INVESTIGATION OF THE AIRY STRESS FUNCTION FOR THE $W=Z^2$

COORDINATE SYSTEM.

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE IN ENGINEERING

in the

MECHANICAL ENGINEERING

Program

______________________________ 8/10/73
Adviser Date

______________________________ 8/13/73
Dean Date

YOUNGSTOWN STATE UNIVERSITY

June, 1973.

ABSTRACT

INVESTIGATION OF THE AIRY STRESS FUNCTION FOR THE $W=Z^2$ COORDINATE SYSTEM.

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At the graduate level in Applied Elasticity, we have developed the expressions for stress equations in Cartesian, Polar, Spherical, Cylindrical and Bipolar coordinate systems.

Here in this thesis, an attempt is made to develop various expressions for stress equations by solving Biharmonic Equation for $W=Z^2$ coordinate system and then to find suitable applications to match our Airy stress function and corresponding stress equations. W and Z are complex numbers, defined as $W=U+iV$ and $Z=X+iY$.

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ACKNOWLEDGEMENTS

It is a great pleasure to make grateful acknowledgement of valuable suggestions and big help from my adviser, Dr. Frank A. D'Isa, Chairman of Mechanical Engineering Department.

I am also indebted to Dr. Frank J. Tarantine who has been kind enough to go through the manuscript of this thesis and has given valuable suggestions.

Youngstown State University, Ishwarlal R. Dave.

Youngstown, Ohio.

June, 1973.

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LIST OF SYMBOLS

SYMBOL	DEFINITION	UNITS OR REFERENCE
ϕ	Airy Stress Function.	Page 1.
u, v	Coordinates.	Unit of length.
x, y	Coordinates.	Unit of length.
$c_1, c_2 \ldots\ldots c_{27}$	Constants	None.
$\int$	Sign of Integration.	None.
M	Displacement in U direction	Unit of length.
N	Displacement in V direction	Unit of length.
ε_u	Strain in U direction	None.
ε_v	Strain in V direction	None.
γ_{uv}	Shear Strain	None.
σ_u	Normal Stress in U direction	Force/Unit area.
σ_v	Normal Stress in V direction	Force/Unit area.
τ_{uv}	Shear Stress	Force/Unit area.

LIST OF FIGURES

CHAPTER I

STATEMENT: Investigation of the Airy Stress function for the $W=Z^2$ coordinate system.

INTRODUCTION: W and Z are defined as follows,

$W=U+iV$,

$Z=X+iY$.

So ultimately we get $U=X^2-Y^2$ and $V=2XY$.

If we plot U and V, by choosing values of X and Y we get a system of coordinates as shown in fig. 1 on page 2.

For above coordinate system, solving Biharmonic equation, we get various expressions for Airy stress function, and from that we can get corresponding sets of stress equations. After that for certain set of stress equation we try to match with a real problem with that stress conditions.

A very effective way of proceeding with the simultaneous solving of compatibility equation and the equilibrium equations was first suggested by the English mathematician George Biddel Airy (1801-1992) 1862*, a stress function called an Airy stress function, is defined so as to satisfy the equilibrium equations and thereby reduce the number of partial differential equations from three to one.

In our case, it is assumed that the body forces are zero.

* G.B. Airy, Brit. Assoc. Advan. Sci. Rept., 1862.

Set of Coordinates.

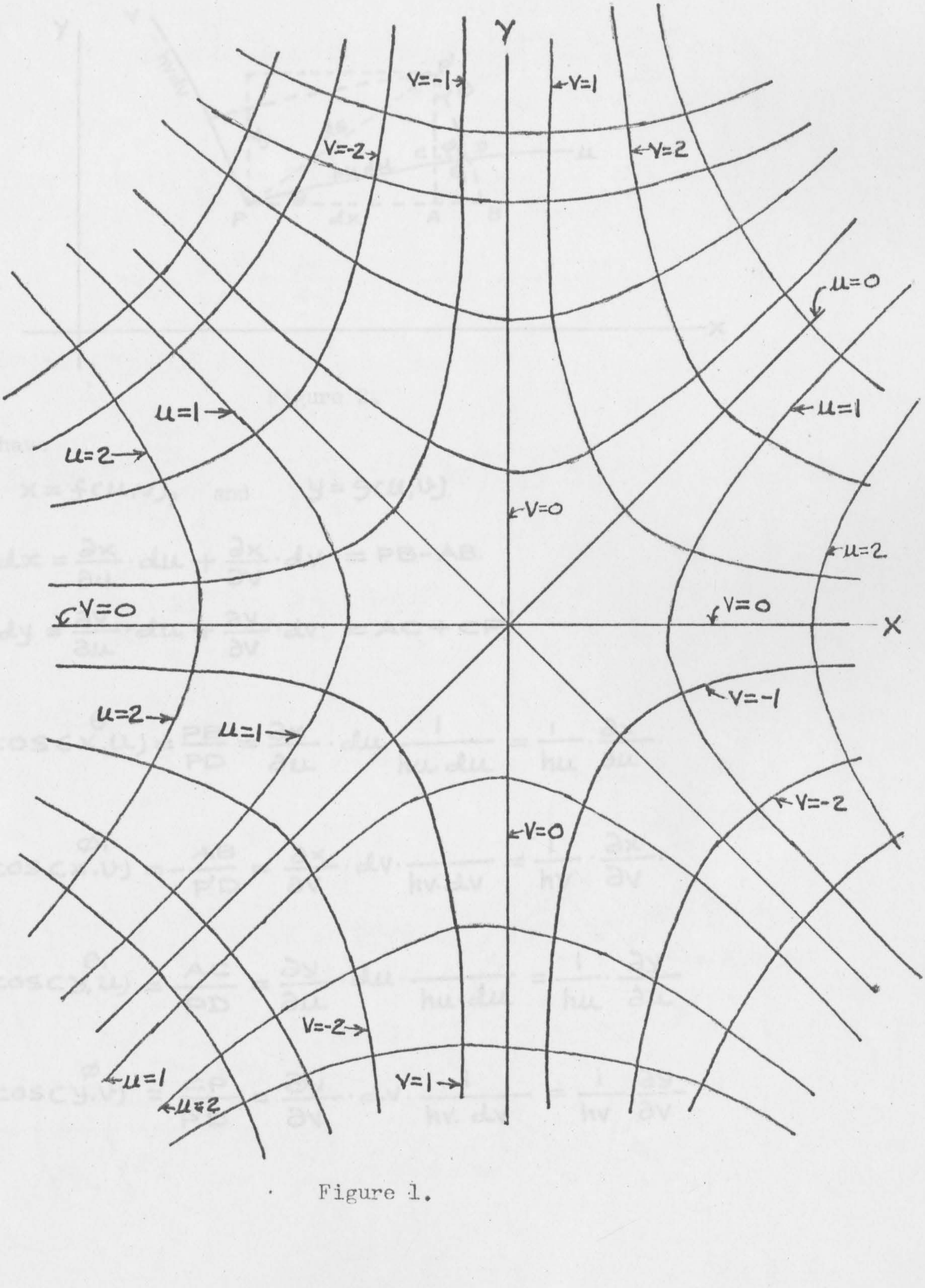

Figure 1.

CHAPTER II

DEVELOPMENT OF $\nabla^4\phi$ AND STRESS EQUATIONS:

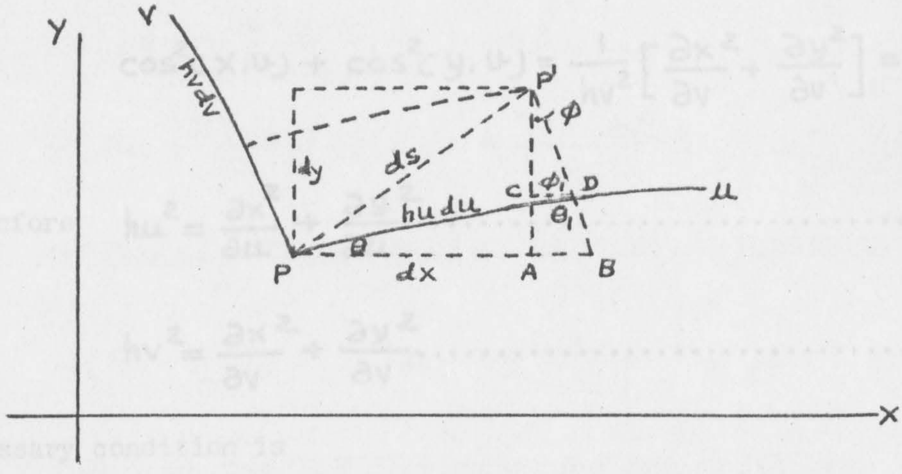

Figure 2.

We have

$$x = f(u,v), \quad \text{and} \quad y = g(u,v).$$

$$\therefore dx = \frac{\partial x}{\partial u}\cdot du + \frac{\partial x}{\partial v}\cdot dv = PB - AB.$$

$$\therefore dy = \frac{\partial y}{\partial u}\cdot du + \frac{\partial y}{\partial v}\cdot dv = AC + CP'.$$

$$\overset{\theta}{\cos(x,u)} = \frac{PB}{PD} = \frac{\partial x}{\partial u}\cdot du \cdot \frac{1}{hu\cdot du} = \frac{1}{hu}\cdot\frac{\partial x}{\partial u}.$$

$$\overset{\phi_1}{\cos(x,v)} = -\frac{AB}{P'D} = \frac{\partial x}{\partial v}\cdot dv \cdot \frac{1}{hv\cdot dv} = \frac{1}{hv}\cdot\frac{\partial x}{\partial v}.$$

$$\overset{\theta_1}{\cos(y,u)} = \frac{AC}{PD} = \frac{\partial y}{\partial u}\cdot du \cdot \frac{1}{hu\cdot du} = \frac{1}{hu}\cdot\frac{\partial y}{\partial u}$$

$$\overset{\phi}{\cos(y,v)} = \frac{CP'}{P'D} = \frac{\partial y}{\partial v}\cdot dv \cdot \frac{1}{hv\cdot dv} = \frac{1}{hv}\cdot\frac{\partial y}{\partial v}$$

Then

$$\cos^2(x,u)+\cos^2(y,u)=\frac{1}{hu^2}\left[\frac{\partial x}{\partial u}^2+\frac{\partial y}{\partial u}^2\right]=1$$

$$\cos^2(x,v)+\cos^2(y,v)=\frac{1}{hv^2}\left[\frac{\partial x}{\partial v}^2+\frac{\partial y}{\partial v}^2\right]=1.$$

Therefore $$hu^2=\frac{\partial x}{\partial u}^2+\frac{\partial y}{\partial u}^2 \qquad (5)$$

$$hv^2=\frac{\partial x}{\partial v}^2+\frac{\partial y}{\partial v}^2 \qquad (6)$$

Necessary condition is

$$\left.\frac{\partial y}{\partial x}\right]_u=\frac{\dfrac{\partial y}{\partial u}}{\dfrac{\partial x}{\partial u}}=-\frac{1}{\left.\dfrac{\partial y}{\partial x}\right]_v}=-\frac{1}{\dfrac{\partial y}{\partial v}\Big/\dfrac{\partial x}{\partial v}}$$

Thus $$\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v}=-\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v}$$

EQUILIBRIUM EQUATIONS:

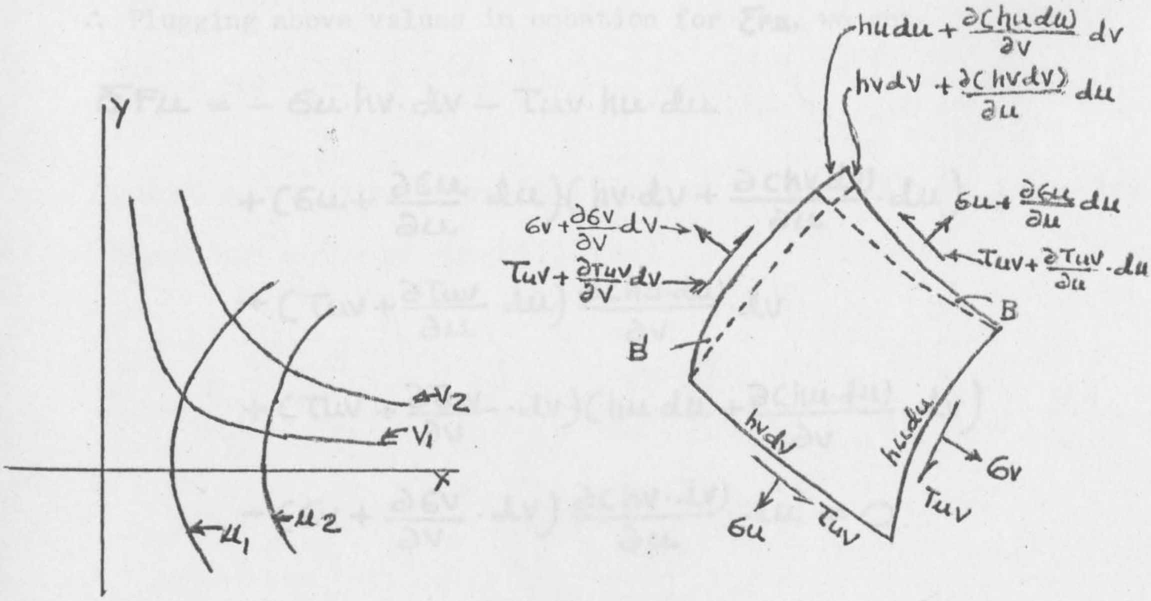

Figure 3.

$$\Sigma F_u = -\sigma_u \cdot h_v \cdot dv - \tau_{uv} \cdot h_u \cdot du + \left(\sigma_u + \frac{\partial \sigma_u}{\partial u} \cdot du\right) \times$$

$$\times \left(h_v \cdot dv + \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du\right) \cos B$$

$$+ \left(\tau_{uv} + \frac{\partial \tau_{uv}}{\partial u} \cdot du\right)\left(h_v dv + \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du\right) \sin B.$$

$$+ \left(\tau_{uv} + \frac{\partial \tau_{uv}}{\partial v} \cdot dv\right)\left(h_u du + \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv\right) \cos B'$$

$$- \left(\sigma_v + \frac{\partial \sigma_v}{\partial v} \cdot dv\right)\left(h_u du + \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv\right) \sin B' = 0.$$

From figure 2 on page 3, following are defined as follows,

$$\sin B \cong \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv \bigg/ h_v \cdot dv + \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du;$$

$$\cos B \cong 1; \qquad \cos B' \cong 1;$$

$$\sin B' \cong \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du \bigg/ h_u \cdot du + \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv;$$

∴ Plugging above values in equation for ΣF_u, we get

$$\Sigma F_u = -\sigma_u \cdot h_v \cdot dv - \tau_{uv} \cdot h_u \cdot du$$

$$+ \left(\sigma_u + \frac{\partial \sigma_u}{\partial u} \cdot du\right)\left(h_v \cdot dv + \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du\right)$$

$$+ \left(\tau_{uv} + \frac{\partial \tau_{uv}}{\partial u} \cdot du\right) \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv$$

$$+ \left(\tau_{uv} + \frac{\partial \tau_{uv}}{\partial v} \cdot dv\right)\left(h_u \cdot du + \frac{\partial (h_u \cdot du)}{\partial v} \cdot dv\right)$$

$$- \left(\sigma_v + \frac{\partial \sigma_v}{\partial v} \cdot dv\right) \frac{\partial (h_v \cdot dv)}{\partial u} \cdot du = 0.$$

$$\Sigma F_u = -\sigma_u \cdot h_v \cdot dv - \tau_{uv} \cdot h_u \cdot du + \sigma_u \cdot h_v \cdot dv + \sigma_u \cdot \frac{\partial h_v}{\partial u} \cdot dv \cdot du$$

$$+ \frac{\partial \sigma_u}{\partial u} \cdot h_v \cdot du \cdot dv + \frac{\partial \sigma_u}{\partial u} \cdot \frac{\partial h_v}{\partial u} \cdot dv \cdot (du)^2$$

$$+ \tau_{uv} \cdot \frac{\partial h_u}{\partial v} \cdot du \cdot dv + \frac{\partial \tau_{uv}}{\partial u} \cdot \frac{\partial h_u}{\partial v} \cdot dv \cdot (du)^2$$

$$+ \tau_{uv} \cdot h_u \cdot du + \tau_{uv} \cdot \frac{\partial h_u}{\partial v} \cdot du \cdot dv + \frac{\partial \tau_{uv}}{\partial v} \cdot h_u \cdot dv \cdot du$$

$$+ \frac{\partial \tau_{uv}}{\partial v} \cdot \frac{\partial h_u}{\partial v} \cdot du \cdot (dv)^2 - \sigma_v \cdot \frac{\partial h_v}{\partial u} \cdot dv \cdot du$$

$$- \frac{\partial \sigma_v}{\partial v} \cdot \frac{\partial h_v}{\partial u} \cdot du \cdot (dv)^2 = 0.$$

Dividing by dudv and eliminating differentials of higher order than one,

$$\Sigma F_u = \sigma_u \cdot \frac{\partial h_v}{\partial u} + \frac{\partial \sigma_u}{\partial u} \cdot h_v + 2\tau_{uv} \cdot \frac{\partial h_u}{\partial v} + \frac{\partial \tau_{uv}}{\partial v} \cdot h_u - \sigma_v \frac{\partial h_v}{\partial u} = 0.$$

Similarly summing forces in V direction we get,

$$\Sigma F_v = \sigma_v \cdot \frac{\partial h_u}{\partial v} + \frac{\partial \sigma_v}{\partial v} \cdot h_u + 2\tau_{uv} \cdot \frac{\partial h_v}{\partial u} + \frac{\partial \tau_{uv}}{\partial u} \cdot h_v - \sigma_u \cdot \frac{\partial h_u}{\partial v} = 0$$

These above two equations are satisfied by an Airy Stress function defined as follows,

$$\sigma_u = \frac{1}{h_v^2} \cdot \frac{\partial^2 \phi}{\partial v^2} + \frac{1}{h_u^2 h_v} \cdot \frac{\partial \phi}{\partial u} \cdot \frac{\partial h_v}{\partial u} - \frac{1}{h_u^3} \cdot \frac{\partial \phi}{\partial v} \cdot \frac{\partial h_v}{\partial v},$$

$$\sigma v = \frac{1}{hu^2}\cdot\frac{\partial^2\phi}{\partial u^2} + \frac{1}{hu\cdot hv^2}\cdot\frac{\partial\phi}{\partial v}\cdot\frac{\partial hu}{\partial v} - \frac{1}{hu^3}\cdot\frac{\partial\phi}{\partial u}\cdot\frac{\partial hu}{\partial u},$$

$$\tau uv = -\frac{1}{hu\cdot hv}\cdot\frac{\partial^2\phi}{\partial u\cdot\partial v} + \frac{1}{hu\cdot hv^2}\cdot\frac{\partial\phi}{\partial v}\cdot\frac{\partial hv}{\partial u} + \frac{1}{hu^2\cdot hv}\cdot\frac{\partial\phi}{\partial u}\cdot\frac{\partial hu}{\partial v}.$$

STRAINS IN CURVILINEAR COORDINATES:

Let M be displacement in U direction,

N be displacement in V direction.

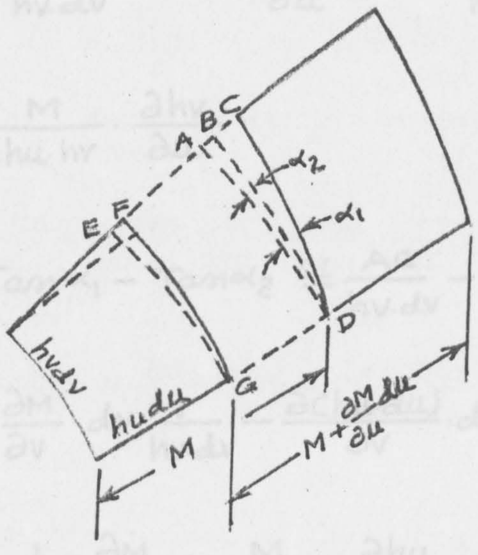

Figure 4.

Case: 1 When N=0.

$$EF = \frac{\partial(hu\cdot du)}{\partial v}\cdot dv$$

$$AB = \frac{\partial(hu\cdot du)}{\partial v}\cdot dv\cdot\frac{M}{hu\cdot du}.$$

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$$FG = hv \cdot dv + \frac{\partial(hv \cdot dv)}{\partial u} \cdot du$$

$$BD = hv \cdot dv + \frac{\partial(hv \cdot dv)}{\partial u} \cdot du \cdot \frac{M}{hu \cdot du}$$

$$AC = \frac{\partial M}{\partial v} \cdot dv$$

$$\varepsilon u = \frac{\partial M}{\partial u} \cdot du \cdot \frac{1}{hu \cdot du} = \frac{1}{hu} \cdot \frac{\partial M}{\partial u} \cdot$$

$$\varepsilon v = \frac{BD - hv \cdot dv}{hv \cdot dv} = \frac{\partial(hv \cdot dv)}{\partial u} \cdot du \cdot \frac{M}{hu \cdot du} \cdot \frac{1}{hv \cdot dv}$$

$$\therefore \varepsilon v = \frac{M}{hu \cdot hv} \cdot \frac{\partial hv}{\partial u}$$

$$\gamma uv = \operatorname{Tan}\alpha_1 - \operatorname{Tan}\alpha_2 \cong \frac{AC}{hv \cdot dv} - \frac{AB}{hv \cdot dv}$$

$$\therefore \gamma uv = \frac{\partial M}{\partial v} \cdot dv \cdot \frac{1}{hv \cdot dv} - \frac{\partial(hu \cdot du)}{\partial v} \cdot dv \cdot \frac{M}{hu \cdot du} \cdot \frac{1}{hv \cdot dv}$$

$$\therefore \gamma uv = \frac{1}{hv} \cdot \frac{\partial M}{\partial v} - \frac{M}{hu \cdot hv} \cdot \frac{\partial hu}{\partial v}$$

In a similar manner, for Case 2, When M=0.

$$\varepsilon v = \frac{1}{hv} \cdot \frac{\partial N}{\partial v}$$

$$\varepsilon u = \frac{N}{hu \cdot hv} \cdot \frac{\partial hu}{\partial v},$$

$$\gamma_{uv} = \frac{1}{h_u} \cdot \frac{\partial N}{\partial u} - \frac{N}{h_u \cdot h_v} \cdot \frac{\partial h_v}{\partial u} \cdot$$

For a general case; M≠0, N≠0, the strains are added.

$$\varepsilon_u = \frac{1}{h_u} \cdot \frac{\partial M}{\partial u} + \frac{N}{h_u \cdot h_v} \cdot \frac{\partial h_u}{\partial v},$$

$$\varepsilon_v = \frac{1}{h_v} \cdot \frac{\partial N}{\partial v} + \frac{M}{h_u \cdot h_v} \cdot \frac{\partial h_v}{\partial u},$$

$$\gamma_{uv} = \frac{1}{h_v} \cdot \frac{\partial M}{\partial v} - \frac{M}{h_u \cdot h_v} \cdot \frac{\partial h_u}{\partial v} + \frac{1}{h_u} \cdot \frac{\partial N}{\partial u} - \frac{N}{h_u \cdot h_v} \cdot \frac{\partial h_v}{\partial u},$$

Eliminating M and N results in the compatibility equation, as follows

$$\frac{\partial}{\partial u}\left[\frac{h_v}{h_u} \cdot \frac{\partial \varepsilon_v}{\partial u} + \frac{\varepsilon_v}{h_u} \cdot \frac{\partial h_v}{\partial u} - \frac{\varepsilon_u}{h_u} \cdot \frac{\partial h_v}{\partial u} - \frac{\gamma_{uv}}{h_u} \cdot \frac{\partial h_u}{\partial v}\right]$$

$$+\frac{\partial}{\partial v}\left[\frac{h_u}{h_v} \cdot \frac{\partial \varepsilon_u}{\partial v} + \frac{\varepsilon_u}{h_v} \cdot \frac{\partial h_u}{\partial v} - \frac{\varepsilon_v}{h_v} \cdot \frac{\partial h_u}{\partial v} - \frac{\gamma_{uv}}{h_v} \cdot \frac{\partial h_v}{\partial u}\right] = \frac{\partial^2 \gamma_{uv}}{\partial u \cdot \partial v}$$

Expressing this in terms of the Airy stress function ϕ for the case of plane stress or for the case of plane strain gives, *

$$\left[\frac{1}{h_u^2} \cdot \frac{\partial^2}{\partial u^2} + \frac{1}{h_v^2 h_u} \cdot \frac{\partial h_u}{\partial v} \cdot \frac{\partial}{\partial v} - \frac{1}{h_u^3} \cdot \frac{\partial h_u}{\partial u} \cdot \frac{\partial}{\partial u} + \frac{1}{h_v^2} \cdot \frac{\partial^2}{\partial v^2}\right.$$

$$\left. + \frac{1}{h_u^2 h_v} \cdot \frac{\partial h_v}{\partial u} \cdot \frac{\partial}{\partial u} - \frac{1}{h_v^3} \cdot \frac{\partial h_v}{\partial v} \cdot \frac{\partial}{\partial v}\right]^2 \phi = 0.$$

*Through the use of Hook's law.

Given equations for our coordinate system:

$$u = x^2 - y^2 \quad \text{...} (1)$$

$$v = 2xy \quad \text{...} (2)$$

From equation (2) we get

$x = \dfrac{v}{2y}$ and plugging this in equation (1) we get

$$u = \frac{v^2}{4y^2} - y^2$$

$$\therefore 4y^2u = v^2 - 4y^4$$

$\therefore 4y^4 + 4y^2u - v^2 = 0.$ Solving this equation we get,

$$y^2 = \frac{-4u \pm (16u^2 + 16v^2)^{1/2}}{8}.$$

$$\therefore y^2 = \frac{-4u \pm 4(u^2+v^2)^{1/2}}{8}.$$

$$\therefore y^2 = \frac{-u \pm (u^2+v^2)^{1/2}}{2}.$$

$$\therefore y = \pm\left[\frac{-u \pm (u^2+v^2)^{1/2}}{2}\right]^{1/2} \quad \text{.......................................} (3)$$

and similarly from equation (2) we have

$y = \dfrac{v}{2x}$ and plugging back in equation (1) we get,

$$u = x^2 - \frac{v^2}{4x^2}$$

$\therefore 4x^4 - 4x^2u - v^2 = 0.$ Solving this equation we get,

$$x^2 = \frac{4u \pm (16u^2 + 16v^2)^{1/2}}{8}$$

$$\therefore x^2 = \frac{u \pm (u^2+v^2)^{1/2}}{2}$$

$$\therefore x = \pm \left[\frac{u \pm (u^2+v^2)^{1/2}}{2}\right]^{1/2} \qquad \ldots\ldots (4)$$

Now $$h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 \qquad \ldots\ldots (5)$$

Take eqn. $$x^2 = \frac{u \pm (u^2+v^2)^{1/2}}{2}$$

$$\therefore 2x \cdot \delta x = \frac{\partial u}{2} \pm \frac{1}{2} \cdot (u^2+v^2)^{-1/2} \cdot 2u \cdot du$$

$$\therefore 2x \cdot \delta x = \frac{\partial u}{2}\left[1 \pm \frac{u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \frac{\partial x}{\partial u} = \frac{1}{4x}\left[1 \pm \frac{u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \frac{\partial x}{\partial u} = \pm \frac{1}{4}\left[\frac{2}{u \pm (u^2+v^2)^{1/2}}\right]^{1/2}\left[\frac{(u^2+v^2)^{1/2} \pm u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \left[\frac{\partial x}{\partial u}\right]^2 = \frac{2u^2+v^2 \pm 2u(u^2+v^2)^{1/2}}{8(u^2+v^2)(u \pm \sqrt{u^2+v^2})} \qquad \ldots\ldots (6)$$

and similarly we have

$$y^2 = \frac{-u \pm (u^2+v^2)^{1/2}}{2}$$

$$\therefore 2y \cdot \partial y = -\frac{\partial u}{2} \pm \frac{1}{2} \cdot (u^2+v^2)^{-1/2} \cdot 2u \cdot \partial u$$

$$\therefore 2y \cdot \partial y = \frac{\partial u}{2}\left[-1 \pm \frac{u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \frac{\partial y}{\partial u} = \frac{1}{4y} \cdot \left[-1 \pm \frac{u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \frac{\partial y}{\partial u} = \pm \frac{1}{4}\left[\frac{2}{-u \pm (u^2+v^2)^{1/2}}\right]^{1/2} \cdot \left[\frac{-u(u^2+v^2)^{1/2} \pm u}{(u^2+v^2)^{1/2}}\right]$$

$$\therefore \left[\frac{\partial y}{\partial u}\right]^2 = \frac{2u^2+v^2 \mp 2u(u^2+v^2)^{1/2}}{8(u^2+v^2)(-u \pm \sqrt{u^2+v^2})} \quad \text{.............................(7)}$$

Adding equations (6) and (7) we get equation (5)

$$hu^2 = \pm \frac{1}{4} \cdot \frac{1}{(u^2+v^2)^{1/2}} \quad \text{.......................................(8)}$$

And also, $$hv^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 \quad \text{...............................(9)}$$

Take equation $$x^2 = \frac{u \pm (u^2+v^2)^{1/2}}{2}$$

$$\therefore 2x \cdot \partial x = \frac{1}{2}\left[0 \pm \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2v \cdot \partial v\right]$$

$$= \frac{1}{2}\left[\pm \frac{v\, \partial v}{(u^2+v^2)^{1/2}}\right].$$

$$\therefore \left[\frac{\partial x}{\partial v}\right]^2 = \frac{1}{16x^2} \cdot \frac{v^2}{u^2+v^2}$$

$$\therefore \left[\frac{\partial x}{\partial v}\right]^2 = \frac{v^2}{8(u^2+v^2)\left[u \pm (u^2+v^2)^{1/2}\right]} \qquad \text{.......(10)}$$

and take equation

$$y^2 = \frac{-u \pm (u^2+v^2)^{1/2}}{2}$$

$$\therefore 2y.\partial y = \frac{1}{2}\left[0 \pm \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2v.\partial v\right]$$

$$\frac{\partial y}{\partial v} = \frac{1}{4y}\left[\frac{v}{(u^2+v^2)^{1/2}}\right]$$

$$\left[\frac{\partial y}{\partial v}\right]^2 = \frac{1}{16y^2} \cdot \frac{v^2}{u^2+v^2}.$$

$$\therefore \left[\frac{\partial y}{\partial v}\right]^2 = \frac{v^2}{8(u^2+v^2)\left[-u \pm (u^2+v^2)^{1/2}\right]} \qquad \text{.......(11)}$$

Adding equations (10) and (11) we get equation (9).

$$\therefore hv^2 = \pm \frac{1}{4} \cdot \frac{1}{(u^2+v^2)^{1/2}} \qquad \text{.......(12)}$$

We have the biharmonic equation as follows,

$$\nabla^4 \phi = \left[\frac{1}{hu^2} \cdot \frac{\partial^2}{\partial u^2} + \frac{1}{hv^2 hu} \cdot \frac{\partial hu}{\partial v} \cdot \frac{\partial}{\partial v} - \frac{1}{hu^3} \cdot \frac{\partial hu}{\partial u} \cdot \frac{\partial}{\partial u} + \frac{1}{hv^2} \cdot \frac{\partial^2}{\partial v^2} \right.$$

$$\left. + \frac{1}{hu^2 \cdot hv} \cdot \frac{\partial hv}{\partial u} \cdot \frac{\partial}{\partial u} - \frac{1}{hv^3} \cdot \frac{\partial hv}{\partial v} \cdot \frac{\partial}{\partial v}\right]^2 \phi = 0$$

Where function ϕ is the Airy stress function and, since hu=hv above equation becomes

$$\left[\frac{1}{hu^2}\cdot\frac{\partial^2}{\partial u^2}+\frac{1}{hu^2}\cdot\frac{\partial^2}{\partial v^2}\right]\left[\frac{1}{hu^2}\cdot\frac{\partial^2}{\partial u^2}+\frac{1}{hu^2}\cdot\frac{\partial^2}{\partial v^2}\right]\phi=0$$

Plugging value of hu and hv above equation becomes

$$\left[4(u^2+v^2)^{1/2}\cdot\frac{\partial^2}{\partial u^2}+4(u^2+v^2)^{1/2}\cdot\frac{\partial^2}{\partial v^2}\right]\left[4(u^2+v^2)^{1/2}\cdot\frac{\partial^2}{\partial u^2}+4(u^2+v^2)^{1/2}\cdot\frac{\partial^2}{\partial v^2}\right]\phi$$

$$=0.$$

$$\therefore 16(u^2+v^2)^{1/2}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]\left\{(u^2+v^2)^{1/2}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]\right\}\phi=0$$

$$\therefore\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]\left\{(u^2+v^2)^{1/2}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]\right\}\phi=0 \quad \text{...........(13)}$$

Equation (13) is the biharmonic differential equation for our coordinate system. Now we find solutions to above differential equation.

CHAPTER III

SOLUTIONS TO $\nabla^4\phi$ EQUATION AND EXPRESSIONS FOR STRESS EQUATIONS.

Refer Appendix section for satisfaction of $\nabla^4\phi$ equation and derivation of stress equations.

(1). $\phi = C_1 \sinh ku \cdot \sin kv,$

$$\sigma_u = 2C_1(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\cdot\sinh ku\cdot\sin kv - ku\cosh ku\sin kv + kv\sinh ku\cdot\cos kv\right],$$

$$\sigma_v = -2C_1(u^2+v^2)^{-1/2}\cdot\left[-2k^2(u^2+v^2)\cdot\sinh ku\sin kv - ku\cosh ku\sin kv + kv\sinh ku\cdot\cos kv\right],$$

$$\tau_{uv} = -2C_1(u^2+v^2)^{-1/2}\cdot\left[2k^2(u^2+v^2)\cosh ku\cdot\cos kv + ku\sinh ku\cdot\cos kv + kv\cosh ku\cdot\sin kv\right].$$

(2). $\phi = C_2 \sinh ku\cdot\cos kv.$

$$\sigma_u = 2C_2(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\sinh ku\cdot\cos kv - ku\cosh ku\cdot\cos kv - kv\sinh ku\cdot\sin kv\right],$$

$$\sigma_v = -2C_2(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\sinh ku\cdot\cos kv - ku\cosh ku\cdot\cos kv - kv\sinh ku\cdot\sin kv\right],$$

$$\tau_{uv} = 2C_2(u^2+v^2)^{-1/2}\left[2k^2(u^2+v^2)\cosh ku\cdot\sin kv + ku\sinh ku\cdot\sin kv - kv\cosh ku\cdot\cos kv\right].$$

(3). $\Phi = C_3 \cosh ku \cdot \cos kv$,

$$\sigma_u = 2C_3(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cdot\cos kv\right.$$
$$\left. - kv\cosh ku\cdot\sin kv\right].$$

$$\sigma_v = -2C_3(u^2+v^2)^{-1/2}\cdot\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cos kv\right.$$
$$\left. - kv\cosh ku\cdot\sin kv\right],$$

$$\tau_{uv} = 2C_3(u^2+v^2)^{-1/2}\cdot\left[2k^2(u^2+v^2)\sinh ku\cdot\sin kv - ku\cosh ku\sin kv\right.$$
$$\left. - kv\sinh ku\cdot\cos kv\right].$$

(4). $\Phi = C_4 \cosh ku \cdot \sin kv$.

$$\sigma_u = 2C_4(u^2+v^2)^{-1/2}\cdot\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cdot\sin kv\right.$$
$$\left. + kv\cosh ku\cdot\cos kv\right].$$

$$\sigma_v = -2C_4(u^2+v^2)^{-1/2}\cdot\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cdot\sin kv\right.$$
$$\left. + kv\cosh ku\cdot\cos kv\right],$$

$$\tau_{uv} = -2C_4(u^2+v^2)^{-1/2}\cdot\left[2k^2(u^2+v^2)\sinh ku\cdot\cos kv + ku\cosh ku\cdot\cos kv\right.$$
$$\left. + kv\sinh ku\cdot\sin kv\right].$$

(5). $\phi = c_5$.

$\sigma_u = 0,$

$\sigma_v = 0,$

$\tau_{uv} = 0.$

(6). $\phi = c_6 u$

$\sigma_u = -2c_6 u (u^2+v^2)^{-1/2}$

$\sigma_v = 2c_6 u (u^2+v^2)^{-1/2}$

$\tau_{uv} = -2c_6 v (u^2+v^2)^{-1/2}.$

(7). $\phi = c_7 v$

$\sigma_u = 2c_7 v (u^2+v^2)^{-1/2},$

$\sigma_v = -2c_7 v (u^2+v^2)^{-1/2},$

$\tau_{uv} = -2c_7 u (u^2+v^2)^{-1/2}.$

(8). $\phi = c_8 uv$

$\sigma_u = 0,$

$\sigma_v = 0,$

$\tau_{uv} = -6c_8(u^2+v^2)^{-1/2}$

(9). $\phi = c_9(u^2-v^2)$

$\sigma_u = -12c_9(u^2+v^2)^{1/2}.$

$\sigma_v = 12c_9(u^2+v^2)^{1/2}.$

$\tau_{uv} = 0.$

(10). $\phi = \dfrac{c_{10}u}{u^2+v^2}$

$\sigma_u = 6c_{10}u(3v^2-u^2)(u^2+v^2)^{-5/2},$

$\sigma_v = -6c_{10}u(3v^2-u^2)(u^2+v^2)^{-5/2},$

$\tau_{uv} = -6c_{10}v(3u^2-v^2)(u^2+v^2)^{-5/2}.$

(11). $\phi = \dfrac{c_{11}v}{u^2+v^2}$

$\sigma_u = 6c_{11}v(v^2-3u^2)(u^2+v^2)^{-5/2},$

$\sigma_v = -6c_{11}v(v^2-3u^2)(u^2+v^2)^{-5/2},$

$\tau_{uv} = -6c_{11}u(3v^2-u^2)(u^2+v^2)^{-5/2}.$

(12). $\phi = c_{12}(3u^2v - v^3)$

$\sigma_u = -30c_{12}v(u^2+v^2)^{1/2},$

$\sigma_v = 30c_{12}v(u^2+v^2)^{1/2},$

$\tau_{uv} = -30c_{12}u(u^2+v^2)^{1/2}.$

(13). $\phi = c_{13}(3uv^3 - u^3)$

$\sigma_u = 30c_{13}u(u^2+v^2)^{1/2},$

$\sigma_v = -30c_{13}u(u^2+v^2)^{1/2},$

$\tau_{uv} = 30c_{13}v(u^2+v^2)^{1/2}.$

(14). $\phi = c_{14}(6u^2v^2 - u^4 - v^4)$

$\sigma_u = 56c_{14}(u^2+v^2)^{-1/2}\cdot(u^4-v^4),$

$\sigma_v = -56c_{14}(u^2+v^2)^{-1/2}\cdot(u^4-v^4),$

$\tau_{uv} = -112c_{14}uv(u^2+v^2)^{1/2}.$

(15). $\phi = c_{15}(v^5+v^3-10v^3u^2-3vu^2-5u^4v)$

$\sigma_u = 30c_{15}(u^2+v^2)^{-1/2}\cdot(3v^5+v^3-9u^4v+u^2v-6u^2v^3),$

$\sigma_v = -30c_{15}(u^2+v^2)^{-1/2}\cdot(3v^5+v^3-9u^4v+u^2v-6u^2v^3),$

$\tau_{uv} = 30c_{15}(u^2+v^2)^{-1/2}\cdot(-3u^5+u^3+9uv^4+uv^2+6u^3v^2).$

(16). $\phi = c_{16}(u^5 + u^3 - 10u^3v^2 - 3uv^2 - 5v^4u)$

$\sigma_u = 30c_{16}(u^2+v^2)^{-1/2}(3u^5 + u^3 - 9uv^4 + uv^2 - 6u^3v^2).$

$\sigma_v = -30c_{16}(u^2+v^2)^{-1/2}(3u^5 + u^3 - 9uv^4 + uv^2 - 6u^3v^2).$

$\tau_{uv} = 30c_{16}(u^2+v^2)^{-1/2}(-3v^5 + v^3 + 9u^4v + u^2v + 6u^2v^3).$

(17). $\phi = c_{17}\, u\,(u^2+v^2)^{1/2}$

$\sigma_u = 0.$

$\sigma_v = 12c_{17}u.$

$\tau_{uv} = -6c_{17}v.$

(18). $\phi = c_{18}\, v\,(u^2+v^2)^{1/2}$

$\sigma_u = 12c_{18}v.$

$\sigma_v = 0.$

$\tau_{uv} = -6c_{18}u.$

(19). $\phi = c_{19}(u^2-v^2)(u^2+v^2)^{1/2}$

$\sigma_u = -10c_{19}(3v^2+u^2).$

$\sigma_v = 10c_{19}(3u^2+v^2).$

$\tau_{uv} = 0.$

(20). $\phi = C_{20}(u^2+v^2)^{1/2}$

$\sigma u = 2C_{20}$,

$\sigma v = 2C_{20}$.

$\tau uv = 0$.

(21). $\phi = C_{21}\,uv(u^2+v^2)^{1/2}$.

$\sigma u = 10C_{21}\,uv$,

$\sigma v = 10C_{21}\,uv$,

$\tau uv = -10C_{21}(u^2+v^2)$.

(22). $\phi = C_{22}(3u^2v - v^3)(u^2+v^2)^{1/2}$

$\sigma u = -56C_{22}v^3$,

$\sigma v = 28C_{22}(3u^2v + v^3)$,

$\tau uv = -42C_{22}\,u(u^2+v^2)$.

(23). $\phi = C_{23}(3v^2u - u^3)(u^2+v^2)^{1/2}$

$\sigma u = 28C_{23}(3uv^2 + u^3)$,

$\sigma v = -56C_{23}u^3$,

$\tau uv = -42C_{23}\,v(u^2+v^2)$.

(24). $\phi = C_{24} \log(u^2+v^2).$

$$\sigma u = 4C_{24}(u^2-v^2)(u^2+v^2)^{-3/2},$$

$$\sigma v = 4C_{24}(v^2-u^2)(u^2+v^2)^{-3/2},$$

$$\tau uv = 8C_{24}uv(u^2+v^2)^{-3/2}.$$

(25). $\phi = C_{25}\left(u \tan^{-1}\frac{u}{v} - \frac{v}{2}\log(u^2+v^2)\right).$

$$\sigma u = -2C_{25}(u^2+v^2)^{-1/2}\cdot\left(3v + u\tan^{-1}\frac{u}{v} + \frac{v}{2}\log(u^2+v^2)\right).$$

$$\sigma v = 2C_{25}(u^2+v^2)^{-1/2}\cdot\left(3v + u\tan^{-1}\frac{u}{v} + \frac{v}{2}\log(u^2+v^2)\right).$$

$$\tau uv = -2C_{25}(u^2+v^2)^{-1/2}\cdot\left(-3u + v\tan^{-1}\frac{u}{v} - \frac{u}{2}\log(u^2+v^2)\right).$$

(26). $\phi = C_{26}\left(v\tan^{-1}\frac{v}{u} - \frac{u}{2}\log(u^2+v^2)\right).$

$$\sigma u = -2C_{26}(u^2+v^2)^{-1/2}\cdot\left(3u + v\tan^{-1}\frac{v}{u} + \frac{u}{2}\log(u^2+v^2)\right).$$

$$\sigma v = 2C_{26}(u^2+v^2)^{-1/2}\cdot\left(3u + v\tan^{-1}\frac{v}{u} + \frac{u}{2}\log(u^2+v^2)\right).$$

$$\tau uv = -2C_{26}(u^2+v^2)^{-1/2}\cdot\left(-3v + u\tan^{-1}\frac{v}{u} - \frac{v}{2}\log(u^2+v^2)\right).$$

(27). $\phi = C_{27}(u^2+v^2)^{1/2}\cdot\log(u^2+v^2)$

$$\sigma u = 2C_{27}\left(4 + 2(v^2-u^2)(u^2+v^2)^{-1} + \log(u^2+v^2)\right).$$

$$\sigma v = 2C_{27}\left(4 + 2(u^2-v^2)(u^2+v^2)^{-1} + \log(u^2+v^2)\right).$$

$$\tau uv = -8C_{27}uv(u^2+v^2)^{-1}.$$

CHAPTER IV

APPLICATIONS:

(1). Consider a bridge like member subjected to load as shown in figure 5. P is load in lbs and b is width of member. Superimpose solution #13, #14 and #20 from Chapter III and obtain a ϕ function as following, adjust constants $c_{13} = B/30;\ c_{14} = A/56;\ c_{20} = C/2$ so following are; ϕ function and stress expressions.

$$\phi = A/56\,(6u^2v^2 - u^4 - v^4) + B/30\,(3uv^2 - u^3) + \frac{C}{2}\,(u^2+v^2)^{1/2}$$

$$\sigma_u = A(u^2+v^2)^{1/2}\cdot(u^2-v^2) + Bu(u^2+v^2)^{1/2} + C,$$

$$\sigma_v = -A(u^2+v^2)^{1/2}\cdot(u^2-v^2) - Bu(u^2+v^2)^{1/2} + C,$$

$$\tau_{uv} = -2Auv(u^2+v^2)^{1/2} + Bv(u^2+v^2)^{1/2}$$

Boundry conditions are as follows,

At $v = 0,\quad \tau_{uv} = 0,$

$u = u_0,\quad \tau_{uv} = 0,$

Applying above conditions we get,

$$\therefore\ 0 = (-2Au_0 + B)\,v\,(u^2+v^2)^{1/2}$$

$$\therefore\ B = 2Au_0.$$

Thus $\sigma_u = A(u^2+v^2)^{1/2}\cdot(u^2-v^2+2u_0u) + C,$

$$\sigma_v = -A(u^2+v^2)^{1/2}\cdot(u^2-v^2+2u_0u) + C,$$

In following figure, the σ_v; τ_{uv} stresses are shown, on next page σ_u stress is shown which is perpendicular to σ_v and also lies in the plane of page.

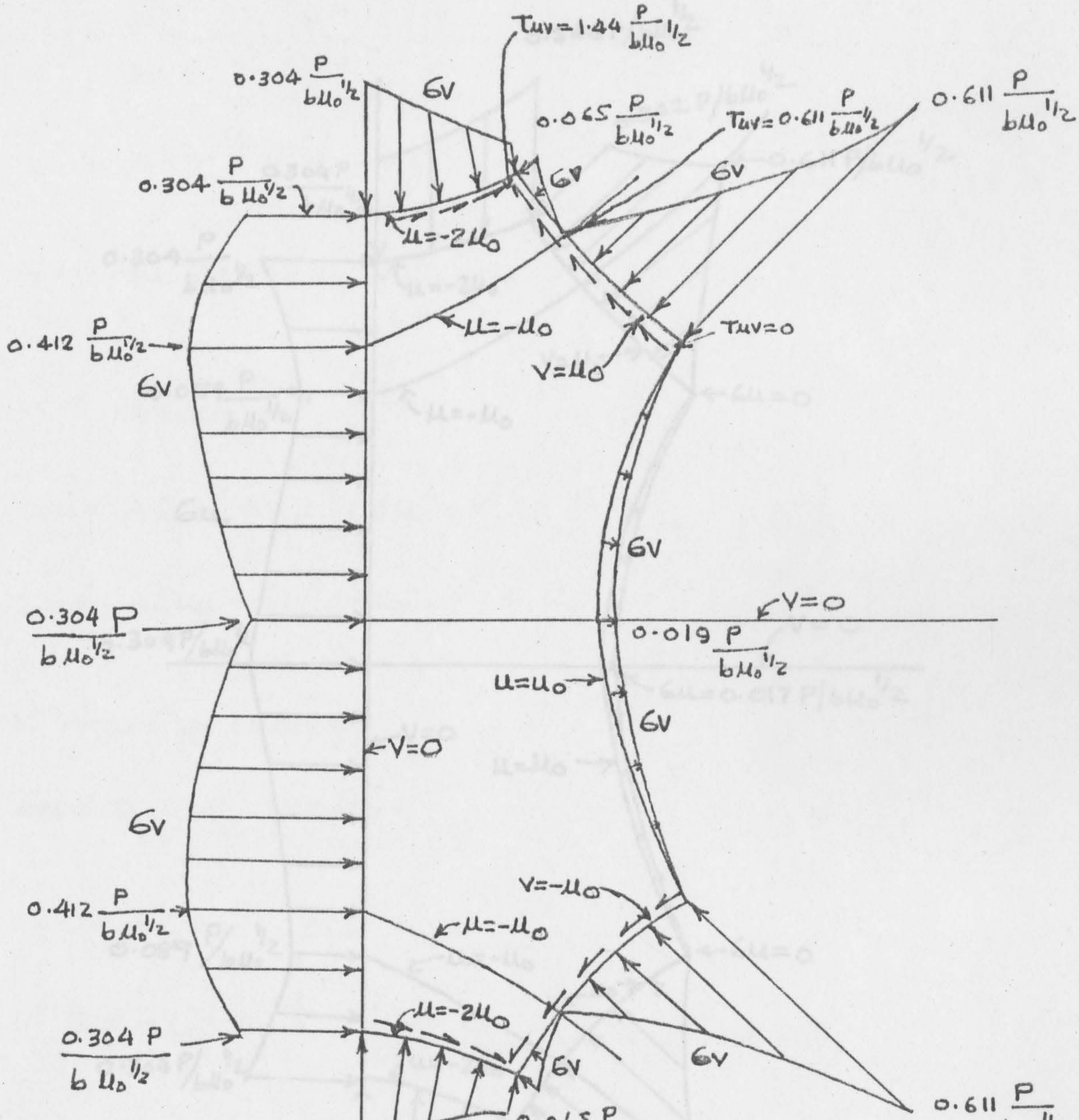

Figure 5

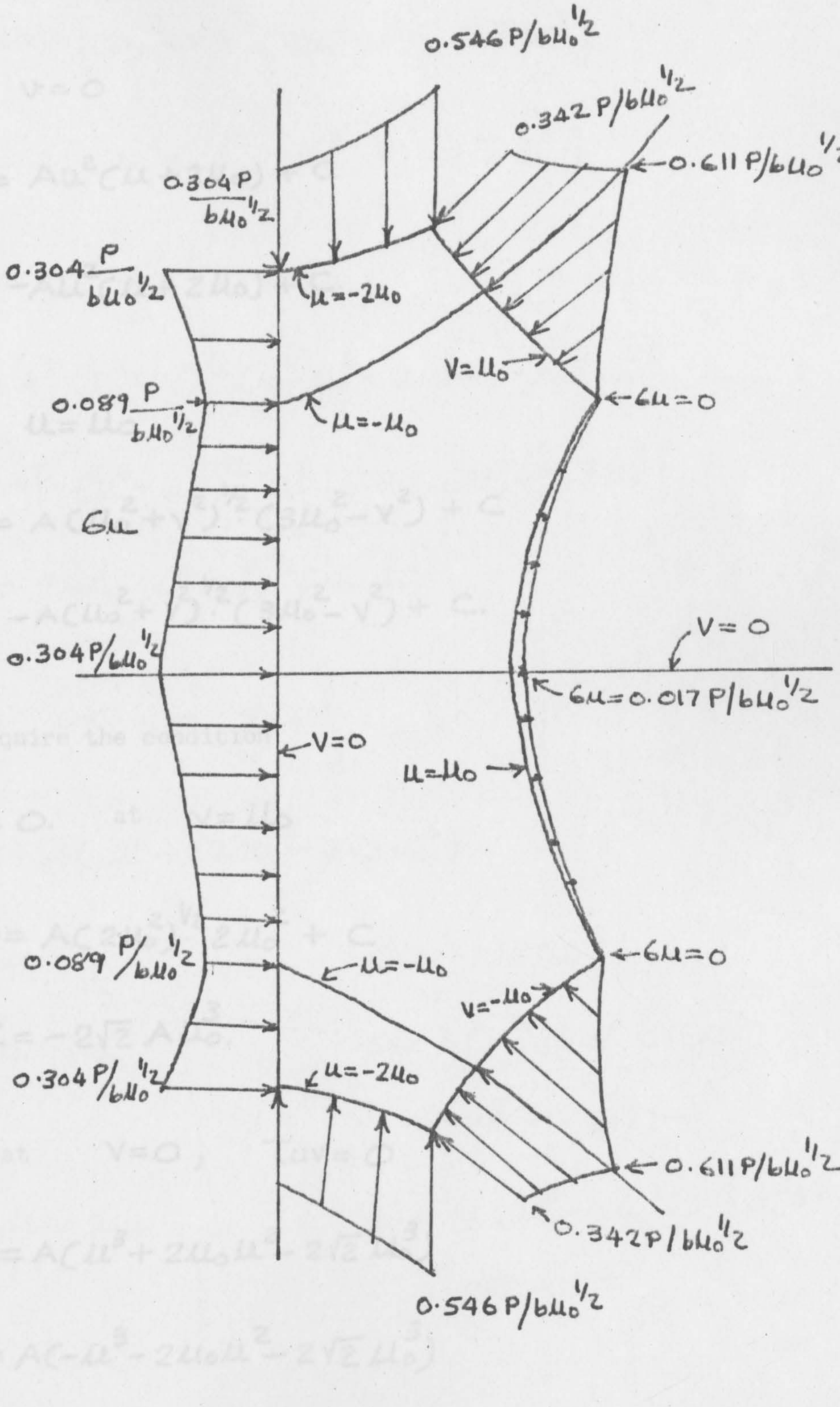
$0.546\,P/bu_0^{1/2}$
$0.342\,P/bu_0^{1/2}$
$0.611\,P/bu_0^{1/2}$
$0.304\,P/bu_0^{1/2}$
$0.304\,P/bu_0^{1/2}$
$u=-2u_0$
$V=u_0$
$\sigma u=0$
$0.089\,P/bu_0^{1/2}$
$u=-u_0$
σu
$V=0$
$0.304\,P/bu_0^{1/2}$
$\sigma u=0.017\,P/bu_0^{1/2}$
$V=0$
$u=u_0$
$\sigma u=0$
$0.089\,P/bu_0^{1/2}$
$u=-u_0$
$V=-u_0$
$0.304\,P/bu_0^{1/2}$
$u=-2u_0$
$0.611\,P/bu_0^{1/2}$
$0.342\,P/bu_0^{1/2}$
$0.546\,P/bu_0^{1/2}$

Figure 6.

$$\therefore \tau_{uv} = 2Av(u^2+v^2)^{1/2}(u_0-u)$$

At $v=0$

$$\sigma_u = Au^2(u+2u_0)+C$$

$$\sigma_v = -Au^2(u+2u_0)+C.$$

At $u=u_0$

$$\sigma_u = A(u_0^2+v^2)^{1/2}(3u_0^2-v^2)+C$$

$$\sigma_v = -A(u_0^2+v^2)^{1/2}(3u_0^2-v^2)+C.$$

We require the condition

$\sigma_u = 0.$ at $v = u_0$

$$\therefore 0 = A(2u_0^2)^{1/2}\,2u_0^2 + C$$

$$\therefore C = -2\sqrt{2}\,Au_0^3.$$

Then at $v=0$; $\tau_{uv}=0$

$$\sigma_u = A(u^3+2u_0u^2-2\sqrt{2}\,u_0^3)$$

$$\sigma_v = A(-u^3-2u_0u^2-2\sqrt{2}\,u_0^3)$$

And at $u = u_o$; $\tau_{uv} = 0$.

$$\therefore \sigma_u = A\left[(u_o^2+v^2)^{1/2}(3u_o^2 - v^2) - 2\sqrt{2}\,u_o^3\right]$$

$$\therefore \sigma_v = A\left[-(u_o^2+v^2)^{1/2}\cdot(3u_o^2 - v^2) - 2\sqrt{2}\,u_o^3\right]$$

Also at $v = u_o$

$$\sigma_u = A\left[(u^2+u_o^2)^{1/2}\cdot(u^2 - u_o^2 + 2u_o u) - 2\sqrt{2}\,u_o^3\right]$$

$$\sigma_v = A\left[-(u^2+u_o^2)^{1/2}\cdot(u^2 - u_o^2 + 2u_o u) + 2\sqrt{2}\,u_o^3\right]$$

$$\tau_{uv} = 2Au_o(u^2+u_o^2)^{1/2}\cdot(u - u_o)$$

Study σ_v along $v = 0$.

$$\sigma_v = A(-u^3 - 2u_o u^2 - 2\sqrt{2}\,u_o^3)$$

At $u = 0$ $\qquad \sigma_v = -2\sqrt{2}\,Au_o^3$

$u = -u_o$. $\qquad \sigma_v = A(u_o^3 - 2u_o^3 - 2\sqrt{2}\,u_o^3)$

$\qquad = Au_o^3(-1 - 2\sqrt{2})$

$u = -2u_o$ $\qquad \sigma_v = A(8u_o^3 - 8u_o^3 - 2\sqrt{2}\,u_o^3)$

$\qquad = -2\sqrt{2}\,Au_o^3$

Study σ_u along $v=0$

$$\sigma_u = A(u^3 + 2u_o u^2 - 2\sqrt{2}\,u_o^3)$$

At $u=0.$ $\qquad \sigma_u = -2\sqrt{2}\,A u_o^3$

At $u=-u_o$ $\qquad \sigma_u = A(-u_o^3 + 2u_o^3 - 2\sqrt{2}\,u_o^3)$

$$= A(1-2\sqrt{2})u_o^3.$$

At $u=-2u_o$ $\qquad \sigma_u = A(-8u_o^3 + 8u_o^3 - 2\sqrt{2}\,u_o^3)$

$$= -2\sqrt{2}\,A u_o^3.$$

Study σ_v along $u=u_o$

$$\sigma_v = A(-(u_o^2+v^2)^{1/2}\cdot(3u_o^2-v^2) - 2\sqrt{2}\,u_o^3)$$

At $v=0$ $\qquad \sigma_v = A u_o^3(-3-2\sqrt{2})$

$v=u_o$ $\qquad \sigma_v = -A u_o^3(4\sqrt{2})$

Study functions along $v=u_o$

$$\sigma_u = A\left[(u^2+u_o^2)^{1/2}\cdot(u^2-u_o^2+2u_o u) - 2\sqrt{2}\,u_o^3\right].$$

At $u=-2u_o$ $\qquad \sigma_u = A u_o^3\left[-\sqrt{5}-2\sqrt{2}\right].$

At $u=-u_o$ $\qquad \sigma_u = A u_o^3\left[-2\sqrt{2}-2\sqrt{2}\right]$

At $u=0$ $\qquad \sigma_u = A u_o^3\left[-1-2\sqrt{2}\right]$

At $u = u_0$ $\qquad \sigma_u = 0.$

$$\sigma_v = A\left[-(u^2+u_0^2)^{1/2}\cdot(u^2-u_0^2+2u_0u) - 2\sqrt{2}u_0^3\right]$$

At $u = -2u_0$ $\qquad \sigma_v = Au_0^3(\sqrt{5}-2\sqrt{2})$

$u = -u_0$ $\qquad \sigma_v = 0$

$u = 0$ $\qquad \sigma_v = (1-2\sqrt{2})Au_0^3$

$u = u_0$ $\qquad \sigma_v = (-2\sqrt{2}-2\sqrt{2})Au_0^3$

$$\tau_{uv} = 2Au_0(u^2+u_0^2)^{1/2}\cdot(u-u_0)$$

At $u = -2u_0$ $\qquad \tau_{uv} = -6\sqrt{5}Au_0^3$

$u = -u_0$ $\qquad \tau_{uv} = -4\sqrt{2}Au_0^3$

$u = 0$ $\qquad \tau_{uv} = -2Au_0^3$

$u = u_0$ $\qquad \tau_{uv} = 0.$

Along $u = -2u_0$

$$\sigma_u = A\left[(4u_0^2+v^2)^{1/2}\cdot(-v^2) - 2\sqrt{2}u_0^3\right]$$

$$\sigma_v = A\left[-(4u_0^2+v^2)^{1/2}\cdot(-v^2) - 2\sqrt{2}u_0^3\right]$$

$$\tau_{uv} = 2Av(4u_0^2+v^2)^{1/2}\cdot 3u_0.$$

At $V=0$ $\quad \sigma_u = -2\sqrt{2}\,A u_0^3$

$$\sigma_v = -2\sqrt{2}\,A u_0^3$$

$$\tau_{uv} = 0.$$

$V = u_0$ $\quad \sigma_u = A u_0^3\left[-\sqrt{5} - 2\sqrt{2}\right]$

$$\sigma_v = A u_0^3\left[\sqrt{5} - 2\sqrt{2}\right]$$

$$\tau_{uv} = 6 A u_0^3 \cdot \sqrt{5}$$

Total load $\quad P = 2\int_0^{+2u_0} \sigma_v \cdot b \cdot h_u \cdot du$

$$\therefore\ P = 2\int_0^{2u_0} A\left(u^3 - 2u_0 u^2 - 2\sqrt{2}\,u_0^3\right) b \cdot \frac{1}{2u^{1/2}}\, du$$

$$= Ab\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u_0 u^{5/2} - 4\sqrt{2}\,u_0^3 u^{1/2}\right]_0^{2u_0}$$

$$= Ab\left[16\sqrt{2}\left(\frac{1}{7} - \frac{1}{5}\right) - 8\right] u_0^{7/2} = -9.293\, b \cdot A u_0^{7/2}$$

$$\therefore\ A = -\frac{P}{9.293\, b \cdot u_0^{7/2}}$$

$$2\int_0^{2u_0} h_u\, du = 2\int_0^{2u_0} \frac{1}{2 \cdot u^{1/2}}\, du = 2\sqrt{2}\, u_0^{1/2}$$

$$\therefore\ P_{AVE.} = \frac{P}{2\sqrt{2}\, u_0^{1/2}} = -3.29\, A b u_0^3.$$

Now let us find out that how much total force is acting at $u = u_0$

The total force can be summed up as follows, Let P' be that force.

$$\therefore P' = 2\int_0^{u_0} \sigma_u \cdot b \cdot hv\,dv \cdot \cos(x, v)$$

$$= 2\int_0^{u_0} \sigma_u \cdot b \cdot \frac{dv}{2(u_0^2+v^2)^{1/4}} \cdot \frac{v \cdot \sqrt{8}\left[(u_0^2+v^2)^{1/2}+u_0^2\right]^{1/2}}{\sqrt{8}\,(u_0^2+v^2)^{1/4}\left[(u_0^2+v^2)^{1/2}+u_0\right]^{1/2}}$$

$$= \int_0^{u_0} \sigma_u \cdot b \cdot v \cdot dv \cdot \frac{1}{(u_0^2+v^2)^{1/2}}$$

$$= b\int_0^{u_0} A(3u_0^2 - v^2)\,v\,dv - 2\sqrt{2}\,A b u_0^3 \int_0^{u_0} \frac{v\,dv}{(u_0^2+v^2)^{1/2}}$$

$$= 1.25\,A b u_0^4 - 0.25\,A b u_0^4 + 2\sqrt{2}\,A b u_0^4$$

$$= 1.23\,A b u_0^4$$

$$\therefore P' = 1.23\,A b u_0^4$$

$$\therefore P' = -1.23\,\frac{P}{9.293\,b\,u_0^{7/2}} \cdot b\,u_0^4$$

$$\therefore P' = -0.134 \cdot P \cdot u_0^{1/2}$$

(2). Consider a long plate with thickness b units, loaded with triangular load uniformly increasing, zero at center o of the plate. Refer figure 7.

Take ϕ function and corresponding stress equations from result (9) from chapter III.

$$\phi = Cq(u^2 - v^2)$$

$$\sigma_u = -12Cq(u^2+v^2)^{1/2},$$

$$\sigma_v = 12Cq(u^2+v^2)^{1/2}, \quad \tau_{uv} = 0.$$

Then at top of plate we V=0, U=anything,

So we have $\sigma_u = -12Cqu$;

which is a compressive stress.

Shear stress τ_{uv} is zero on boundry and anywhere in the plate.

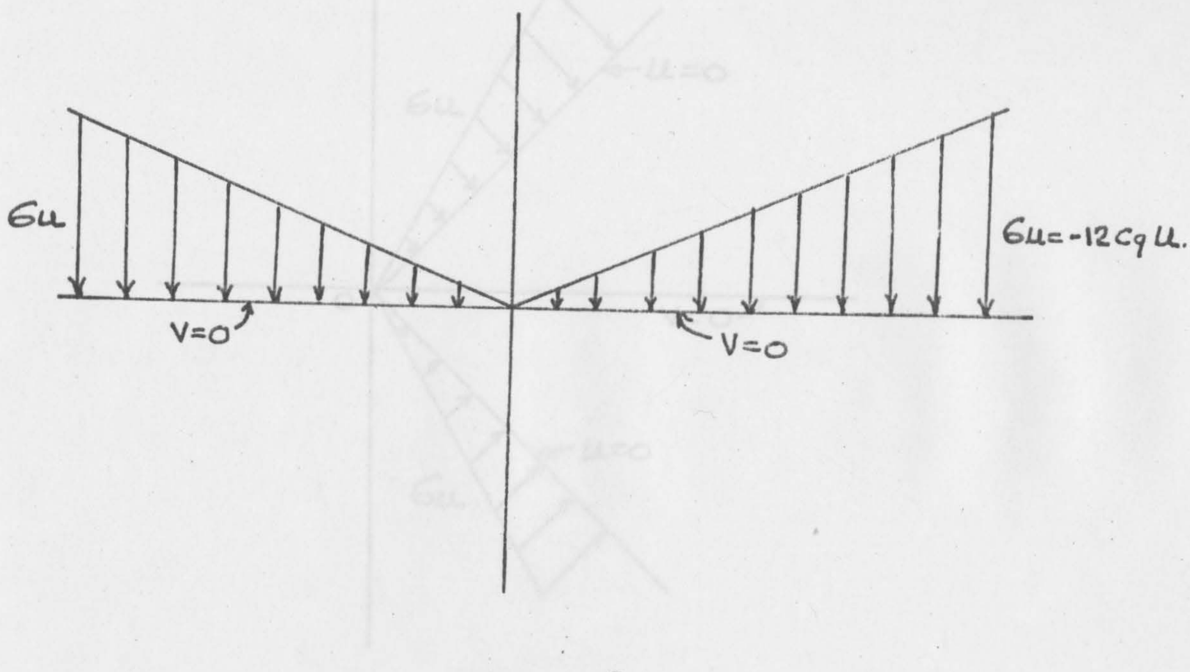

Figure 7.

(3). Consider a wedge having 90 corner with thickness b units loaded with triangular load uniformly increasing, zero at center O of the wedge. Refer figure 8.

Take ϕ function and corresponding stress equations from result (9) from chapter III.

$$\phi = Cq(u^2 - v^2)$$

$$\sigma_u = -12Cq(u^2+v^2)^{1/2},$$

$$\sigma_v = 12Cq(u^2+v^2)^{1/2},$$

$$\tau_{uv} = 0.$$

Then at surface of the wedge we have U=0, V=anything.

So we have $\sigma_u = -12CqV$ which is a compressive stress.

Shear stress τ_{uv} is zero on surface and anywhere in wedge.

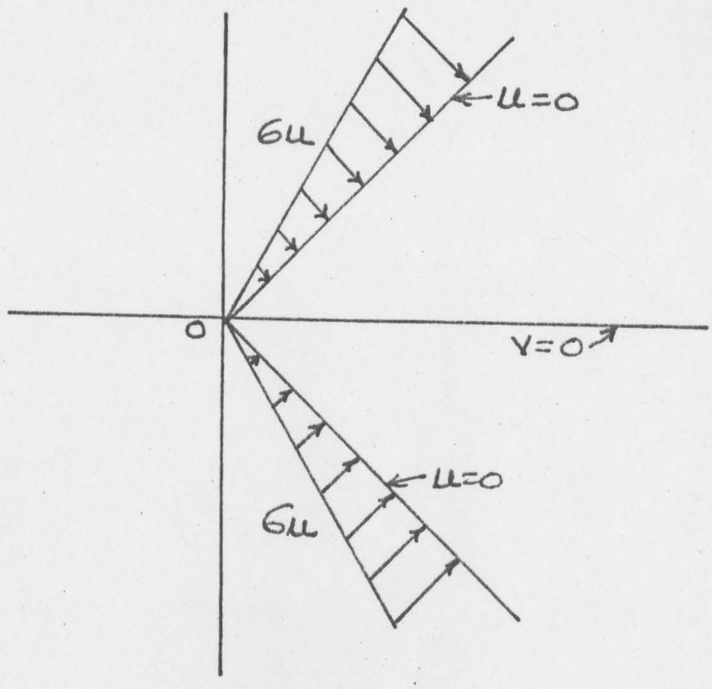

Figure 8.

CHAPTER V

CONCLUSION AND RECOMMENDATION:

After studying all possible ϕ functions and corresponding stress equations related to our coordinate system we can conclude that one can solve a real problem related to aforesaid coordinate system, but we might as well say that this coordinate system is not usable very much for practical purposes due to complexity which we have experienced while working on it and developing all ϕ functions and stress equations.

APPENDIX

In this section, it is shown that for particular chosen ϕ function, it satisfies the $\nabla^4\phi$ equation and then a detailed derivation of stress equations are given.

(1). Try $\phi = C_1 \sinh ku \cdot \sin kv$

$$\therefore \frac{\partial \phi}{\partial u} = C_1 k \cdot \sin kv \cdot \cosh ku.$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = C_1 k^2 \cdot \sin kv \cdot \sinh ku.$$

$$\therefore \frac{\partial \phi}{\partial v} = C_1 k \cos kv \cdot \sinh ku.$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = -C_1 k^2 \cdot \sin kv \cdot \sinh ku.$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = C_1 k^2 \cdot \cos kv \cdot \cosh ku.$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= C_1 k^2 \cdot \sin kv \cdot \sinh ku - C_1 k^2 \cdot \sin kv \cdot \sinh ku$$

$= 0.$ Therefore $\nabla^4 \phi$ equation is satisfied.

Now take the expressions for stress equations from page 6 and 7.

$$\sigma_u = \frac{1}{h_v^2} \cdot \frac{\partial^2 \phi}{\partial v^2} + \frac{1}{h_u^2 h_v} \cdot \frac{\partial \phi}{\partial u} \cdot \frac{\partial h_v}{\partial u} - \frac{1}{h_v^3} \cdot \frac{\partial \phi}{\partial v} \cdot \frac{\partial h_v}{\partial v}.$$

$h_u^2 = h_v^2 = \frac{1}{4} \cdot (u^2+v^2)^{-1/2}$ Substituting this in above equation,

$$\therefore \sigma_u = 2(u^2+v^2)^{-1/2} \left[2(u^2+v^2) \frac{\partial^2 \phi}{\partial v^2} - u \frac{\partial \phi}{\partial u} + v \frac{\partial \phi}{\partial v} \right].$$

$$\sigma_v = \frac{1}{h_u^2}\cdot\frac{\partial^2\phi}{\partial u^2} + \frac{1}{h_v^2 h_u}\cdot\frac{\partial h_u}{\partial v}\cdot\frac{\partial\phi}{\partial v} - \frac{1}{h_u^3}\cdot\frac{\partial\phi}{\partial u}\cdot\frac{\partial h_u}{\partial u}.$$

$$\therefore \sigma_v = 2(u^2+v^2)^{2^{-1/2}}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\tau_{uv} = -\frac{1}{h_u^2}\cdot\frac{\partial^2\phi}{\partial u\partial v} + \frac{1}{h_u^3}\cdot\frac{\partial\phi}{\partial v}\cdot\frac{\partial h_v}{\partial u} + \frac{1}{h_u^3}\cdot\frac{\partial\phi}{\partial u}\cdot\frac{\partial h_u}{\partial v}.$$

$$\therefore \tau_{uv} = -2(u^2+v^2)^{2^{-1/2}}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right].$$

$$\sigma_u = 2C_1\cdot(u^2+v^2)^{2^{-1/2}}\Big[-2K^2(u^2+v^2)\cdot\sinh Ku\cdot\sin Kv - Ku\cosh Ku\cdot\sin Kv$$
$$+ Kv\sinh Ku\cdot\cos Kv\Big].$$

$$\sigma_v = -2C_1\cdot(u^2+v^2)^{2^{-1/2}}\Big[-2K^2(u^2+v^2)\cdot\sinh Ku\cdot\sin Kv - Ku\cosh Ku\cdot\sin Kv$$
$$+ Kv\sinh Ku\cdot\cos Kv\Big].$$

$$\tau_{uv} = -2C_1\cdot(u^2+v^2)^{2^{-1/2}}\Big[2K^2(u^2+v^2)\cosh Ku\cdot\cos Kv + Ku\sinh Ku\cdot\cos Kv$$
$$+ Kv\cosh Ku\cdot\sin Kv\Big].$$

(2). Try $\phi = C_2 \sinh ku \cdot \cos kv.$

$$\therefore \frac{\partial \phi}{\partial u} = C_2 k \cdot \cosh ku \cdot \cos kv,$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = -C_2 k^2 \cdot \cosh ku \cdot \sin kv,$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = C_2 k^2 \cdot \sinh ku \cdot \cos kv,$$

$$\therefore \frac{\partial \phi}{\partial v} = -C_2 k \cdot \sinh ku \cdot \sin kv,$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = -C_2 k^2 \cdot \sinh ku \cdot \cos kv.$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2},$$

$$= C_2 k^2 \cdot \sinh ku \cdot \cos kv - C_2 k^2 \cdot \sinh ku \cdot \cos kv$$

$= 0.$ Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-\frac{1}{2}}\left[2(u^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right],$$

$$\therefore \sigma_u = 2C_2(u^2+v^2)^{-\frac{1}{2}}\left[-2k^2(u^2+v^2)\sinh ku \cdot \cos kv - ku \cosh ku \cdot \cos kv - kv \sinh ku \cdot \sin kv\right].$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \sigma_v = -2C_2(u^2+v^2)^{-1/2}\Big[-2k^2(u^2+v^2)\sinh ku\cdot\cos kv - ku\cosh ku\cdot\cos kv$$

$$-kv\sinh ku\cdot\sin kv\Big].$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right],$$

$$\therefore \tau_{uv} = 2C_2(u^2+v^2)^{-1/2}\Big[2k^2(u^2+v^2)\cosh ku\cdot\sin kv + ku\sinh ku\cdot\sin kv$$

$$-kv\cosh ku\cdot\cos kv\Big].$$

(3). $\phi = C_3 \cosh ku \cdot \cos kv$. Similarly it can be shown that it satisfies the $\nabla^4\phi$ equation and corresponding stress equations are as follows.

$$\sigma_u = 2C_3(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cdot\cos kv - kv\cosh ku\cdot\sin kv\right].$$

$$\sigma_v = -2C_3(u^2+v^2)^{-1/2}\left[-2k^2(u^2+v^2)\cosh ku\cdot\cos kv - ku\sinh ku\cdot\cos kv - kv\cosh ku\cdot\sin kv\right],$$

$$\tau_{uv} = 2C_3(u^2+v^2)^{-1/2}\left[2k^2(u^2+v^2)\sinh ku\cdot\sin kv - ku\cosh ku\cdot\sin kv - kv\sinh ku\cdot\cos kv\right].$$

(4). Similarly $\phi = C_4 \cosh ku \cdot \sin kv$ also satisfies the $\nabla^4\phi$ equation and corresponding stress equations are as follows.

$$\sigma u = 2C_4 (u^2+v^2)^{2^{-1/2}} \Big[-2k^2(u^2+v^2)\cosh ku \cdot \cos kv - ku \sinh ku \cdot \sin kv$$

$$+ kv \cosh ku \cdot \cos kv\Big].$$

$$\sigma v = -2C_4 (u^2+v^2)^{2^{-1/2}} \Big[-2k^2(u^2+v^2)\cosh ku \cdot \cos kv - ku \sinh ku \cdot \sin kv$$

$$+ kv \cosh ku \cdot \cos kv\Big].$$

$$\tau uv = -2C_4 (u^2+v^2)^{2^{-1/2}} \Big[2k^2(u^2+v^2)\sinh ku \cdot \cos kv + ku \cosh ku \cdot \cos kv$$

$$+ kv \sinh ku \cdot \sin kv\Big].$$

(5). When $\phi = C_5$.

Since C_5 is a constant, differentiation of ϕ with respect to u and v will be zero and so it is obvious that all stress expressions will be as follows:

$$\sigma_u = 0,$$

$$\sigma_v = 0,$$

$$\tau_{uv} = 0.$$

(6). Try $\phi = C_6 u$

$$\therefore \frac{\partial\phi}{\partial u} = C_6 .$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = 0.$$

$$\therefore \frac{\partial\phi}{\partial v} = 0.$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = 0.$$

$$\therefore \frac{\partial^2\phi}{\partial u \partial v} = 0.$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= 0 + 0.$$

$= 0.$ Therefore $\nabla^4\phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore \sigma_u = -2C_6 u \cdot (u^2+v^2)^{-1/2}.$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \sigma_v = 2C_6 u \cdot (u^2+v^2)^{-1/2}$$

$$\tau_{uv} = -2(u^2+v^2)^{-\frac{1}{2}}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\,\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \tau_{uv} = -2C_6 v\cdot(u^2+v^2)^{-\frac{1}{2}}.$$

(7). Try $\phi = C_7 v$

$$\therefore \frac{\partial \phi}{\partial v} = C_7 .$$

$$\therefore \frac{\partial \phi}{\partial \mu} = 0.$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = 0.$$

$$\therefore \frac{\partial^2 \phi}{\partial \mu^2} = 0.$$

$$\therefore \frac{\partial^2 \phi}{\partial \mu \partial v} = 0.$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial \mu^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= 0 + 0.$$

$= 0.$ Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma_\mu = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - \mu\frac{\partial \phi}{\partial \mu} + v\frac{\partial \phi}{\partial v}\right],$$

$$\therefore \sigma_\mu = 2C_7 v \cdot (\mu^2+v^2)^{-1/2}$$

$$\sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2 \phi}{\partial \mu^2} - v\frac{\partial \phi}{\partial v} + \mu\frac{\partial \phi}{\partial \mu}\right].$$

$$\therefore \sigma_v = -2C_7 v \cdot (\mu^2+v^2)^{-1/2}.$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\,\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right],$$

$$\therefore\ \tau_{uv} = -2C_7 u\cdot(u^2+v^2)^{-1/2}.$$

(8). Try $\phi = c_8 \mu v$

$\therefore \dfrac{\partial \phi}{\partial \mu} = c_8 v.$

$\therefore \dfrac{\partial^2 \phi}{\partial \mu^2} = 0.$

$\therefore \dfrac{\partial \phi}{\partial v} = c_8 \mu.$

$\therefore \dfrac{\partial^2 \phi}{\partial v^2} = 0.$

$\therefore \dfrac{\partial^2 \phi}{\partial \mu \partial v} = c_8.$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial \mu^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= 0 + 0.$$

$= 0.$ Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma_\mu = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - \mu\frac{\partial \phi}{\partial \mu} + v\frac{\partial \phi}{\partial v}\right].$$

$\therefore \sigma_\mu = 0.$

$$\sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2 \phi}{\partial \mu^2} - v\frac{\partial \phi}{\partial v} + \mu\frac{\partial \phi}{\partial \mu}\right].$$

$\therefore \sigma_v = 0.$

$$\tau_{\mu v} = -2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right],$$

$$\therefore \tau_{\mu v} = -6C_8(\mu^2+v^2)^{-1/2}$$

(9). Try $\phi = Cq(u^2 - v^2)$.

$$\therefore \frac{\partial \phi}{\partial u} = 2Cqu,$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = 2Cq.$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = 0,$$

$$\therefore \frac{\partial \phi}{\partial v} = -2Cqv,$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = -2Cq.$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= 2Cq - 2Cq$$

$= 0$ Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right],$$

$$\therefore \sigma u = -12\,Cq\,(u^2+v^2)^{1/2}$$

$$\sigma v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2 \phi}{\partial u^2} - v\frac{\partial \phi}{\partial v} + u\frac{\partial \phi}{\partial u}\right]$$

$$\therefore \sigma_v = 12 C_9 (u^2+v^2)^{1/2}$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u \partial v} + u\frac{\partial \phi}{\partial v} + v\frac{\partial \phi}{\partial u}\right]$$

$$\therefore \tau_{uv} = 0.$$

(10). Try $\phi = \dfrac{c_{10}u}{u^2+v^2}$

$$\therefore \frac{\partial \phi}{\partial u} = \frac{c_{10}(v^2-u^2)}{(u^2+v^2)^2},$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = \frac{c_{10}(2u^3-6uv^2)}{(u^2+v^2)^3},$$

$$\therefore \frac{\partial \phi}{\partial v} = -\frac{2c_{10}uv}{(u^2+v^2)^2},$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = \frac{c_{10}(6uv^2-2u^3)}{(u^2+v^2)^3},$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = \frac{c_{10}(6u^2v-2v^3)}{(u^2+v^2)^3},$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= \frac{c_{10}(2u^3-6uv^2)}{(u^2+v^2)^3} + \frac{c_{10}(6uv^2-2u^3)}{(u^2+v^2)^3}$$

$= 0$. Therefore $\nabla^4 \phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right]$$

$$\therefore 6u = 6c_{10}u(3v^2-u^2)(u^2+v^2)^{-5/2}.$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right]$$

$$\therefore \sigma_v = -6C_{10}u(3v^2-u^2)(u^2+v^2)^{-5/2}.$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right]$$

$$\therefore \tau_{uv} = -6C_{10}v(3u^2-v^2)(u^2+v^2)^{-5/2}.$$

(11). Similarly $\phi = \dfrac{C_{11} v}{u^2 + v^2}$ also satisfies the $\nabla^4 \phi$ equation and corresponding stress equations are as follows.

$$\sigma_u = 6 C_{11} v (v^2 - 3u^2)(u^2 + v^2)^{-5/2},$$

$$\sigma_v = -6 C_{11} v (v^2 - 3u^2)(u^2 + v^2)^{-5/2},$$

$$\tau_{uv} = -6 C_{11} u (3v^2 - u^2)(u^2 + v^2)^{-5/2}.$$

(12). Try $\phi = C_{12}(3u^2v - v^3)$

$\therefore \dfrac{\partial\phi}{\partial u} = 6C_{12}uv,$

$\therefore \dfrac{\partial^2\phi}{\partial u^2} = 6C_{12}v,$

$\therefore \dfrac{\partial\phi}{\partial v} = 3C_{12}(u^2 - v^2),$

$\therefore \dfrac{\partial^2\phi}{\partial v^2} = -6C_{12}v$

$\therefore \dfrac{\partial^2\phi}{\partial u\partial v} = 6C_{12}u.$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= 6C_{12}v - 6C_{12}v$$

$= 0.$ Therefore $\nabla^4\phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore 6u = -30C_{12}v(u^2+v^2)^{1/2}.$$

$$\sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu^2} - v\frac{\partial\phi}{\partial v} + \mu\frac{\partial\phi}{\partial\mu}\right],$$

$$\therefore \sigma_v = 30C_{12}v(\mu^2+v^2)^{1/2},$$

$$\tau_{\mu v} = -2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right],$$

$$\therefore \tau_{\mu v} = -30C_{12}\mu(\mu^2+v^2)^{1/2}.$$

(13). Similarly $\phi = C_{13}(3uv^2 - u^3)$. also satisfies the $\nabla^4\phi$ equation and corresponding stress equations are as follows.

$$\sigma_u = 30 C_{13}\, u (u^2+v^2)^{1/2}$$

$$\sigma_v = -30 C_{13}\, u (u^2+v^2)^{1/2}$$

$$\tau_{uv} = 30 C_{13}\, v (u^2+v^2)^{1/2}$$

(14). Try $\phi = c_{14}(6u^2v^2 - u^4 - v^4)$,

$$\therefore \frac{\partial\phi}{\partial u} = c_{14}(12uv^2 - 4u^3),$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = 12c_{14}(v^2 - u^2),$$

$$\therefore \frac{\partial\phi}{\partial v} = c_{14}(12u^2v - 4v^3),$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = 12c_{14}(u^2 - v^2),$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = 24c_{14}uv,$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= 12c_{14}(v^2 - u^2) + 12c_{14}(u^2 - v^2)$$

$= 0$. Therefore $\nabla^4\phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore \sigma_u = 56c_{14}(u^2+v^2)^{-1/2}(u^4 - v^4)$$

$$\therefore \sigma_u = 56c_{14}(u^2+v^2)^{1/2}(u^2 - v^2).$$

$$\therefore \sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu^2} - v\frac{\partial\phi}{\partial v} + \mu\frac{\partial\phi}{\partial\mu}\right].$$

$$\therefore \sigma_v = -56C_{14}(\mu^2+v^2)^{-1/2}(\mu^4-v^4)$$

$$\therefore \sigma_v = -56C_{14}(\mu^2+v^2)^{1/2}(\mu^2-v^2).$$

$$\tau_{\mu v} = -2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right].$$

$$\therefore \tau_{\mu v} = -112C_{14}\cdot\mu v(\mu^2+v^2)^{1/2}.$$

(15). Try $\quad \phi = C_{15}(v^5 + v^3 - 10v^3u^2 - 3vu^2 - 5u^4v).$

$$\therefore \frac{\partial\phi}{\partial u} = C_{15}(-20v^3u - 6uv + 20u^3v),$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = C_{15}(-20v^3 - 6v + 60u^2v),$$

$$\therefore \frac{\partial\phi}{\partial v} = C_{15}(5v^4 + 3v^2 - 30u^2v^2 - 3u^2 + 5u^4),$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = C_{15}(20v^3 + 6v - 60u^2v),$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = C_{15}(-60uv^2 - 6u + 20u^3),$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= C_{15}(-20v^3 - 6v + 60u^2v) + C_{15}(20v^3 + 6v - 60u^2v),$$

$= 0.$ Therefore $\nabla^4\phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore 6u = 30C_{15}(u^2+v^2)^{-1/2}(-6u^2v^3 + u^2v - 9u^4v + 3v^5 + v^3).$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right],$$

$$\therefore \sigma_v = -30C_{15}(u^2+v^2)^{-1/2}\cdot(-6u^2v^3 + u^2v - 9u^4v + 3v^5 + v^3).$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right],$$

$$\therefore \tau_{uv} = 30C_{15}(u^2+v^2)^{-1/2}\cdot(-3u^5 + u^3 + 9uv^4 + uv^2 + 6u^3v^2).$$

(16). Similarly $\phi = C_{16}(u^5 + u^3 - 10u^3v^2 - 3uv^2 - 5v^4u)$.
also satisfies the $\nabla^4\phi$ equation and corresponding stress equations are as follows.

$$\sigma_u = 30C_{16}(u^2+v^2)^{-1/2} \cdot (3u^5 + u^3 - 9uv^4 + uv^2 - 6u^3v^2).$$

$$\sigma_v = -30C_{16}(u^2+v^2)^{-1/2} \cdot (3u^5 + u^3 - 9uv^4 + uv^2 - 6u^3v^2).$$

$$\tau_{uv} = 30C_{16}(u^2+v^2)^{-1/2} \cdot (-3v^5 + v^3 + 9u^4v + u^2v + 6u^2v^3).$$

(17). Try $\phi = C_{17}u(u^2+v^2)^{1/2}$

$$\therefore \frac{\partial\phi}{\partial u} = C_{17}\left[(u^2+v^2)^{1/2} + u^2(u^2+v^2)^{-1/2}\right].$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = C_{17}\left[3u(u^2+v^2)^{-1/2} - u^3(u^2+v^2)^{-3/2}\right].$$

$$\therefore \frac{\partial\phi}{\partial v} = C_{17}uv(u^2+v^2)^{-1/2}.$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = C_{17}\left[u(u^2+v^2)^{-1/2} - uv^2(u^2+v^2)^{-3/2}\right].$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = C_{17}\left[v(u^2+v^2)^{-1/2} - u^2v(u^2+v^2)^{-3/2}\right].$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= 3C_{17}u(u^2+v^2)^{-1/2}$$

$$\therefore \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[(u^2+v^2)^{1/2}\nabla^2\phi\right] = \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[3C_{17}u\right] = 0.$$

Therefore $\nabla^4\phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right]$$

$$\therefore 6u = 0.$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right],$$

$$\therefore \sigma_v = 12C_{17}u.$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right]$$

$$\therefore \tau_{uv} = -6C_{17}v.$$

(18). Similarly $\phi = C_{18} V (u^2+v^2)^{1/2}$ also satisfies the $\nabla^4\phi$ equation and corresponding stress equations are as follows.

$$\sigma_u = 12 C_{18} V.$$

$$\sigma_v = 0.$$

$$\tau_{uv} = -6C_{18}u.$$

(19). Try $\phi = C_{19}(u^2 - v^2)(u^2+v^2)^{1/2}$

$$\therefore \frac{\partial\phi}{\partial u} = C_{19}u(u^2-v^2)(u^2+v^2)^{-1/2} + 2C_{19}u(u^2+v^2)^{1/2}$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = C_{19}(5u^2-v^2)(u^2+v^2)^{-1/2} - C_{19}u^2(u^2-v^2)(u^2+v^2)^{-3/2} + 2C_{19}(u^2+v^2)^{1/2}$$

$$\therefore \frac{\partial\phi}{\partial v} = C_{19}v(u^2-v^2)(u^2+v^2)^{-1/2} - 2C_{19}v(u^2+v^2)^{1/2}.$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = C_{19}(u^2-5v^2)(u^2+v^2)^{-1/2} - C_{19}v^2(u^2-v^2)(u^2+v^2)^{-3/2} - 2C_{19}(u^2+v^2)^{1/2},$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = -C_{19}v(u^2-v^2)(u^2+v^2)^{-3/2}.$$

$$\therefore \nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$

$$= 5C_{19}(u^2-v^2)\cdot(u^2+v^2)^{-1/2}$$

$$\therefore (u^2+v^2)^{1/2}\cdot\nabla^2\phi = 5C_{19}(u^2-v^2).$$

$$\therefore \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\left((u^2+v^2)^{1/2}\cdot\nabla^2\phi\right) = 10C_{19} - 10C_{19} = 0.$$

Therefore $\nabla^4\phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore 6u = -10C_{19}(3v^2+u^2).$$

$$\sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu^2} - v\frac{\partial\phi}{\partial v} + \mu\frac{\partial\phi}{\partial\mu}\right].$$

$$\therefore \sigma_v = 10C_{19}(3\mu^2+v^2).$$

$$\tau_{\mu v} = -2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right],$$

$$\therefore \tau_{\mu v} = 0.$$

(20). Try $\phi = C_{20}(u^2+v^2)^{1/2}$

$$\therefore \frac{\partial\phi}{\partial u} = C_{20}u(u^2+v^2)^{-1/2}$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = -C_{20}u^2(u^2+v^2)^{-3/2} + C_{20}(u^2+v^2)^{-1/2},$$

$$\therefore \frac{\partial\phi}{\partial v} = C_{20}v(u^2+v^2)^{-1/2},$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = -C_{20}v^2(u^2+v^2)^{-3/2} + C_{20}(u^2+v^2)^{-1/2},$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = -C_{20}uv(u^2+v^2)^{-3/2}.$$

$$\therefore \nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2},$$

$$= C_{20}(u^2+v^2)^{-1/2}$$

$$\therefore (u^2+v^2)^{1/2}\cdot\nabla^2\phi = C_{20}.$$

$$\therefore \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[(u^2+v^2)^{1/2}\cdot\nabla^2\phi\right] = \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[C_{20}\right] = 0.$$

Therefore $\nabla^4\phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right]$$

$$\therefore \sigma_u = 2C_{20}.$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \sigma_v = 2C_{20},$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \tau_{uv} = 0.$$

(21). Try $\phi = C_{21}\, uv(u^2+v^2)^{1/2}$

$$\therefore \frac{\partial \phi}{\partial u} = C_{21} v\left[(u^2+v^2)^{1/2} + u^2(u^2+v^2)^{-1/2}\right],$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = 3C_{21} uv(u^2+v^2)^{-1/2} - C_{21} u^3 v(u^2+v^2)^{-3/2},$$

$$\therefore \frac{\partial \phi}{\partial v} = C_{21} u\left[(u^2+v^2)^{1/2} + v^2(u^2+v^2)^{-1/2}\right],$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = 3C_{21} uv(u^2+v^2)^{-1/2} - C_{21} uv^3(u^2+v^2)^{-3/2},$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = 2C_{21}(u^2+v^2)^{1/2} - C_{21} u^2 v^2 (u^2+v^2)^{-3/2}.$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2},$$

$$= 5C_{21} uv(u^2+v^2)^{-1/2}$$

$$\therefore (u^2+v^2)^{-1/2} \cdot \nabla^2 \phi = 5C_{21} uv,$$

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[(u^2+v^2)^{1/2} \cdot \nabla^2 \phi\right] = \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right] \cdot 5C_{21} uv = 0.$$

Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right].$$

$$\therefore \sigma_u = 10\,C_{21}\,uv.$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \sigma_v = 10\,C_{21}\,uv$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\,\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right]$$

$$\therefore \tau_{uv} = -10\,C_{21}\,(u^2+v^2).$$

(22). Try $\Phi = C_{22}(3u^2v - v^3)(u^2+v^2)^{1/2}$

$$\therefore \frac{\partial \phi}{\partial u} = 6C_{22}uv(u^2+v^2)^{1/2} + C_{22}u(3u^2v - v^3)(u^2+v^2)^{-1/2}$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = 6C_{22}v(u^2+v^2)^{1/2} + C_{22}(15u^2v - v^3)(u^2+v^2)^{-1/2}$$
$$- C_{22}u^2(3u^2v - v^3)(u^2+v^2)^{-3/2},$$

$$\therefore \frac{\partial \phi}{\partial v} = C_{22}(3u^2 - 3v^2)(u^2+v^2)^{1/2} + C_{22}v(3u^2v - v^3)(u^2+v^2)^{-1/2}$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = -6C_{22}v(u^2+v^2)^{1/2} + C_{22}(9u^2v - 7v^3)(u^2+v^2)^{-1/2}$$
$$- C_{22}v^2(3u^2v - v^3)(u^2+v^2)^{-3/2},$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = 9C_{22}u(u^2+v^2)^{1/2} - C_{22}uv(3u^2v - v^3)(u^2+v^2)^{-3/2},$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$
$$= C_{22}(u^2+v^2)^{-1/2} \cdot (21u^2v - 7v^3).$$

$$\therefore (u^2+v^2)^{1/2} \cdot \nabla^2 \phi = 7C_{22}(3u^2v - v^3),$$

$$\therefore \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[(u^2+v^2)^{1/2} \cdot \nabla^2 \phi\right] = \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]\left[7C_{22}(3u^2v - v^3)\right] = 0.$$

Therefore $\nabla^4 \phi$ equation is satisfied.

$$\sigma_\mu = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial v^2} - \mu\frac{\partial\phi}{\partial\mu} + v\frac{\partial\phi}{\partial v}\right].$$

$$\therefore \sigma_\mu = -56C_{22}v^3.$$

$$\sigma_v = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu^2} - v\frac{\partial\phi}{\partial v} + \mu\frac{\partial\phi}{\partial\mu}\right]$$

$$\therefore \sigma_v = 28C_{22}(3\mu^2v+v^3),$$

$$\tau_{\mu v} = -2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right].$$

$$\therefore \tau_{\mu v} = -42C_{22}(\mu^2+v^2)\mu.$$

(23). Similarly $\phi = C_{23}(3v^2u - u^3)(u^2+v^2)^{1/2}$

also satisfies the $\nabla^4\phi$ equation and following are the corresponding stress equations.

$$\sigma u = 28\, C_{23}(3uv^2 + u^3)$$

$$\sigma v = -56\, C_{23}u^3,$$

$$\tau uv = -42\, C_{23}v(u^2+v^2).$$

(24). Try $$\phi = c_{24} \log(u^2+v^2)$$

$$\therefore \frac{\partial \phi}{\partial u} = c_{24} \cdot \frac{2u}{u^2+v^2};$$

$$\therefore \frac{\partial^2 \phi}{\partial u^2} = 2c_{24} \frac{v^2-u^2}{(u^2+v^2)^2},$$

$$\therefore \frac{\partial^2 \phi}{\partial u \partial v} = -2c_{24} \frac{uv}{(u^2+v^2)^2};$$

$$\therefore \frac{\partial \phi}{\partial v} = 2c_{24} \frac{v}{u^2+v^2};$$

$$\therefore \frac{\partial^2 \phi}{\partial v^2} = 2c_{24} \frac{u^2-v^2}{(u^2+v^2)^2},$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$$= 2c_{24} \frac{v^2-u^2}{(u^2+v^2)^2} + 2c_{24} \frac{u^2-v^2}{(u^2+v^2)^2}$$

$$= 0.$$

Therefore $\nabla^4 \phi$ equation is satisfied.

$$6u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2 \phi}{\partial v^2} - u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right]$$

$$\therefore 6u = \frac{4c_{24}(u^2-v^2)}{(u^2+v^2)^{3/2}};$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right].$$

$$\therefore \sigma_v = \frac{4c_{24}(v^2-u^2)}{(u^2+v^2)^{3/2}};$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right];$$

$$\therefore \tau_{uv} = \frac{8c_{24}uv}{(u^2+v^2)^{3/2}},$$

(25). Try $\phi = c_{25}\left(u\tan^{-1}\frac{u}{v} - \frac{v}{2}\operatorname{Log}(u^2+v^2)\right).$

$$\therefore \frac{\partial\phi}{\partial u} = c_{25}\cdot\tan^{-1}\frac{u}{v};$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = \frac{c_{25}\cdot v}{u^2+v^2};$$

$$\therefore \frac{\partial\phi}{\partial v} = -c_{25}\left[1+\frac{1}{2}\operatorname{Log}(u^2+v^2)\right],$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = -\frac{c_{25}\cdot v}{u^2+v^2};$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = -\frac{c_{25}\cdot u}{u^2+v^2};$$

$$\therefore \nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2};$$

$$= \frac{c_{25}v}{u^2+v^2} + \frac{-c_{25}v}{u^2+v^2}$$

$$= 0.$$

Therefore $\nabla^4\phi$ equation is satisfied.

$$\sigma_u = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial v^2} - u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right],$$

$$\therefore \sigma_u = 2c_{25}(u^2+v^2)^{-1/2}\left[-3v - u\tan^{-1}\frac{u}{v} - \frac{v}{2}\operatorname{Log}(u^2+v^2)\right];$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + u\frac{\partial\phi}{\partial u}\right];$$

$$\therefore \sigma_v = 2C_{25}(u^2+v^2)^{-1/2}\left[3v + u\tan^{-1}\frac{u}{v} + \frac{v}{2}\operatorname{Log}(u^2+v^2)\right];$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial u}\right];$$

$$\therefore \tau_{uv} = -2C_{25}(u^2+v^2)^{-1/2}\left[-3u + v\tan^{-1}\frac{u}{v} - \frac{u}{2}\operatorname{Log}(u^2+v^2)\right],$$

(26). Similarly $\phi = C_{26}(v \tan^{-1}\frac{v}{u} - \frac{u}{2}\log(u^2+v^2))$.
also satisfies the $\nabla^4\phi$ equation and following are the corresponding stress equations.

$$\sigma u = -2C_{26}(u^2+v^2)^{-1/2}\cdot\left[3u + v\tan^{-1}\frac{v}{u} + \frac{u}{2}\log(u^2+v^2)\right].$$

$$\sigma v = 2C_{26}(u^2+v^2)^{-1/2}\cdot\left[3u + v\tan^{-1}\frac{v}{u} + \frac{u}{2}\log(u^2+v^2)\right].$$

$$\tau uv = -2C_{26}(u^2+v^2)^{-1/2}\cdot\left[-3v + u\tan^{-1}\frac{v}{u} - \frac{v}{2}\log(u^2+v^2)\right].$$

(27). Try $\quad \phi = C_{27}(u^2+v^2)^{1/2}\log(u^2+v^2).$

$$\therefore \frac{\partial\phi}{\partial u} = C_{27}\,u(u^2+v^2)^{-1/2}\left[\frac{1}{2}\log(u^2+v^2)+1\right],$$

$$\therefore \frac{\partial^2\phi}{\partial u^2} = C_{27}\left[\frac{1}{2}\log(u^2+v^2)\cdot(u^2+v^2)^{-1/2} - u^2(u^2+v^2)^{-3/2}\cdot\frac{1}{2}\log(u^2+v^2)\right.$$
$$\left. + (u^2+v^2)^{-1/2}\right].$$

$$\therefore \frac{\partial\phi}{\partial v} = C_{27}\,v(u^2+v^2)^{-1/2}\left[\frac{1}{2}\log(u^2+v^2)+1\right],$$

$$\therefore \frac{\partial^2\phi}{\partial v^2} = C_{27}\left[\frac{1}{2}\log(u^2+v^2)\cdot(u^2+v^2)^{-1/2} - v^2(u^2+v^2)^{-3/2}\cdot\frac{1}{2}\log(u^2+v^2)\right.$$
$$\left. + (u^2+v^2)^{-1/2}\right].$$

$$\therefore \frac{\partial^2\phi}{\partial u\partial v} = -C_{27}\cdot\frac{uv}{2}(u^2+v^2)^{-3/2}\log(u^2+v^2),$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$$
$$= \log(u^2+v^2)\cdot(u^2+v^2)^{-1/2} - (u^2+v^2)^{-1/2}\cdot\frac{1}{2}\log(u^2+v^2)$$
$$+ 2(u^2+v^2)^{-1/2}$$
$$= \frac{1}{2}\log(u^2+v^2)\cdot(u^2+v^2)^{-1/2} + 2(u^2+v^2)^{-1/2}$$

$$\therefore \nabla^2\phi\cdot(u^2+v^2)^{1/2} = \frac{1}{2}\log(u^2+v^2) + 2.$$

$$\therefore \left[\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial v^2}\right]\left[\nabla^2\phi \cdot (u^2+v^2)^{1/2}\right] =$$

$$\left[\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial v^2}\right]\left[\frac{1}{2}\operatorname{Log}(\mu^2+v^2) + 2\right] = 0.$$

Therefore the $\nabla^4\phi$ equation is satisfied.

$$\sigma_\mu = 2(\mu^2+v^2)^{-1/2}\left[2(\mu^2+v^2)\frac{\partial^2\phi}{\partial v^2} - \mu\frac{\partial\phi}{\partial\mu} + v\frac{\partial\phi}{\partial v}\right]$$

$$\therefore \sigma_\mu = C_{27}(4 + 2(v^2-\mu^2)(\mu^2+v^2)^{-1} + \log(\mu^2+v^2)].$$

$$\sigma_v = 2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial u^2} - v\frac{\partial\phi}{\partial v} + \mu\frac{\partial\phi}{\partial\mu}\right],$$

$$\sigma_v = C_{27}\left[4 + 2(v^2-\mu^2)(\mu^2+v^2)^{-1} + \operatorname{Log}(\mu^2+v^2)\right].$$

$$\tau_{uv} = -2(u^2+v^2)^{-1/2}\left[2(u^2+v^2)\frac{\partial^2\phi}{\partial\mu\partial v} + \mu\frac{\partial\phi}{\partial v} + v\frac{\partial\phi}{\partial\mu}\right],$$

$$\tau_{uv} = -4C_{27}\,\mu v(u^2+v^2)^{-1}.$$