2, ···n. Defining the matrix $[U_s]$ whose columns contain the eigenvector $\{u_s\}_j$, j = 1, 2, ···n, it follows that

$$[U_s]^T[K][U_s] = [\Lambda_{ks}], \quad \text{and} \tag{14a}$$

$$[U_s]^T[\hat{P}][U_s] = [\Lambda_p], \tag{14b}$$

where $[\Lambda_{ks}]$, and $[\Lambda_p]$ are diagonal matrices. Referring to equation (12), it follows that

$$[K][U_s] = [\hat{P}][U_s][\Lambda_{cr}], \tag{15}$$

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where $[\Lambda_{cr}]$ is a diagonal matrix with terms $(p_{cr})_j$, j = 1, 2, ⋯n. Premultiplying equation (15) by $[U_S]^T$ and noting equations (14a) and (14b), one obtains

$$[\Lambda_{KS}] = [\Lambda_p][\Lambda_{cr}] \; . \tag{16}$$

3.3 THE CONDITIONS FOR SIMULTANEOUS DIAGONALIZATION OF THE MASS, STIFFNESS, AND AXIAL FORCE MATRICES.

Referring to equations (9) and (15) the question is asked, "Under what condition does the matrix equality $[U_f] = [U_s]$ hold". The mathematical requirement is interpreted as follows, "What mathematical constraints must apply, if the eigenvectors of the free vibration problem are identical to the eigenvectors of static stability problem". Noting that the matrix product $[M]^{-1}[K]$ and $[\hat{P}]^{-1}[K]$ are non-symmetric, it follows that two non-symmetric matrices are simultaneously diagonalized by the same non-singular eigenvector matrix $[U]$ if and only if the matrix product commutes[6]. This condition takes the mathematical form

$$\left[[M]^{-1}[K]\right]\left[[\hat{P}]^{-1}[K]\right] = \left[[\hat{P}]^{-1}[K]\right]\left[[M]^{-1}[K]\right]. \tag{17}$$

Assuming the stiffness matrix [K] is non-singular, equation (17) reduces to the form

$$[M]^{-1}[K][\hat{P}]^{-1} = [\hat{P}]^{-1}[K][M]^{-1} \tag{18}$$

The form of equation (18) requires that simultaneously the mass, stiffness, and axial force matrices be non-singular. This condition holds only if the rigid body motions are neglected. For convenience, the following notation is defined:

$$[U_f] = [U_s] = [U] \ . \tag{19}$$

Also from equations (8b) and (14a),

$$[\Lambda_{Kf}] = [\Lambda_{KS}] = [\Lambda_K] \ . \tag{20}$$

3.4 GENERAL SOLUTION OF THE FREE VIBRATION PROBLEM INCLUDING THE EFFECT OF AXIAL FORCE

Referring to equation (4c), and making the substitution

$$\{X\} = [U]\{y\} \ , \tag{21}$$

it follows that,

$$[M][U]\{\ddot{y}\} + \left[[K][U] - p[\hat{P}][U]\right]\{y\} = \{0\} . \tag{22}$$

Premultiplying equation (22) by $[U]^T$ and noting equations (8a), (8b), (14a), (14b), (19), and (20), one obtains

$$[\Lambda_m]\{\ddot{y}\} + \left[[\Lambda_K] - p[\Lambda_p]\right]\{y\} = \{0\} . \tag{23}$$

Substituting equations (10), (16), and (20) into equation (23) and premultiplying the result by $[\Lambda_m]^{-1}$ yields

$$[I]\{\ddot{y}\} + [\Lambda_\Omega]\{y\} = \{0\} , \tag{24}$$

where,

$$[\Lambda_\Omega] \equiv [\Lambda_\omega]\left[[I] - p[\Lambda_{cr}]^{-1}\right] . \tag{25}$$

The matrix $[\Lambda_\Omega]$ is a diagonal matrix with Ω_j^2, j = 1, 2, $\cdots$n, where Ω_j is the natural frequency of free vibration including the effect of axial force.

3.5 GENERAL SOLUTION OF THE FORCED VIBRATION PROBLEM INCLUDING THE EFFECT OF AXIAL FORCE

Substituting equation (11) into equation (3), premultiplying the result by $[U]^T$, and noting equations (8a), (8b), (14a), (14b), (19), (20), and (21), yields

$$[\Lambda_m]\{\ddot{y}\} + \left[[\Lambda_K] - P[\Lambda_p]\right]\{y\} = [U]^T\{f(t)\} . \tag{26}$$

Substituting equations (10), (16), (20), and (25) into equation (26), it follows that,

$$[I]\{\ddot{y}\} + [\Lambda_\Omega]\{y\} = \{g(t)\} , \tag{27}$$

where,
$$\{g(t)\} = [\Lambda_m]^{-1}[U]^T\{f(t)\} . \tag{28}$$

The form of equation (26) represents a total uncoupling of equations of motion. The jth scalar equation of matrix equation (27) takes the form

$$\ddot{y}_j(t) + \Omega_j^2\, y_j(t) = g_j(t) . \tag{29}$$

Using Lagrange variation of parameters, the solution of equation (29) becomes

$$y_j(t) = a_j \cos\Omega_j t + \frac{b_j}{\Omega_j} \sin\Omega_j t + \frac{1}{\Omega_j}\int_{t'=0}^{t'=t} g_j(t') \sin\Omega_j(t-t')\,dt' . \tag{30}$$

Equation (30) is written in the matrix form as follows:

$$\{y\} = [A]\{a\} + [\Lambda_\Omega]^{-1/2}[B]\{b\} + [\Lambda_\Omega]^{-1/2}\int_{t'=0}^{t'=t}[E]\{g(t')\}dt', \tag{31}$$

where, matrices $[A]$, $[B]$, and $[E]$ are diagonal matrices with terms $\cos\Omega_j t$, $\sin\Omega_j t$, and $\sin\Omega_j(t-t')$ respectively. Substituting equation

(31) into equation (21) yields

$$\{X(t)\} = [U][A]\{a\} + [U][\Lambda_\Omega]^{-\frac{1}{2}}[B]\{b\} + [U][\Lambda\Omega]^{-1/2}\int_{t'=0}^{t'=t}[E]\{g(t')\,dt'. \quad (32)$$

Similarly, $\{\dot{X}(t)\}$ is written as follows:

$$\{\dot{X}(t)\} = -[U][\Lambda\Omega]^{\frac{1}{2}}[B]\{a\} + [U][A]\{b\} + [U]\int_{t'=0}^{t'=t}[F]\{g(t')\,dt' , \quad (33)$$

where, [F] is a diagonal matrix with terms $\cos\Omega_j(t-t')$.

Using the following prescribed initial conditions

$$\left.\begin{array}{ll} \text{(i) @ } t=0, & \{X(t)\}=\{X(0)\}, \text{ and} \\ \text{(ii) @ } t=0, & \{\dot{X}(t)\}=\{\dot{X}(0)\} \end{array}\right\} , \quad (34)$$

it follows that,

$$\{X(0)\} = [U]\{a\} , \quad \text{and}$$

$$\{\dot{X}(0)\} = [U]\{b\} .$$

Noting equation (8a) and (19), one obtains

$$\{a\} = [\Lambda_m]^{-1}[U]^T[M]\{X(0)\} , \quad \text{and} \quad (35a)$$

$$\{b\} = [\Lambda_m]^{-1}[U]^T[M]\{X(0)\} . \quad (35b)$$

The general solution of equation (3) then takes the following form:

$$\{X(t)\} = [U][A][\Lambda_m]^{-1}[U]^T[M]\{X(0)\} + [U][\Lambda\Omega]^{-\frac{1}{2}}[B][\Lambda_m]^{-1}[U][M]\{\dot{X}(0)\} + [U][\Lambda\Omega]^{-\frac{1}{2}}[\Lambda_m]^{-1}\int_{t'=0}^{t'=t}[E][U]^T\{f(t')\}\,dt' . \quad (36)$$

Equation (36) is investigated for the special case of the externally applied force (i.e. $\{f(t)\}$). Taking the initial condition as zero,

$\{X(0)\} = \{\dot{X}(0)\} = \{0\}$, and $\{f(t')\} = \{f(0)\}$, where the external forces are assumed as constants, equation (36) reduces to the form

$$\{X(t)\} = [U][\Lambda_\Omega]^{-1}[\Lambda_m]^{-1}\big[[I]-[A]\big][U]^T\{f(0)\}. \qquad (37)$$

Also, assuming the initial conditions as zero and the externally applied forces as harmonic variation of time in the form

$$\{f(t')\} = \begin{bmatrix} f_1 \sin\alpha_1 t' \\ f_2 \sin\alpha_2 t' \\ \cdot \\ \cdot \\ \cdot \\ f_n \sin\alpha_n t' \end{bmatrix}$$

it follows that, for steady state motion only, equation (36) reduces to the form

$$\{X(t)\} = [U][\Lambda_\Omega]^{-\frac{1}{2}}[\Lambda_m]^{-1}[\hat{U}]\{f(t)\} ,$$

where

$$[\hat{U}] = \begin{bmatrix} u_{11}\dfrac{\Omega_1}{\Omega_1^2-\alpha_1^2} & \cdot & \cdot & \cdot & u_{n1}\dfrac{\Omega_1}{\Omega_1^2-\alpha_n^2} \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ u_{1n}\dfrac{\Omega_n}{\Omega_n^2-\alpha_1^2} & \cdot & \cdot & \cdot & u_{nn}\dfrac{\Omega_n}{\Omega_n^2-\alpha_n^2} \end{bmatrix} \qquad (38)$$

If any of the impressed frequencies, $\alpha_1, \alpha_2, \cdots \alpha_n$ is equal to any of the natural frequencies, $\Omega_1, \Omega_2, \ldots \Omega_n$, then the resulting motion is unstable, that is at least one of the generalized displacements $x_j(t)$ takes on an infinite value.

3.6 NUMERICAL EXAMPLE OF FORCED VIBRATION PROBLEM NEGLECTING THE EFFECT OF VISCOUS DAMPING

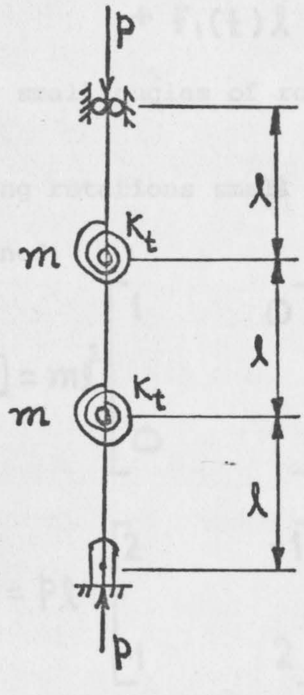

Fig. 3.1a
Initial Configuration

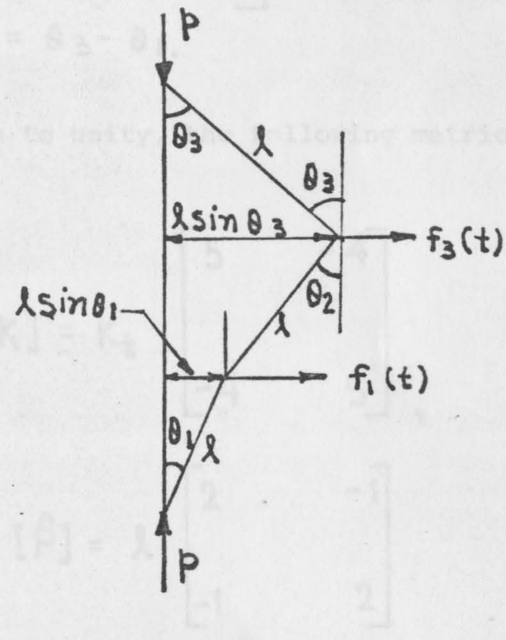

Fig. 3.1b
Displaced Configuration

The following three scalar functions are defined for the mathematical model shown above:

$$\left.\begin{aligned} T &= \tfrac{1}{2}\, m(\ell\dot{\theta}_3)^2 + \tfrac{1}{2}\, m(\ell\dot{\theta}_1)^2, \\ V_i &= \tfrac{1}{2} K_t(\theta_2-\theta_1)^2 + \tfrac{1}{2}K_t(\theta_2+\theta_3)^2, \\ V_e &= P(3\ell - \ell\cos\theta_1 - \ell\cos\theta_2 - \ell\cos\theta_3) \\ &\quad + f_1(t)\ell\sin\theta_1 + f_3(t)\ell\sin\theta_3, \end{aligned}\right] \tag{39}$$

where for small angles of rotation $\theta_2 = \theta_3 - \theta_1$.

Considering rotations small in comparison to unity, the following matrices are obtained:

$$[M] = m\ell^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad [K] = K_t\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix},$$

$$[P] = P\ell\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}, \qquad \text{where } [\hat{P}] = \ell\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$$

$$\{f(t)\} = \begin{bmatrix}\ell f_1(t)\\ \ell f_3(t)\end{bmatrix}, \quad \text{and} \quad \{X(t) = \begin{bmatrix}\theta_1(t)\\ \theta_3(t)\end{bmatrix}.$$

From equation (18), it follows that

$$[M]^{-1}[K][\hat{P}]^{-1} = [\hat{P}]^{-1}[K][M]^{-1} = \frac{K_t}{m\ell^4}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}.$$

Carrying out the eigenvalue - eigenvector problem, one obtains

$$[U_f] = [U_s] = [U] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

where $[U]$ is an orthonormal matrix.

In addition, the following diagonal matrices are obtained:

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{K_t}{\ell} & 0 \\ 0 & \frac{3K_t}{\ell} \end{bmatrix}, \qquad [\Lambda_m] = m\ell^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$[\Lambda_\omega] = \begin{bmatrix} \frac{K_t}{m\ell^2} & 0 \\ 0 & \frac{9K_t}{m\ell^2} \end{bmatrix}, \quad \text{and } [\Lambda_\Omega] = \begin{bmatrix} \frac{K_t}{m\ell^2}(1-\frac{P\ell}{K_t}) & 0 \\ 0 & \frac{9K_t}{m\ell^2}(1-\frac{P\ell}{3K_t}) \end{bmatrix}$$

For special case of constant external transverse forces

$$\{f(t)\} = \begin{bmatrix} \ell f_1(0) \\ \ell f_3(0) \end{bmatrix} = \{f(0)\},$$

and neglecting the effect of initial conditions, equation (37) yields

$$\begin{bmatrix} \theta_1(t) \\ \\ \theta_3(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{K_t(1-\frac{P\ell}{K_t})} & 0 \\ \\ 0 & \dfrac{1}{3K_t(3-\frac{P\ell}{K_t})} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{(1-\cos\Omega_1 t)}{K_t(1-\frac{P\ell}{K_t})} & 0 \\ \\ 0 & \dfrac{(1-\cos\Omega_2 t)}{3K_t(3-\frac{P\ell}{K_t})} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \ell f_1(0) \\ \\ \ell f_3(0) \end{bmatrix}.$$

Defining the external forcing function in the form

$$\{f(t)\} = \begin{bmatrix} \ell f_1 \sin\alpha_1 t \\ \\ \ell f_3 \sin\alpha_2 t \end{bmatrix},$$

and neglecting initial conditions, equation (38) for steady state motion becomes

$$\begin{bmatrix} \theta_1(t) \\ \\ \theta_3(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{K_t(1-\frac{P\ell}{K_t})} & 0 \\ \\ 0 & \dfrac{1}{3K_t(3-\frac{P\ell}{K_t})} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\Omega_1}{\Omega_1^2-\alpha_1^2} & \dfrac{\Omega_1}{\Omega_1^2-\alpha_2^2} \\ \\ \dfrac{\Omega_2}{\Omega_2^2-\alpha_1^2} & -\dfrac{\Omega_2}{\Omega_2^2-\alpha_2^2} \end{bmatrix} \begin{bmatrix} \ell f_1 \sin\alpha_1 t \\ \\ \ell f_3 \sin\alpha_2 t \end{bmatrix}.$$

CHAPTER IV

GENERAL SOLUTION OF THE EQUATIONS OF MOTION INCLUDING THE EFFECTS OF AXIAL FORCE AND VISCOUS DAMPING

Premultiplying equation (2) by $[U]^T$ and noting equations (8a), (8b), (11), (14a), (14b), (19), (20), and (21), one obtains

$$[\Lambda_m]\{\ddot{y}\} + [\hat{C}]\{\dot{y}\} + \big[[\Lambda_K] - P[\Lambda_p]\big]\{y\} = [U]^T\{f(t)\}, \tag{40}$$

where

$$[\hat{C}] = [U]^T[C][U]. \tag{41}$$

Substituting equations (10), (16), and (20) into equation (40) and premultiplying the result by $[\Lambda_m]^{-1}$ yields

$$[I]\{\ddot{y}\} + [\Lambda_m]^{-1}[\hat{C}]\{\dot{y}\} + [\Lambda\Omega]\{y\} = \{g(t)\}, \tag{42}$$

where $\{g(t)\}$ is defined by equation (28). Equation (42) is rewritten in the following partitioned matrix form:

$$[\overset{*}{R}]\{\dot{\overset{*}{y}}\} + [\overset{*}{S}]\{\overset{*}{y}\} = \{\overset{*}{g}(t)\}, \tag{43}$$

$$\overset{*}{[S]} = \begin{bmatrix} -[I] & [0] \\ [0] & [\Lambda\Omega] \end{bmatrix},$$

$$\{\overset{*}{\dot{y}}\} = \begin{bmatrix} \{\dot{y}_2\} \\ \{\dot{y}_1\} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \dot{y}_{2n} \\ \dot{y}_{2n-1} \\ \vdots \\ \dot{y}_{n+1} \end{bmatrix} \\ \begin{bmatrix} \dot{y}_n \\ \dot{y}_{n-1} \\ \vdots \\ \dot{y}_1 \end{bmatrix} \end{bmatrix}, \text{ and}$$

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of equation (43) is assumed as

$$e^{qt}\{\overset{*}{u}\}, \qquad (44)$$

eigenvalue and the vector

tor. Substituting equation (44)

ion (43) yields

$$\} = \{\overset{*}{0}\}, \qquad (45)$$

which for non-trivial solution of the vector $\{\overset{*}{u}\}$ requires that,

$$\det\left(q[\overset{*}{R}] + [\overset{*}{S}]\right) = 0. \tag{46}$$

Equation (46) yields 2n values of the parameter q_j, $j = 1,2, \cdots 2n$. Corresponding to each value of q_j equation (45) yields a single eigenvector $\{\overset{*}{u}_j\}$, $j = 1,2, \cdots 2n$. The matrix $[\overset{*}{U}]$ is defined as the one whose columns contain the eigenvectors $\{\overset{*}{u}_j\}$. The first n columns contain the eigenvectors $\{u_{2n}\} \ldots \{u_{n+1}\}$, the remaining n columns contain the additional n eigenvectors $\{u_n\} \ldots \{u_1\}$. Noting the partitioned form of equation (45), it follows that,

$$\{\overset{*}{u}_j\} = \begin{bmatrix} \{u_2\} \\ \\ \{u_1\} \end{bmatrix} = \begin{bmatrix} q\{u_1\} \\ \\ \{u_1\} \end{bmatrix}. \tag{47}$$

For convenience the matrix $[\overset{*}{U}]$ is defined in the following partitioned matrix form:

$$[\overset{*}{U}] = \begin{bmatrix} [U_2][\Lambda_2] & [U_1][\Lambda_1] \\ \\ [U_2] & [U_1] \end{bmatrix}, \tag{48}$$

where the matrix $[U_2]$ contains the eigenvectors $\{\mathcal{U}_j\}$, j = n + 1, n + 2, ⋯2n, and the matrix $[U_1]$ contains the eigenvectors $\{\mathcal{U}_j\}$, j = n, n - 1, ⋯1. Also, the eigenvalue matrices are defined as follows:

$$[\Lambda_2] = \begin{bmatrix} q_{2n} & 0 & \cdots & 0 \\ 0 & q_{2n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n+1} \end{bmatrix}, \quad [\Lambda_1] = \begin{bmatrix} q_n & 0 & \cdots & 0 \\ 0 & q_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_1 \end{bmatrix} \tag{49}$$

Noting equations (48), and (49), equation (45) is rewritten in the following partitioned matrix form:

$$[\overset{*}{S}][\overset{*}{U}] + [\overset{*}{R}][\overset{*}{U}][\overset{*}{\Lambda}] = [\overset{*}{O}] \tag{50}$$

where
$$[\overset{*}{\Lambda}] = \begin{bmatrix} [\Lambda_2] & [0] \\ [0] & [\Lambda_1] \end{bmatrix}. \tag{51}$$

Since $[\overset{*}{U}]$ simultaneously diagonalizes both $[\overset{*}{S}]$ and $[\overset{*}{R}]$, the following matrix equations hold:

$$[\overset{*}{U}]^T[\overset{*}{S}][\overset{*}{U}] = [\overset{*}{\Lambda}_S], \tag{52}$$

and
$$[\overset{*}{U}]^T[\overset{*}{R}][\overset{*}{U}] = [\overset{*}{\Lambda}_R], \tag{53}$$

where $[\overset{*}{\Lambda}_S]$ and $[\overset{*}{\Lambda}_R]$ are diagonal, partitioned matrices defined as follows:

$$[\overset{*}{\Lambda}_S] = \begin{bmatrix} [\Lambda_{S2}] & [0] \\ [0] & [\Lambda_{S1}] \end{bmatrix}, \quad [\overset{*}{\Lambda}_R] = \begin{bmatrix} [\Lambda_{R2}] & [0] \\ [0] & [\Lambda_{R1}] \end{bmatrix}.$$

Noting equations (50), (52), and (53), it follows that,

$$[\Lambda_S] = -[\overset{*}{\Lambda}_R][\overset{*}{\Lambda}] \tag{54}$$

4.2 SOLUTION OF THE FORCED VIBRATION PROBLEM

The non-homogeneous solution of equation (43) is assumed as

$$\{\overset{*}{y}(t)\} = [\overset{*}{U}]\{\overset{*}{Z}(t)\}, \tag{55}$$

where

$$\{\overset{*}{Z}\} = \begin{bmatrix} \{Z_2\} \\ \\ \{Z_1\} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} Z_{2n} \\ Z_{2n-1} \\ \vdots \\ Z_{n+1} \end{bmatrix} \\ \hline \begin{bmatrix} Z_n \\ Z_{n-1} \\ \vdots \\ Z_1 \end{bmatrix} \end{bmatrix}.$$

Substituting equation (55) into equation (43), and premultiplying the result by $[U]^T$, and noting equations (52), (53), (54), one obtains

$$[I]\{\dot{\overset{*}{Z}}\} - [\overset{*}{\Lambda}]\{\overset{*}{Z}\} = \{\overset{*}{h}(t)\}\ , \tag{56}$$

where

$$\{\overset{*}{h}(t)\} = [\overset{*}{\Lambda}_R]^{-1}[\overset{*}{\hat{U}}]^T\{\overset{*}{g}(t)\}\ . \tag{57}$$

The form of equation (56) represents a total uncoupling of equations of motion. The jth scalar equation of matrix equation (56) takes the following form:

$$\dot{Z}_j - q_j Z_j = h_j(t)\ ,\quad j = 1, 2, \cdots 2n. \tag{58}$$

The homogeneous solution of equation (58) is assumed as

$$Z_j = a_j e^{\alpha_j t}\ . \tag{59}$$

Substituting equation (59) into equation (58), and using Lagrange variation of parameters, the solution of equation (58) becomes

$$Z_j(t) = e^{q_j t} a_j(0) + \int_{t'=0}^{t'=t} h_j(t')\, e^{q_j(t-t')}\, dt'\ . \tag{60}$$

Equation (60) is rewritten in the following partitioned matrix form:

$$\{\overset{*}{Z}\} = [\exp.[\overset{*}{\Lambda} t]]\{\overset{*}{a}(0)\} + \int_{t'=0}^{t'=t} [\exp.[\overset{*}{\Lambda}(t-t')]]\{\overset{*}{h}(t')\}\, dt' \tag{61}$$

where

$$\{\overset{*}{a}(0)\} = \begin{bmatrix} \{a_2(0)\} \\ \\ \{a_1(0)\} \end{bmatrix} . \tag{62}$$

Substituting equation (55) into equation (61) yields

$$\{\overset{*}{y}(t)\} = [\overset{*}{U}][\exp.[\overset{*}{\Lambda} t]]\{\overset{*}{a}(0)\} + [\overset{*}{U}]\int_{t'=0}^{t'=t}[\exp.[\Lambda(t-t')]]\{\overset{*}{h}(t')\}\,dt' . \quad (63)$$

From equation (21), it follows that,

$$\{\overset{*}{X}\} = [\overset{*}{\hat{U}}]\{\overset{*}{y}\} , \quad (64)$$

where

$$\{\overset{*}{X}\} = \begin{bmatrix} \{\dot{X}\} \\ \\ \{X\} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \dot{x}_n \\ \dot{x}_{n-1} \\ \vdots \\ \dot{x}_1 \end{bmatrix} \\ \hline \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix} \end{bmatrix}, \quad [\overset{*}{\hat{U}}] = \begin{bmatrix} [U] & [0] \\ & \\ [0] & [U] \end{bmatrix} . \quad (65)$$

Substituting equation (63) into equation (64) yields

$$\{\overset{*}{X}\} = [\overset{*}{\hat{U}}][\overset{*}{U}][\exp.[\overset{*}{\Lambda} t]\{\overset{*}{a}(0)\} + [\overset{*}{\hat{U}}][\overset{*}{U}]\int_{t'=0}^{t'=t}[\exp.[\overset{*}{\Lambda}(t-t')]]\{\overset{*}{h}(t')\}\,dt' . \quad (66)$$

Using the following prescribed initial conditions

$$@\ t = 0 , \quad \{\overset{*}{X}(t)\} = \{\overset{*}{X}(0)\} ,$$

it follows that,

$$\{\overset{*}{X}(0)\} = [\overset{*}{\hat{U}}][\overset{*}{U}]\{\overset{*}{a}(0)\} , \quad (67)$$

or

$$\{\overset{*}{a}(0)\} = [\overset{*}{U}]^{-1}[\overset{*}{\hat{U}}]^{-1}\{\overset{*}{X}(0)\} . \quad (68)$$

Noting the form of the matrix $[\overset{*}{U}]^{-1}$ from equation (53), it follows that,

$$\{\overset{*}{a}(0)\} = [\overset{*}{\Lambda}_R]^{-1}[\overset{*}{U}]^T[\overset{*}{R}][\overset{*}{\hat{U}}]^{-1}\{\overset{*}{X}(0)\} . \tag{69}$$

Substituting equation (69) into equation (66) and noting (57), one obtains

$$\{\overset{*}{X}\} = [\overset{*}{\hat{U}}][\overset{*}{U}][\exp[\overset{*}{\Lambda}t]][\overset{*}{\Lambda}_R]^{-1}[\overset{*}{U}]^T[\overset{*}{R}][\overset{*}{\hat{U}}]^{-1}\{X(0)\}$$

$$+ [\overset{*}{\hat{U}}][\overset{*}{U}]\int_{t'=0}^{t'=t}[\exp[\overset{*}{\Lambda}(t-t')]][\overset{*}{\Lambda}_R]^{-1}[\overset{*}{U}]^T\{\overset{*}{g}(t')\}dt' , \tag{70}$$

where

$$[\overset{*}{\hat{U}}]^{-1} = \begin{bmatrix} [\Lambda_m]^{-1}[U]^T[M] & [0] \\ [0] & [\Lambda_m]^{-1}[U]^T[M] \end{bmatrix} .$$

Equation (70) represents the general solution of the equation (2). Recasting equation (70) in partitioned matrix form yields

$$
\begin{bmatrix} \{\dot{X}\} \\ \\ \{X\} \end{bmatrix} = \begin{bmatrix} [U] & [0] \\ \\ [0] & [U] \end{bmatrix} \begin{bmatrix} [U_2][\Lambda_2] & [U_1][\Lambda_1] \\ \\ [U_2] & [U_1] \end{bmatrix}
$$

$$
\begin{bmatrix} [\exp[\Lambda_2 t]] & [0] \\ \\ [0] & [\exp[\Lambda_1 t]] \end{bmatrix} \begin{bmatrix} [\Lambda_{R2}]^{-1} & [0] \\ \\ [0] & [\Lambda_{R1}]^{-1} \end{bmatrix}
$$

$$
\begin{bmatrix} [\Lambda_2][U_2]^T & [U_2]^T \\ \\ [\Lambda_1][U_1]^T & [U_1]^T \end{bmatrix} \begin{bmatrix} [0] & [I] \\ \\ [I] & [\Lambda_m]^{-1}[\hat{C}] \end{bmatrix}
$$

$$
\begin{bmatrix} [\Lambda_m]^{-1}[U]^T[M] & \\ \\ [0] & [\Lambda_m]^{-1}[U]^T[M] \end{bmatrix} \begin{bmatrix} \{\dot{X}(0)\} \\ \\ \{X(0)\} \end{bmatrix}
$$

$$
+ \int_{t'=0}^{t'=t} \begin{bmatrix} [\exp[\Lambda_2(t-t')]] & [0] \\ \\ [0] & [\exp[\Lambda_1(t-t')]] \end{bmatrix} \begin{bmatrix} [\Lambda_{R2}]^{-1} & [0] \\ \\ [0] & [\Lambda_{R1}]^{-1} \end{bmatrix}
$$

$$
\begin{bmatrix} [\Lambda_2][U]^T & [U_2]^T \\ \\ [\Lambda_1][U_1]^T & [U_1]^T \end{bmatrix} \begin{bmatrix} \{0\} \\ \\ \{g(t')\} \end{bmatrix} dt'. \qquad (71)
$$

Simplification of equation (71) yields the following displacement vector $\{X\}$ in the form:

$$
\begin{aligned}
\{X\} = {} & [U][U_2][\exp.[\Lambda_2 t]] \Big[[\Lambda_{R2}]^{-1}[U_2]^T[\Lambda_m]^{-1}[U]^T[M]\{\dot{X}(0)\} \\
& + [\Lambda_{R2}]^{-1}[\Lambda_2][U_2]^T[\Lambda_m]^{-1}[U]^T[M]\{X(0)\} \\
& + [\Lambda_{R2}]^{-1}[U_2]^T[\Lambda_m]^{-1}[\hat{C}][\Lambda_m]^{-1}[U]^T[M]\{X(0)\} \Big] \\
& + [U][U_1][\exp.[\Lambda_1 t]] \Big[[\Lambda_{R1}]^{-1}[U_1]^T[\Lambda_m]^{-1}[U]^T[M]\{\dot{X}(0)\} \\
& + [\Lambda_{R1}]^{-1}[\Lambda_1][U_1]^T[\Lambda_m]^{-1}[U]^T[M]\{X(0)\} \\
& + [\Lambda_{R1}]^{-1}[U_1]^T[\Lambda_m]^{-1}[\hat{C}][\Lambda_m]^{-1}[U]^T[M]\{X(0)\} \Big] \\
& + [U] \int_{t'=0}^{t'=t} [U_2][\exp.[\Lambda_2(t-t')]][\Lambda_{R2}]^{-1}[U_2]^T[\Lambda_m]^{-1}[U]^T\{g(t')\} \\
& + [U_1]\big[\exp.[\Lambda_1(t-t')]\big][\Lambda_{R1}]^{-1}[U_1]^T[\Lambda_m]^{-1}[U]^T\{f(t')\}\,dt'. \qquad (72)
\end{aligned}
$$

4.3 THE SPECIAL CASE OF AN UNDERDAMPED SYSTEM

For an underdamped system, the 2n eigenvalues and the corresponding eigenvectors appear as n complex conjugate pairs, that is, the following conjugate matrices are defined:

$$
\begin{aligned}
& [\Lambda_2] = [\overleftrightarrow{\Lambda}_1], \quad [U_2] = [\overleftrightarrow{U}_1], \\
& [\Lambda_{R2}] = [\overleftrightarrow{\Lambda}_{R1}], \text{ and } [\Lambda_{S2}] = [\overleftrightarrow{\Lambda}_{S1}], \qquad (73)
\end{aligned}
$$

where the symbol $(\leftrightarrow)$ denotes the complex conjugate form.

Noting equations (73), the following matrix reductions are noted:

$$[\overset{*}{U}] = \begin{bmatrix} [\tilde{U}_1][\tilde{\Lambda}_1] & [U_1][\Lambda_1] \\ [\tilde{U}_1] & [U_1] \end{bmatrix}, \quad [\overset{*}{\Lambda}] = \begin{bmatrix} [\tilde{\Lambda}_1] & [0] \\ [0] & [\Lambda_1] \end{bmatrix},$$

$$[\overset{*}{\Lambda}_R] = \begin{bmatrix} [\tilde{\Lambda}_{R1}] & [0] \\ [0] & [\Lambda_{R1}] \end{bmatrix}, \text{ and } [\overset{*}{\Lambda}_S] = \begin{bmatrix} [\tilde{\Lambda}_{S1}] & [0] \\ [0] & [\Lambda_{S2}] \end{bmatrix}. \qquad (74)$$

Noting equations (73) and (74), one can separate the following complex matrices into their real and imaginary parts in the following form:

$$\begin{aligned} &[U_1] = [V] + i[W], \quad [\Lambda_1] = [G] + i[H], \quad [\Lambda_{R1}] = [G_r] + i[H_r] \\ &[\tilde{U}_1] = [V] - i[W], \quad [\tilde{\Lambda}_1] = [G] - i[H], \quad [\tilde{\Lambda}_{R1}] = [G_r] - i[H_r]. \end{aligned} \qquad (75)$$

Substituting equation (75) into equation (72) and noting equation (73), one obtains

$$
\begin{aligned}
\{X(t)\} = {} & 2[U]\Big[[V][\exp.[Gt]][\hat{A}] - [W][\exp.[Gt]][\hat{B}]\Big] \\
& \Big[\big[[G_r]^{-1}[V]^T - [H_r]^{-1}[W]^T\big][\Lambda_m]^{-1}[U]^T[M]\{\dot{X}(0)\} \\
& + \big[[G_r]^{-1}[G][V]^T - [H_r]^{-1}[H][V]^T - [H_r]^{-1}[G][W]^T - [G_r]^{-1}[H][W]^T\big] \\
& \qquad [\Lambda_m]^{-1}[U]^T[M]\{X(0)\} \\
& + \big[[G_r]^{-1}[V]^T - [H_r]^{-1}[W]^T\big][\Lambda_m]^{-1}[\hat{C}][\Lambda_m]^{-1}[U]^T[M]\{X(0)\}\Big] \\
& - 2[U]\Big[[W][\exp.[Gt]][\hat{A}] + [V][\exp.[Gt]][\hat{B}]\Big] \\
& \Big[\big[[H_r]^{-1}[V]^T + [G_r]^{-1}[W]^T\big][\Lambda_m]^{-1}[U]^T[M]\{\dot{X}(0)\} \\
& - \big[[H_r]^{-1}[G][V]^T + [G_r]^{-1}[H][V]^T + [G_r]^{-1}[G][W]^T - [H_r]^{-1}[H][W]^T\big] \\
& \qquad [\Lambda_m]^{-1}[U][M]\{X(0)\} \\
& - \big[[H_r]^{-1}[V]^T + [G_r]^{-1}[W]^T\big][\Lambda_m]^{-1}[\hat{C}][\Lambda_m]^{-1}[U]^T[M]\{X(0)\}\Big] \\
& + 2[U]\int_{t'=0}^{t'=t}\Big[\big[[V][\exp.[G(t-t')]][\hat{D}] - [W][\exp.[G(t-t')]][\hat{E}]\big] \\
& \qquad \big[[G_r]^{-1}[V]^T - [H_r]^{-1}[W]^T\big] \\
& - \big[[W][\exp.[G(t-t')]][\hat{D}] + [V][\exp.[G(t-t')]][\hat{E}]\big] \\
& \qquad \big[[H_r]^{-1}[V]^T + [G_r]^{-1}[W]^T\big]\Big][\Lambda_m]^{-1}[U]^T\{f(t')\}\,dt'.
\end{aligned}
\tag{76}
$$

The algebraic operations from which equation (76) is obtained from equation (72) is shown in Appendix A. In equation (76), $[\hat{A}]$, $[\hat{B}]$, $[\hat{D}]$, and $[\hat{E}]$ are diagonal matrices with terms $\cos(s_j t)$, $\sin(s_j t)$, $\cos(s_j(t-t'))$, and $\sin(s_j(t-t'))$, respectively. Also, the matrices $[\exp[Gt]]$, and $[\exp[G(t-t')]]$ are diagonal matrices with terms $e^{r_j t}$, and $e^{r_j(t-t')}$, respectively. The terms r_j and s_j are the real and imaginary parts of the eigenvalues described as follows:

$$q_j = r_j + i s_j \quad , \text{and} \quad \bar{q}_j = r_j - i s_j .$$

Equation (76) represents the general solution of equation (2) for the special case of an underdamped system. This complex form of the equation of motion is reduced to the special case when damping is omitted, (i.e. [C] = [O]) by noting the following simplifications:

$$\left.\begin{aligned}
&[V] = [I] \ , \quad [W] = [O] \ , \\
&[G_r]^{-1} = [O] \ , \quad [H_r]^{-1} = \tfrac{1}{2}[\Lambda\Omega]^{-\frac{1}{2}} , \\
&[H] = -[\Lambda\Omega]^{\frac{1}{2}} , \quad [G] = [O] , \\
&r_j = 0, \text{and } q_j = i s_j = -i\Omega_j
\end{aligned}\right] \quad (77)$$

Substituting equation (77) into equation (76), one obtains

$$\begin{aligned}
\{X(t)\} = {} & [U][\hat{A}][U]^T[M]\{X(0)\} \\
& - [U][\Lambda\Omega]^{-1}[\hat{B}][\Lambda_m]^{-1}[U]^T[M]\{\dot{X}(0)\} \\
& - [U][\Lambda\Omega]^{-\frac{1}{2}}[\Lambda_m]^{-1}\int_{t'=0}^{t'=t}[\hat{E}][U]^T\{f(t')\}dt' . \quad (78)
\end{aligned}$$

Noting equation (77), it follows that,

$$[\hat{A}] = [A]\,,\ [\hat{B}] = -[B],\ \text{and}\ [\hat{E}] = -[E]\,, \tag{79}$$

where, matrices [A], [B], and [E] are defined in the article 3.5. Combination of equations (78) and (79) yields equation (36) in the article 3.5.

Equation (78) is investigated for the special case of the externally applied force (i.e. $\{f(t)\}$). Taking the initial condition as zero, $\{X(0)\} = \{\dot{X}(0)\} = \{0\}$, and $\{f(t')\} = \{f(0)\}$ where the external forces are assumed as constants, equation (78) reduces to the form:

$$\begin{aligned}\{X(t)\} = 2[U]\Big[\Big[[V]\big[[\hat{G}][-[I] + [\hat{A}][\exp.[Gt]]] + [\hat{H}][\hat{B}][\exp.[Gt]]\big]\\ -[W]\big[[\hat{H}][[I] - [\hat{A}][\exp.[Gt]]] + [\hat{G}][\hat{B}][\exp.[Gt]]\big]\\ \big[[G_r]^{-1}[V]^T - [H_r]^{-1}[W]^T\big]\\ -\big[[W]\big[[\hat{G}][-[I] + [\hat{A}][\exp.[Gt]]] + [\hat{H}][\hat{B}][\exp.[Gt]]\big]\\ +[V]\big[[\hat{H}][[I] - [\hat{A}][\exp.[Gt]]] + [\hat{G}][\hat{B}][\exp.[Gt]]\big]\\ \big[[H_r]^{-1}[V]^T + [G_r]^{-1}[W]^T\big]\Big][\Lambda_m]^{-1}[U]^T\{f(0)\}\,,\end{aligned} \tag{80}$$

where $[\hat{G}]$ and $[\hat{H}]$ are diagonal matrices with terms $\frac{r_j}{r_j^2 + s_j^2}$ and $\frac{s_j}{r_j^2 + s_j^2}$ respectively, j = 1,2, ···n. Equation (80) reduces to the special case when damping is omitted (i.e. [C] = [0]) by noting the following simplifications:

$$[\hat{G}] = [0]\,,\ \text{and}\ \ [\hat{H}] = -[\Lambda\Omega]^{-\frac{1}{2}}\,. \tag{81}$$

Combination of equations (77), (79), and (81) yields equation (37) in the article 3.5.

Also, assuming the initial conditions as zero and the externally applied forces as harmonic variation of time in the form

$$\{f(t')\} = \begin{bmatrix} f_1 \sin \alpha_1 t' \\ f_2 \sin \alpha_2 t' \\ \cdot \\ \cdot \\ \cdot \\ f_n \sin \alpha_n t' \end{bmatrix},$$

equation (76) reduces to the form:

$$\begin{aligned}
\{X(t)\} = 2[U]\Big[& [V][[F_1]+[F_2]][J][-[L][Q]-[N][T]+[\text{exp.}[Gt]][N] \\
& + 2[\text{exp.}[G(t)]][\hat{N}][\hat{B}]]\{f_n\} \\
& -[W][[F_1]+[F_2]][J][[\hat{L}][Q]+2[\hat{N}][T]+[\text{exp.}[Gt]][N][\hat{B}] \\
& -2[\text{exp.}[Gt]][\hat{A}]]\{f_n\} \\
& -[W][[F_3]+[F_4]][J][-[L][Q]-[N][T]+[\text{exp.}[Gt]][N] \\
& + 2[\text{exp.}[Gt]][\hat{N}][\hat{B}]]\{f_n\} \\
& -[V][[F_3]+[F_4]][J][[\hat{L}][Q]+2[\hat{N}][T]+[\text{exp.}[Gt]][N][\hat{B}] \\
& -2[\text{exp.}[Gt]][\hat{N}][\hat{A}]]\{f_n\}\Big] . \qquad (82)
\end{aligned}$$

In equation (82), the following diagonal matrices are defined as under

[J] is a matrix containing terms $\dfrac{1}{(\gamma_j^2 - s_j^2 + \alpha_j^2) + 4\gamma_j^2 s_j^2}$

[L] is a matrix containing terms $\gamma_j(\gamma_j^2 + s_j^2 + \alpha_j^2)$

[N] is a matrix containing terms $(\gamma_j^2 - s_j^2 + \alpha_j^2)$

$[\hat{L}]$ is a matrix containing terms $s_j(\gamma_j^2 + s_j^2 - \alpha_j^2)$

$[\hat{N}]$ is a matrix containing terms $\gamma_j s_j$

[Q] is a matrix containing terms $\sin \alpha_j t$

[T] is a matrix containing terms $\cos \alpha_j t$

$\{f_n\}$ is a column matrix containing terms $f_1, f_2, \cdots f_n$.

In addition the following matrices are defined:

$$[F_1] = \begin{bmatrix} (g_{r_1}^{-1} v_{11} - h_{r_1}^{-1} w_{11})\hat{m}_1^{-1} u_{11} & \cdot & \cdot & \cdot & (g_{r_1}^{-1} v_{n1} - h_{r_1}^{-1} w_{n1})\hat{m}_n^{-1} u_{nn} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ (g_{r_n} v_{nn} - h_n^{-1} w_{nn})\hat{m}_n^{-1} u_{1n} & \cdot & \cdot & \cdot & (g_{r_n}^{-1} v_{1n} - h_{r_n}^{-1} w_{1n})\hat{m}_1^{-1} u_{n1} \end{bmatrix},$$

$$[F_2] = \begin{bmatrix} (g_{r_1}^{-1} v_{nn} - h_{r_n} w_{nn})\hat{m}_n^{-1} u_{1n} & \cdot & \cdot & \cdot & (g_{r_1}^{-1} v_{11} - h_{r_1}^{-1} w_{11})\hat{m}_1^{-1} u_{n1} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ (g_{r_n}^{-1} v_{nn} - h_{r_n}^{-1} w_{nn})\hat{m}_n^{-1} u_{1n} & \cdot & \cdot & \cdot & (g_{r_n}^{-1} v_{1n} - h_{r_n}^{-1} w_{1n})\hat{m}_1^{-1} u_{n1} \end{bmatrix},$$

$$[F_3] = \begin{bmatrix} (h_{r_1}^{-1} v_{11} + g_{r_1}^{-1} w_{11}) \hat{m}_1^{-1} u_{11} & \cdot & \cdot & \cdot & (h_{r_1}^{-1} v_{n1} + g_{r_1}^{-1} w_{n1}) \hat{m}_n^{-1} u_{nn} \\ \cdot & & \cdot & & \cdot \\ & & & & \cdot \\ \cdot & & \cdot & & \cdot \\ (h_{r_n}^{-1} v_{1n} + g_{r_n}^{-1} w_{1n}) \hat{m}_1^{-1} u_{11} & \cdot & \cdot & \cdot & (h_{r_n}^{-1} v_{nn} + g_{r_n}^{-1} w_{nn}) \hat{m}_n^{-1} u_{nn} \end{bmatrix},$$

$$[F_4] = \begin{bmatrix} (h_{r_1}^{-1} v_{n1} + g_{r_1}^{-1} w_{n1}) \hat{m}_n^{-1} u_{1n} & \cdot & \cdot & \cdot & (h_{r_1}^{-1} v_{11} + g_{r_1}^{-1} w_{11}) \hat{m}_1^{-1} u_{n1} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ (h_{r_n}^{-1} v_{nn} + g_{r_n}^{-1} w_{nn}) \hat{m}_n^{-1} u_{1n} & \cdot & \cdot & \cdot & (h_{r_n}^{-1} v_{1n} + g_{r_n}^{-1} w_{1n}) \hat{m}_1^{-1} u_{n1} \end{bmatrix},$$

where $g_{r_j}^{-1}$, $h_{r_j}^{-1}$, $\hat{m}_j^{-1}$, v_{ij}, w_{ij}, and u_{ij} are the terms of the matrices $[G_r]^{-1}$, $[H_r]^{-1}$, $[\Lambda_m]^{-1}$, $[V]$, $[W]$, and $[U]$, respectively. Equation (82) reduces to the special case when damping is omitted (i.e. $[C] = [0]$) by noting equations (77), and (79), and noting the following simplifications:

$$[F_1] = [F_2] = [L] = [\hat{N}] = [0]. \tag{83}$$

Substituting equation (83) into equation (82) and noting equation (79) yields

$$\{X(t)\} = 2[U][J]\big[[F_3][F_4]\big]\big[[\hat{L}][Q] + [N][B]\big]\{f_n\}. \tag{84}$$

For steady state motion (i.e. [B] = [0]), and noting the following simplifications:

$$\left.\begin{aligned} \left[[F_3]+[F_4]\right][J][\hat{L}] &= \tfrac{1}{2}\,[\Lambda\Omega]^{-\frac{1}{2}}[\Lambda_m]^{-1}[\hat{U}], \\ \text{and} \qquad [Q]\{f_n\} &= \{f(t)\} \end{aligned}\right], \qquad (85)$$

equation (84) reduces to the equation (38) in the article 3.5.

4.4 NUMERICAL EXAMPLE OF FORCED VIBRATION PROBLEM INCLUDING THE EFFECTS OF VISCOUS DAMPING

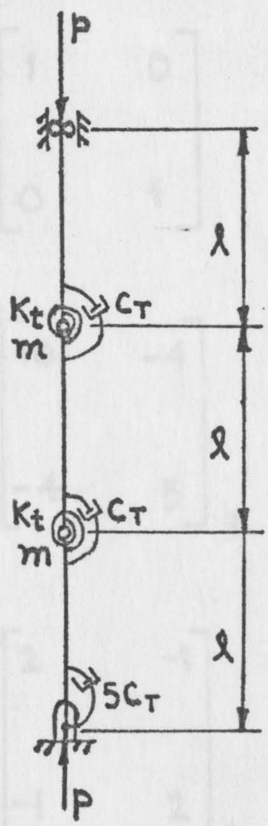

Fig. 4.1a
Initial Configuration

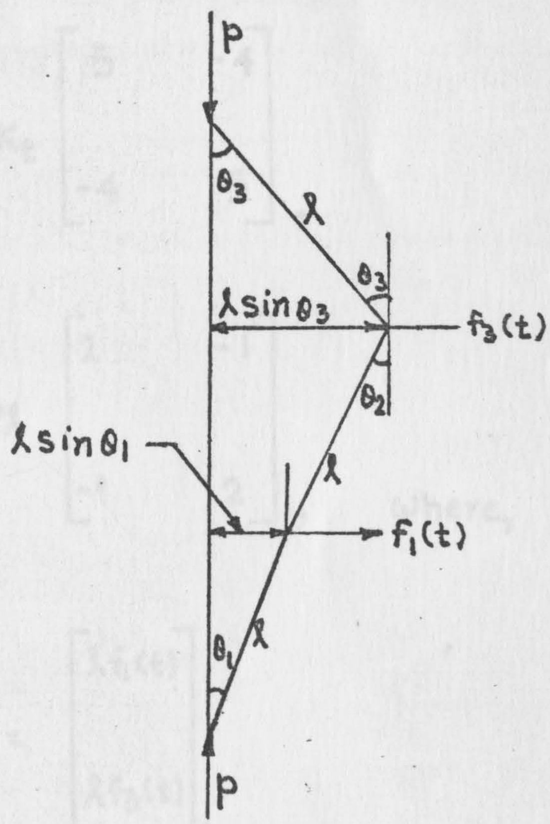

Fig. 4.1b
Displaced Configuration

The following four scalar functions are defined for the mathematical model shown above:

$$\left.\begin{aligned}
T &= \tfrac{1}{2} m(l\dot{\theta}_3)^2 + \tfrac{1}{2} m(l\dot{\theta}_1)^2 \\
D &= \tfrac{1}{2} 5C_T\dot{\theta}_1^2 + \tfrac{1}{2} C_T(\dot{\theta}_2-\dot{\theta}_1)^2 + \tfrac{1}{2} C_T(\dot{\theta}_2+\dot{\theta}_3)^2 \\
V_i &= \tfrac{1}{2} K_t(\theta_2-\theta_1)^2 + \tfrac{1}{2} K_t(\theta_2+\theta_3)^2 \\
V_e &= P(3l - l\cos\theta_1 - l\cos\theta_2 - l\cos\theta_3) \\
&\qquad + f_1(t)\, l\sin\theta_1 + f_3(t)\, l\sin\theta_3
\end{aligned}\right\} \quad (86)$$

where for small angles of rotation $\theta_2 = \theta_3 - \theta_1$ Considering rotations small in comparison to unity the following matrices are obtained:

$$[M] = ml^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [K] = K_t\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix},$$

$$[C] = C_T\begin{bmatrix} 10 & -4 \\ -4 & 5 \end{bmatrix}, \qquad [P] = Pl\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{where,}$$

$$[\hat{P}] = l\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \{f(t)\} = \begin{bmatrix} l f_1(t) \\ l f_3(t) \end{bmatrix},$$

$$\text{and} \quad \{X(t)\} = \begin{bmatrix} \theta_1(t) \\ \theta_3(t) \end{bmatrix}.$$

From equation (18), it follows that

$$[M]^{-1}[K][\hat{P}]^{-1} = [\hat{P}]^{-1}[K][M]^{-1} = \frac{K_t}{m\ell^4}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Carrying out the eigenvalue-eigenvector problem one obtains

$$[U_f] = [U_s] = [U] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

where [U] is an orthonormal matrix. In addition, the following diagonal matrices are obtained:

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{K_t}{\ell} & 0 \\ 0 & \frac{3K_t}{\ell} \end{bmatrix}, \qquad [\Lambda_m] = m\ell^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$[\Lambda_\omega] = \begin{bmatrix} \frac{K_t}{m\ell^2} & 0 \\ 0 & \frac{9K_t}{m\ell^2} \end{bmatrix}, \qquad [\Lambda_\Omega] = \begin{bmatrix} \frac{K_t}{m\ell^2}\left(1-\frac{P\ell}{K_t}\right) & 0 \\ 0 & \frac{9K_t}{m\ell^2}\left(1-\frac{P\ell}{3K_t}\right) \end{bmatrix}.$$

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{k_t}{\ell} & 0 \\ 0 & \frac{3k_t}{\ell} \end{bmatrix}, \qquad [\Lambda_m] = m\ell^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$[\Lambda_\omega] = \begin{bmatrix} \frac{k_t}{m\ell^2} & 0 \\ 0 & \frac{9k_t}{m\ell^2} \end{bmatrix}, \qquad [\Lambda_\Omega] = \begin{bmatrix} \frac{k_t}{m\ell^2}\left(1-\frac{p\ell}{k_t}\right) & 0 \\ 0 & \frac{9k_t}{m\ell^2}\left(1-\frac{p\ell}{3k_t}\right) \end{bmatrix}.$$

Referring to equation (46), and selecting $m = k_t = \ell = 1$, and $p = 1/2$, where $P < (Pcr)$ min., the following characteristic equation holds:

$$q^4 + 15C_T q^3 + (34C_T^2 + 8)q^2 + 32C_T + 3\cdot 75 = 0. \quad (87)$$

Choosing $C_T = 0\cdot 2$, the following complex conjugate pairs of eigenvalues are obtained:

$$\left.\begin{aligned} q_1 &= -0\cdot 363 + i0\cdot 625 \\ q_2 &= -1\cdot 137 + i2\cdot 428 \\ q_3 &= -0\cdot 363 - i0\cdot 625 \\ q_4 &= -1\cdot 137 - i2\cdot 428 \end{aligned}\right] . \quad (88)$$

From equation (45), and (88), the following matrices are obtained:

$$[\Lambda_1] = \begin{bmatrix} -1.137 + i2.428 & 0 \\ 0 & -0.363 + i0.625 \end{bmatrix},$$

$$[U_1] = \begin{bmatrix} 1 & 1 \\ 1.033 - i4.518 & 0.020 - i0.052 \end{bmatrix},$$

$$[V] = \begin{bmatrix} 1 & 1 \\ 1.033 & 0.020 \end{bmatrix}, \qquad [W] = \begin{bmatrix} 0 & 0 \\ -4.518 & -0.052 \end{bmatrix}$$

For special case of the constant external transverse forces

$$\{f(t)\} = \begin{bmatrix} \ell f_1(0) \\ \ell f_3(0) \end{bmatrix} = \{f(0)\},$$

and neglecting the effect of initial conditions, and referring to the equation (80), the following matrices are obtained:

$$[\hat{G}] = \begin{bmatrix} -0.158 & 0 \\ 0 & -0.694 \end{bmatrix}, \qquad [\hat{H}] = \begin{bmatrix} 0.338 & 0 \\ 0 & 1.199 \end{bmatrix},$$

$$[\hat{A}] = \begin{bmatrix} \cos(2.428)t & 0 \\ 0 & \cos(.625)t \end{bmatrix}, \qquad [\hat{B}] = \begin{bmatrix} \sin(2.428)t & 0 \\ 0 & \sin(.625)t \end{bmatrix};$$

$$[\exp.[Gt]] = \begin{bmatrix} e^{(-1.137)t} & 0 \\ 0 & e^{(-.363)t} \end{bmatrix}, \qquad [G_r] = \begin{bmatrix} 44.299 & 0 \\ 0 & -0.007 \end{bmatrix},$$

$$[H_r] = \begin{bmatrix} 93.840 & 0 \\ 0 & -1.187 \end{bmatrix}, \qquad [G_r]^{-1} = \begin{bmatrix} 0.022 & 0 \\ 0 & -142.8 \end{bmatrix},$$

$$[H_r]^{-1} = \begin{bmatrix} 0.011 & 0 \\ 0 & -0.842 \end{bmatrix}.$$

Equation (80) is solved for $\{x(t)\}$ by substituting the above matrices and simplifying.

Defining the external forcing function in the form

$$\{f(t)\} = \begin{bmatrix} \lambda f_1 \sin \alpha_1 t \\ \\ \ell f_3 \sin \alpha_2 t \end{bmatrix},$$

and neglecting initial conditions, and referring to equation (82), the following matrices are obtained:

$$[J] = \begin{bmatrix} \dfrac{1}{\alpha_2^4 - 9.204\alpha_2^2 + 51.667} & 0 \\ 0 & \dfrac{1}{\alpha_1^4 - 0.518\alpha_1^2 + 0.235} \end{bmatrix},$$

$$[L] = \begin{bmatrix} -8.173 - 1.137\alpha_2^2 & 0 \\ 0 & -0.190 - 0.363\alpha_1^2 \end{bmatrix},$$

$$[N] = \begin{bmatrix} -4.602 + \alpha_2^2 & 0 \\ 0 & -0.259 + \alpha_1^2 \end{bmatrix}, \qquad [\hat{L}] = \begin{bmatrix} 17.452 - 2.428\alpha_2^2 & 0 \\ 0 & -.327 - .625\alpha_1^2 \end{bmatrix},$$

$$[\hat{N}] = \begin{bmatrix} -2.761 & 0 \\ 0 & -0.227 \end{bmatrix}, \qquad [Q] = \begin{bmatrix} \sin\alpha_1 t & 0 \\ 0 & \sin\alpha_2 t \end{bmatrix},$$

$$[T] = \begin{bmatrix} \cos\alpha_1 t & 0 \\ 0 & \cos\alpha_2 t \end{bmatrix}, \qquad \{f_n\} = \begin{bmatrix} f_1 \\ f_3 \end{bmatrix},$$

$$[F_1] = \begin{bmatrix} 0{\cdot}016 & -0{\cdot}0516 \\ 100{\cdot}960 & 2{\cdot}050 \end{bmatrix}, \qquad [F_2] = \begin{bmatrix} 0{\cdot}0516 & 0{\cdot}016 \\ -2{\cdot}050 & -100{\cdot}96 \end{bmatrix},$$

$$[F_3] = \begin{bmatrix} 0{\cdot}008 & 0{\cdot}062 \\ -0{\cdot}595 & -5{\cdot}238 \end{bmatrix}, \qquad [F_4] = \begin{bmatrix} -0{\cdot}062 & 0{\cdot}008 \\ 5{\cdot}238 & -0{\cdot}595 \end{bmatrix}.$$

Equation (82) is solved for $\{x(t)\}$ by substituting the above matrices and simplifying.

DISCUSSION

The use of the matrix type form for the equations of motion is proven more efficient than the series or algebraic type form. Its efficiency arises due to the fact that matrix type solution is easily programmed for computer use. The formal matrix type solution presented in this thesis requires inversion of diagonal matrices which is extremely important for large scale system, since the inversion process requires a large amount of memory core in the computer. Since the solution is given in Duhamel's integral form, one can use any type of time-varying external forcing functions. In this particular thesis, constant external and harmonic time-varying forces are solved as special cases.

If damping is included, the n-degree of freedom is converted into a 2n degree system and solved in partitioned matrix form. This yields a matrix system which is much larger in scale than the original system, however, mathematically the system of differential equations uncouples much more uniquely.

The concept of overdamping and underdamping is readily understood by use of the plot of the eigenvalues on the complex plane. For example, for an underdamped system, all the eigenvalues appear in complex conjugate sets with negative real parts. For an overdamped system, all the roots appear as negative real values with no imaginary parts.

The forced vibration problem neglecting the effect of viscous damping is solved first in order to investigate the concept of critical buckling load, and to establish a condition which simultaneously diagonalizes the mass, stiffness, and stability matrices.

It is not the intention of this thesis to develop directly a design procedure to convert a physical dynamic problem into a mathematical model as illustrated in this thesis. This ability is obtained only by considerable experience both in the design office and under actual field conditions. In an actual field situation, the magnitude of the stiffness and damping parameters (i.e. K_t and C_T) are usually obtained from comparison with actual field testing and measurements of the actual existing structures.

CONCLUSION

For an underdamped n-degree of freedom system, simultaneous diagonalization of the mass, stiffness, and axial force matrices by a non-singular matrix takes place only under specific mathematical operations which are based on matrix commutivity conditions. The most efficient mathematical approach to the problem is a matrix-type solution of order (n x n). In general, the matrix which simultaneously diagonalizes the mass, stiffness, and axial force matrices needs not diagonalize the damping matrix. If damping is included, the matrix solution becomes more complex since partitioned matrices of order (2n x 2n) are required. However, this approach is still mathematically more convenient. Whether damping is considered or not, the matrix-type solution is comparatively more efficient than series-type solution, since the matrix form can be uniquely and efficiently computerized in compact program form.

The basic matrix computations present in the general solution involve typical matrix addition and multiplication. For the case when matrix inversion occurs, it is only necessary to invert diagonal matrices. Also present in the mathematical solution is matrix eigenvalue-eigenvector problem. Programs for this special type solution exist in most computer libraries.

A closed form solution for the cases of constant external forces or harmonic time-varying forces considering steady-state motion is obtained in a closed form using this matrix-type solution, and may be readily programmed in an efficient compact form. For both problems, exact closed form solution is presented in matrix form which may be easily programmed.

APPENDIX A

Noting equation (75), the terms of equation (72) are expanded as follows:

$$\begin{aligned}
[U_1][\exp.[\Lambda_1 t]] &= [[V]+i[W]][\exp.[Gt]][\exp.[iHt]] \\
&= [[V][\exp.[Gt]]+i[W][\exp.[Gt]]][[\hat{A}]+i[\hat{B}]] \\
&= [[V][\exp.[Gt]][\hat{A}]-[W][\exp.[Gt]][\hat{B}]] \\
&\qquad +i[[W][\exp.[Gt]][\hat{A}]+[V][\exp.[Gt]][\hat{B}]]
\end{aligned}$$

$$\begin{aligned}
[U_2][\exp.[\Lambda_2 t]] &= [[V][\exp.[Gt]][\hat{A}]-[W][\exp.[Gt]][\hat{B}]] \\
&\qquad -i[[W][\exp.[Gt]][\hat{A}]+[V][\exp.[Gt]][\hat{B}]]
\end{aligned}$$

$$\begin{aligned}
[\Lambda_{R1}]^{-1}[U_1]^T &= [[G_r]^{-1}+i[H_r]^{-1}][[V]^T+i[W]^T] \\
&= [[G_r]^{-1}[V]^T-[H_r]^{-1}[W]^T]+i[[G_r]^{-1}[W]^T+[H_r]^{-1}[V]^T]
\end{aligned}$$

$$[\Lambda_{R2}]^{-1}[U_2]^T = [[G_r]^{-1}[V]^T-[H_r]^{-1}[W]^T]-i[[G_r]^{-1}[W]^T+[H_r]^{-1}[V]^T]$$

$$\begin{aligned}
[\Lambda_{R1}]^{-1}[\Lambda_1][U_1]^T &= [[G_r]^{-1}+i[H_r]^{-1}][[G]+i[H]][[V]^T+i[W]^T] \\
&= [[G_r]^{-1}[G][V]^T-[H_r]^{-1}[H][V]^T-[H_r]^{-1}[G][W]^T-[G_r]^{-1}[H][W]^T] \\
&\qquad +i[[H_r]^{-1}[G][V]^T+[G_r]^{-1}[H][V]^T+[G_r]^{-1}[G][W]^T-[H_r]^{-1}[H][W]^T]
\end{aligned}$$

$$\begin{aligned}
[\Lambda_{R2}]^{-1}[\Lambda_2][U_2]^T &= [[G_r]^{-1}[G][V]^T-[H_r]^{-1}[H][V]^T-[H_r]^{-1}[G][W]^T-[G_r]^{-1}[H][W]^T] \\
&\qquad -i[[H_r]^{-1}[G][V]^T+[G_r]^{-1}[H][V]^T+[G_r]^{-1}[G][W]^T-[H_r]^{-1}[H][W]^T] .
\end{aligned}$$

Substituting the above terms into equation (72) in proper order, the imaginary parts cancel with each other and the real parts add with each other making the solution to be real as shown in equation (76).

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