MODAL SHAPES AND FREQUENCIES

OF THE DYNAMIC STIFFNESS MATRIX OF A BEAM-COLUMN

by

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ABSTRACT

MODAL SHAPES AND FREQUENCIES
OF THE DYNAMIC STIFFNESS MATRIX OF A BEAM-COLUMN

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The purpose of this thesis is to investigate the characteristics of the normal mode shapes and frequencies associated with the general stiffness matrix for a beam-column element, which is derived from the Bernoulli-Euler differential equation with the inclusion of the axial force and transverse inertial loading. The components of the general stiffness matrix contain hyperbolic and harmonic functions.

Three separate problems are analyzed. These include the statical beam-column bending problem, the dynamic beam problem in free vibration, and the dynamical beam-column problem in free vibration. In each case, the orthogonality conditions of the modal shapes and frequencies are established.

For each of the above three problems, the special case of rigid body motion is investigated. Each possess at most two rigid body modes, the remaining two families of modes shapes being associated with deformed geometry including, the resonant frequency of free vibration of a free-free beam, and the critical buckling load of a simply supported column, and the resonant frequency of a free-free beam-column.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Area of cross-section
A_1, A_2, A_3, A_4	Constants of integration
B	$= \dfrac{1}{2 p_1 p_2 - 2 p_1 p_2 Cc + (p_1^2 - p_2^2) Ss}$
C	$= \cosh p_1 L$
c	$= \cos p_2 L$
d	$= 2uv(1 - Cc) + (u^2 - v^2) Ss$
d_S	$= \dfrac{1}{2 - 2c - qs}$
d_ω	$= \dfrac{1}{1 - Cc}$
G	$= \dfrac{1}{K(2 - 2c - qs)}$
I	$= Ar^2$, Cross-section moment of inertia
K^2	$= \dfrac{P}{EI}$
L	Length of beam-column element
M_{AB}, M_{BA}	Bending moment at the end of beam column element
P	Axial force
P_{cr}	Critical buckling load
P_1	$= \left[-\dfrac{K^2}{2} + \sqrt{\left(\dfrac{K^2}{2}\right)^2 + \beta^4} \right]^{1/2}$
P_2	$= \left[\dfrac{K^2}{2} + \sqrt{\left(\dfrac{K^2}{2}\right) + \beta^4} \right]^{1/2}$
q	$= KL$
r	Radius of gyration
S	$= \sinh p_1 L$
s	$= \sin p_1 L$
t	Time variable

SYMBOL	DEFINITION
u	$= p_1 L$
v	$= p_2 L$
V_{AB}, V_{BA}	Shear forces at the ends of beam element
x , y	Co-ordinates along the axes of a deflected beam-column
$\bar{Y}(x)$	Mode shape
ω	Natural frequency of beam
Ω	Natural frequency of the beam-column
β^4	$= \frac{\rho A \Omega^2}{EI}$
ρ	Mass density per unit volume
δ_{AB} , δ_{BA}	Transverse deflection at the ends of the beam-column
λ	Eigenvalues
θ_{AB} , θ_{BA}	Angular deflection at the ends of the beam column
$\{f\}$	Vector of element forces in dimensionless form
$\{\delta\}$	Vector of displacement in dimensionless form
$[I]$	The identity matrix
$[K_D]$	Dynamic stiffness matrix in dimensionless form
$[K_S]$	Stability stiffness matrix in dimensionless form
$[S]$	General stiffness matrix in dimensionless form

SYMBOL	DEFINITION
$\left[U\right]$	Modal Matrix
$\left[\Delta\right]$	Diagonal matrix of eigenvalues

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CHAPTER I

INTRODUCTION

1.1 Historical Background

The general stiffness matrix for a beam and/or a beam-column element is derived from the Bernoulli-Euler differential equation with the inclusion of the axial force for the beam-column. Rubenstein(1)* derived the required stiffness, mass, and axial force matrix utilizing static displacement functions for the beam-column element. Henshell(2)* used the exact dynamic equations in obtaining the dynamic stiffness coefficients (i.e., mass matrix) for a beam element. Wang(3)* used the "exact method" in deriving the geometric stiffness or axial force matrix for a beam-column element. Paz(4)* used the "exact method" based on the corresponding differential equation for the vibrating beam-column which leads to transcendental components including both trigonometric and hyperbolic functions relating the forces (moments) and displacements (rotations) at the ends of beam element.

* Number in parenthesis refers to literature cited in the Bibliography.

1.2 General Matrix Form

The matrix form of the above relationships is written as

$$\{f\} = [S]\{\delta\} \tag{1-1a}$$

where

$$\{f\} = \frac{L}{EI}\begin{Bmatrix} V_{AB}L \\ M_{AB} \\ V_{BA}L \\ M_{BA} \end{Bmatrix} , \qquad \{\delta\} = \begin{Bmatrix} \frac{\delta_{AB}}{L} \\ \theta_{AB} \\ \frac{\delta_{BA}}{L} \\ \theta_{BA} \end{Bmatrix} \tag{1-1b,1c}$$

The general stiffness matrix $[S]$ is symmetric, but not necessarily positive definite. In general, it is positive indefinite, that is, its eigenvalues are positive, but also may include zero. These particular zero eigenvalues are associated with rigid body mode shapes.

By transforming this general stiffness matrix $[S]$ into diagonal form (i.e., spectral decomposition), that is, performing the eigenvalue-eigenvector problem, a complete set of mode shapes are obtainable. This process requires the calculation of a matrix $[U]$ called the eigenvector matrix which satisfies the conditions

$$[U]^T[S][U] = [\Lambda] \tag{1-2a}$$

and

$$[U][U]^T = [U]^T[U] = [I] \tag{1-2b}$$

that is, $[U]$ is orthonormal. The matrix $[\Lambda]$ is a diagonal matrix of eigenvalues whose zeros are associated with rigid body mode shapes. Nonzero terms of the matrix $\left[\Lambda\right]$ when equated to zero yield values of critical buckling load and natural frequency. Since $\left[S\right]$ is a symmetric, it is diagonalized by an orthogonal matrix $\left[U\right]$. This condition is shown in equation (1-2b).

The modal shape problem is defined by the condition that the force vector is proportional to the displacement vector, that is,

$$\left\{f\right\} = \left[S\right]\left\{\delta\right\} = \lambda\left\{\delta\right\} \tag{1-3a}$$

where λ's are defined as eigenvalues. Equation (1-3a) is rewritten in the form

$$\left[\left[S\right] - \lambda\left[I\right]\right]\left\{\delta\right\} = \left\{0\right\} \tag{1-4}$$

For non-zero value of $\left\{\delta\right\}$, it follows that

$$\left|\left[S\right] - \lambda\left[I\right]\right| = 0 \tag{1-5a}$$

which yields the characteristic equation of this matrix $\left[S\right]$ which is solved directly for the eigenvalues. The general form of equation (1-5a) becomes

$$\lambda^4 - I_1\lambda^3 + I_2\lambda^2 - I_3\lambda + I_4 = 0 \tag{1-5b}$$

with the coefficients I_1, I_2, I_3, and I_4 as the matrix invariants,

where I_1 = trace of the matrix $\left[S\right]$ (1-5c)

I_2 = sum of all (2 x 2) determinant minors formed by successively eliminating all possible combinations of any two rows and the corresponding two columns (1-5d)

I_3 = sum of the (3 x 3) determinant minors of the principal diagonal elements (1-5e)

I_4 = the determinant of $\left[S\right]$ (1-5f)

The roots of the equation (1-5b), λ_1, λ_2, λ_3 and λ_4 are the eigenvalues of $\left[S\right]$.

The eigenvalues of equation (1-5b) are individually substituted into equations (1-4) and the corresponding eigenvectors $\left\{\delta\right\}$ are obtained which directly define the modal shapes. These vectors are then combined to form the columns of the modal matrix $\left[U\right]$.

CHAPTER II

EQUATIONS OF MOTION

2.1 Differential Equation

The exact general stiffness matrix for a beam-column element is derived from the Bernoulli-Euler differential equation with the inclusion of the axial force for the beam-column, the components of the general stiffness matrix are composed of terms containing hyperbolic and trigonometric functions.

When the effect of rotary inertia and shear deformation are neglected, the differential equation of motion for a uniform beam-column is given as

$$\frac{\partial^4 y}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (2\text{-}1)$$

with x & y - the co-ordinates axes of the beam as shown in Fig. I.

P - the axial forces.

EI - the flexural stiffness.

ρA - the mass per unit length.

t - the time variable.

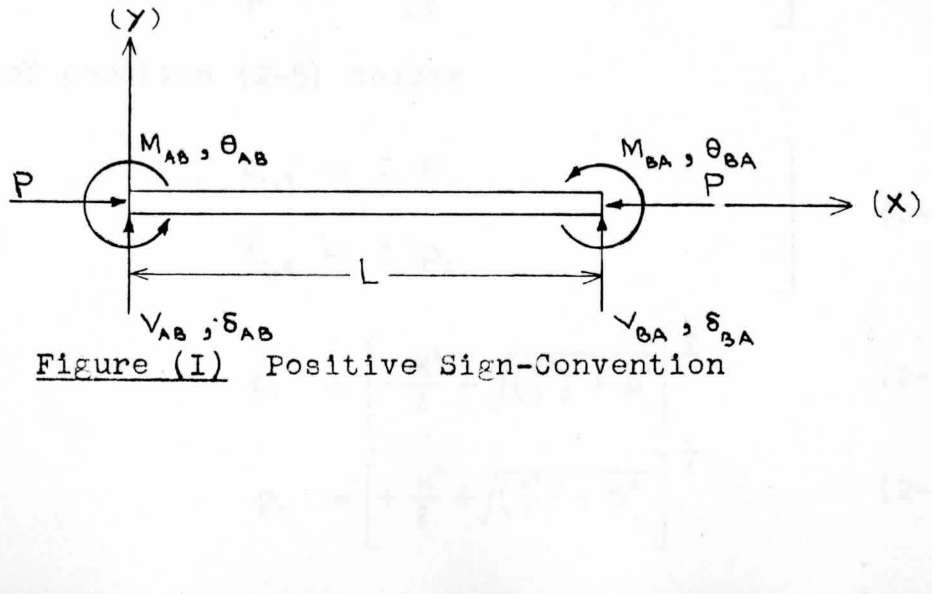


Figure (I) Positive Sign-Convention

If the beam is not subjected to external forces, the assumed solution of equation (2-1) is a harmonic function of time:

$$Y(x,t) = \overline{Y}(x)\, e^{i\Omega t} \tag{2-2}$$

where Ω - natural frequency (radians/sec.)

$\overline{Y}(x)$ - the corresponding mode shape.

Substituting equation (2-2) into (2-1) yields the ordinary differential equation

$$\frac{d^4}{dx^4}\overline{Y} + \frac{P}{EI}\frac{d^2}{dx^2}\overline{Y} - \frac{\rho A \Omega^2}{EI}\overline{Y} = 0 \tag{2-3}$$

Assuming the solution of equation (2-3) takes the general form

$$\overline{Y}(x) = A_m e^{\gamma x} \tag{2-4}$$

it follows that

$$\gamma^4 + k^2\gamma^2 - \beta^4 = 0 \tag{2-5}$$

where

$$\left.\begin{aligned} k^2 &= \frac{P}{EI} \\ \beta^4 &= \frac{\rho A \Omega^2}{EI} \end{aligned}\right] \tag{2-6}$$

The roots of equation (2-5) become

$$\left.\begin{aligned} \gamma_{1,2} &= \pm p_1 \\ \gamma_{3,4} &= \pm p_2 \end{aligned}\right] \tag{2-7a}$$

where

$$p_1 = \left[-\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \beta^4}\right]^{\frac{1}{2}} \tag{2-7b}$$

$$p_2 = \left[+\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \beta^4}\right]^{\frac{1}{2}} \tag{2-7c}$$

The general solution of the differential equation (2-3) is

$$\overline{Y}(x) = A_1 \sin p_2 x + A_2 \cos p_2 x + A_3 \sinh p_1 x + A_4 \cosh p_1 x \quad (2\text{-}8)$$

with A_1, A_2. A_3 and A_4 as constants.

2.2 Boundary Conditions

The eight boundary conditions associated with the beam-column are

$$\left.\begin{aligned} \overline{Y}(0) &= \delta_{AB} & \overline{Y}(L) &= \delta_{BA} \\ \frac{d}{dx}\overline{Y}(0) &= \theta_{AB} & \frac{d}{dx}\overline{Y}(L) &= \theta_{BA} \end{aligned}\right] \quad (2\text{-}9a)$$

$$\left.\begin{aligned} \frac{d^3}{dx^3}\overline{Y}(0) &= \frac{V_{AB}}{EI} - \frac{P\theta_{AB}}{EI} & \frac{d^3}{dx^3}\overline{Y}(L) &= -\frac{V_{BA}}{EI} - \frac{P\theta_{BA}}{EI} \\ \frac{d^2}{dx^2}\overline{Y}(0) &= -\frac{M_{AB}}{EI} & \frac{d^2}{dx^2}\overline{Y}(L) &= \frac{M_{BA}}{EI} \end{aligned}\right] \quad (2\text{-}9b)$$

where δ_{AB}, δ_{BA} and θ_{AB}, θ_{BA} are respectively the transverse displacements and angular rotations at the ends of the beam-column. V_{AB}, V_{BA} and M_{AB}, M_{BA} are correspondingly shear forces and moments at the boundaries (see Figure I). Substituting the boundary conditions given by equation (2-9a) and (1-9b) into equation (2-8) yields respectively the matrix forms.

$$\begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ p_2 & 0 & p_1 & 0 \\ s & c & S & C \\ p_2 c & -p_2 s & p_1 C & p_1 S \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (2\text{-}10)$$

and

$$\begin{Bmatrix} \frac{V_{AB}}{EI} - \frac{P}{EI}\theta_{AB} \\ -\frac{M_{AB}}{EI} \\ -\frac{V_{BA}}{EI} - \frac{P}{EI}\theta_{BA} \\ \frac{M_{BA}}{EI} \end{Bmatrix} = \begin{bmatrix} -p_2^3 & 0 & p_1^3 & 0 \\ 0 & -p_2^2 & 0 & p_1^2 \\ -p_2^3 c & p_2^3 s & p_1^3 C & p_1^3 S \\ -p_2^2 s & -p_2^2 c & p_1^2 S & p_1^2 C \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (2\text{-}11)$$

For convenience purposes

$$s = \sin p_2 L$$

$$c = \cos p_2 L$$

$$S = \sinh p_1 L$$

$$C = \cosh p_1 L$$

2.3 Stiffness Matrix

Performing the inverse operation on equation (2-10) gives

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} \qquad (2\text{-}12)$$

where

$$\left.\begin{aligned}
a_{11} &= -p_1 p_2 C s - p_1^2 S e \\
a_{12} &= p_1 - p_1 C e - p_2 S s \\
a_{13} &= p_1^2 S + p_1 p_2 s \\
a_{14} &= -p_1 C + p_1 e \\
a_{21} &= -p_1 p_2 (S^2 + C^2) - p_1 p_2 C e - p_1^2 S s \\
a_{22} &= p_1 C s - p_2 S e \\
a_{23} &= -p_1 p_2 C + p_1 p_2 e \\
a_{24} &= p_2 S - p_1 s \\
a_{31} &= p_1 p_2 S e + p_2^2 C s \\
a_{32} &= -p_2 C e + p_2 (s^2 - e^2) - p_1 S s
\end{aligned}\right] \qquad (2\text{-}13)$$

$$
\left.
\begin{aligned}
a_{33} &= -p_2^2 s - p_1 p_2 S \\
a_{34} &= -p_2 e + p_2 C \\
a_{41} &= -p_1 p_2 C e + p_1 p_2 (s^2 - e^2) + p_2^2 S s \\
a_{42} &= p_2 S e - p_1 C s \\
a_{43} &= -p_1 p_2 e + p_1 p_2 C \\
a_{44} &= p_1 s - p_2 S
\end{aligned}
\right]
$$

Substituting equation (2-12) into equation (2-11), the final form of the general stiffness matrix relating and harmonic forces and moments to displacements and rotations is obtained in the dimensionless form

$$
\frac{L}{EI}\begin{Bmatrix} V_{AB}L \\ M_{AB} \\ V_{BA}L \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} S_{11} & & \text{SYMMETRIC} & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{Bmatrix} \frac{\delta_{AB}}{L} \\ \theta_{AB} \\ \frac{\delta_{BA}}{L} \\ \theta_{BA} \end{Bmatrix} \tag{2-14}
$$

where

$$
S_{11} = S_{33} = B\left[(p_1^2 p_2^3 - p_1^4 p_2) S e + (p_1 p_2^4 + p_1^3 p_2^2) C s\right] L^3 \tag{2-15}
$$

$$
S_{21} = -S_{43} = B\left[(p_1 p_2^3 - p_1^3 p_2) + (p_1^3 p_2 - p_1 p_2^3) C e + 2 p_1^2 p_2^2 S s\right] L^2 \tag{2-16}
$$

$$S_{22} = S_{44} = B\left[(p_1 p_2^2 + p_1^3) C s - (p_1^2 p_2 + p_2^3) S c\right] L \qquad (2\text{-}17)$$

$$S_{32} = - S_{41} = B\left[(p_1 p_2^3 + p_1^3 p_2)(c - C)\right] L^2 \qquad (2\text{-}18)$$

$$S_{31} = B\left[(- p_1^2 p_2^3 - p_1^4 p_2) S - (p_1^3 p_2^2 + p_1 p_2^4) s\right] L^3 \qquad (2\text{-}19)$$

$$S_{42} = B\left[(p_1^2 p_2 + p_2^3) S - (p_1 p_2^2 + p_1^3) s\right] L \qquad (2\text{-}20)$$

and

$$B = \frac{1}{2 p_1 p_2 - 2 p_1 p_2 C c + (p_1^2 - p_2^2) S s} \qquad (2\text{-}21)$$

subject to the condition that

$$2 p_1 p_2 - 2 p_1 p_2 C c + (p_1^2 - p_2^2) S s \neq 0 \qquad (2\text{-}22)$$

It is convenient to introduce the following change in notation

$$\left.\begin{aligned} u &= p_1 L \qquad \text{and} \\ v &= p_2 L \end{aligned}\right] \qquad (2\text{-}23)$$

Substituting equation (2-23) in equation (2-15), (2-16), (2-17), (2-18), (2-19) and (2-20) the elements of the general stiffness matrix of equation (2-14) become

$$S_{11} = S_{33} = \frac{\left[uv(u^2+v^2)(u\,Se + v\,Cs)\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-24}$$

$$S_{21} = -S_{43} = \frac{uv\left[-(u^2-v^2)(1-Ce) + 2uv\,Ss\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-25}$$

$$S_{22} = S_{44} = \frac{\left[(u^2+v^2)(u\,Cs - v\,Se)\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-26}$$

$$S_{32} = -S_{41} = \frac{uv\left[(u^2+v^2)(C-e)\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-27}$$

$$S_{31} = \frac{-uv\left[(u^2+v^2)(uS + v\,s)\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-28}$$

$$S_{42} = \frac{\left[(u^2+v^2)(vS - u\,s)\right]}{\left[2uv(1-Ce)+(u^2-v^2)\,Ss\right]} \tag{2-29}$$

The stiffness matrix $\left[S\right]$ is

$$\left[S\right] = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & -S_{41} & S_{42} \\ S_{31} & -S_{41} & S_{11} & -S_{21} \\ S_{41} & S_{42} & -S_{21} & S_{22} \end{bmatrix} \tag{2-30}$$

2.4 Invariants of Stiffness Matrix

Noting equations (1-5b), (1-5c), (1-5d), (1-5e) and (1-5f) together with equation (2-30) the four invariant conditions are written as

$$I_1 = 2\left[S_{11} + S_{22}\right] \tag{2-31a}$$

$$\begin{aligned} I_2 = &\begin{vmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{vmatrix} + \begin{vmatrix} S_{11} & S_{31} \\ S_{31} & S_{11} \end{vmatrix} + \begin{vmatrix} S_{11} & S_{41} \\ S_{41} & S_{22} \end{vmatrix} \\ &+ \begin{vmatrix} S_{11} & -S_{21} \\ -S_{21} & S_{22} \end{vmatrix} + \begin{vmatrix} S_{22} & S_{42} \\ S_{42} & S_{22} \end{vmatrix} + \begin{vmatrix} S_{22} & -S_{41} \\ -S_{41} & S_{11} \end{vmatrix} \end{aligned} \tag{2-31b}$$

$$\begin{aligned} I_3 = &\begin{vmatrix} S_{11} & S_{21} & S_{31} \\ S_{21} & S_{22} & -S_{41} \\ S_{31} & -S_{41} & S_{11} \end{vmatrix} + \begin{vmatrix} S_{11} & S_{21} & S_{41} \\ S_{21} & S_{22} & S_{42} \\ S_{41} & S_{42} & S_{22} \end{vmatrix} \\ &+ \begin{vmatrix} S_{11} & S_{31} & S_{41} \\ S_{31} & S_{11} & -S_{21} \\ S_{41} & S_{21} & S_{22} \end{vmatrix} + \begin{vmatrix} S_{22} & -S_{41} & S_{42} \\ -S_{41} & S_{11} & -S_{21} \\ S_{42} & -S_{21} & S_{22} \end{vmatrix} \end{aligned} \tag{2-31c}$$

$$I_4 = \begin{vmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & -S_{41} & S_{42} \\ S_{31} & -S_{41} & S_{11} & -S_{21} \\ S_{41} & S_{42} & -S_{21} & S_{22} \end{vmatrix} \tag{2-31d}$$

CHAPTER III

VIBRATING BEAM PROBLEM

3.1 The Stiffness Matrix and its Eigenvalues

The dynamic stiffness matrix is obtained from equation (2-14) by letting the axial force equal zero (i.e. k = 0) which is equivalent in the condition that

$$p_1 = p_2 \qquad (3\text{-}1)$$

The dynamic stiffness matrix then becomes

$$\left[K_D\right] = d_\omega \begin{bmatrix} u^3(Cs+Sc) & & & \text{SYMMETRIC} \\ u^2 Ss & u(Cs-Sc) & & \\ -u^3(S+s) & -u^2(C-c) & u^3(Cs+Sc) & \\ u^2(C-c) & u(S-s) & -u^2 Ss & u(Cs-Sc) \end{bmatrix} \qquad (3\text{-}2a)$$

$$\left.\begin{aligned} u &= \beta L = p_1 L = p_2 L \\ C &= \cosh u \\ S &= \sinh u \\ c &= \cos u \\ s &= \sin u \end{aligned}\right] \qquad (3\text{-}2b)$$

$$\left.\begin{aligned} &\text{and} \qquad d_\omega = \frac{1}{1-Cc} \\ &\text{with} \qquad 1-Cc \neq 0 \end{aligned}\right] \qquad (3\text{-}3)$$

The four matrix invariants of $\left[K_D\right]$ matrix in equation (3-6) are

$$
\left.\begin{aligned}
I_1 &= \frac{-2u\left[Se(1-u^2) - Cs(1+u^2)\right]}{(1-Ce)} \\
I_2 &= \frac{2u^2\left[Ss(1-u^4) + (1+3Ce)u^4\right]}{(1-Ce)} \\
I_3 &= \frac{2u^5\left[Se(1-u^2) - Cs(1+u^2)\right]}{(1-Ce)} \\
I_4 &= u^8
\end{aligned}\right] \qquad (3\text{-}4)
$$

The characteristic equation becomes

$$
\lambda^4 + \frac{2u\left[Se(1-u^2) - Cs(1+u^2)\right]}{(1-Ce)}\lambda^3 + \frac{2u^2\left[Ss(1-u^4) + (1+3Ce)u^4\right]}{(1-Ce)}\lambda^2 - \frac{2u^5\left[Se(1-u^2) - Cs(1+u^2)\right]}{(1-Ce)}\lambda + u^8 = 0 \qquad (3\text{-}5a)
$$

or in quadratic factored form as

$$
\left[\lambda^2 + \frac{u\left[(Se-s)(1-u^2) - (Cs-S)(1+u^2)\right]}{(1-Ce)}\lambda - u^4\right] \cdot \left[\lambda^2 + \frac{u\left[(Se+s)(1-u^2) - (Cs+S)(1+u^2)\right]}{(1-Ce)}\lambda - u^4\right] = 0 \qquad (3\text{-}5b)
$$

with the four roots determined as

$$
\lambda_1 = \frac{-u\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)}
$$

$$
\lambda_2 = \frac{-u\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)}
$$

$$
\left.
\begin{aligned}
\lambda_3 &= \frac{-u\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]}{2(1-Ce)} \\
&\quad + \frac{\sqrt{u^2\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)} \\
\lambda_4 &= \frac{-u\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]}{2(1-Ce)} \\
&\quad + \frac{\sqrt{u^2\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)}
\end{aligned}
\right] \qquad (3\text{-}5c)
$$

The eigenvalue matrix takes the form

$$
\left[\Lambda_\omega\right] = \begin{bmatrix} \lambda_1 & & & \text{SYMMETRIC} \\ 0 & \lambda_2 & & \\ 0 & 0 & \lambda_3 & \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \qquad (3\text{-}5d)
$$

3.2 The Eigenvector Matrix

Utilizing equation (1-4), one obtains

$$
\left[\left[K_D\right] - \lambda\left[I\right]\right]\left\{\delta\right\} = \left\{0\right\} \qquad (3\text{-}6)
$$

Substituting the four roots of λ individually into equation (3-6), the eigenvector matrix is constructed as

$$
\left[U\right] = \frac{1}{\sqrt{2}} \begin{bmatrix} n_1/d_1 & n_3/d_2 & n_5/d_3 & n_6/d_4 \\ n_2/d_1 & n_4/d_2 & n_7/d_3 & n_8/d_4 \\ n_1/d_1 & -n_3/d_2 & n_5/d_3 & -n_6/d_4 \\ -n_2/d_1 & n_4/d_2 & -n_7/d_3 & n_8/d_4 \end{bmatrix} \qquad (3\text{-}6a)
$$

where

$$
\left.\begin{aligned}
n_1 &= \frac{u\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)} \\
n_2 &= \frac{u^2(Ss+e-C)}{(1-Ce)} \\
n_3 &= \frac{u\left[(Se-Cs+s-S)+u^2(Se+Cs+s+S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se-Cs+s-S)+u^2(Se+Cs+s+S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)} \\
n_4 &= \frac{u^2(Ss-e+C)}{(1-Ce)} \\
n_5 &= \frac{u\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]}{2(1-Ce)} + \frac{\sqrt{u^2\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)} \\
n_6 &= \frac{u\left[(Se-Cs+s-S)+u^2(Se+Cs+s+S)\right]}{2(1-Ce)} + \frac{\sqrt{u^2\left[(Se-Cs+s-S)+u^2(Se+Cs+s+S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)} \\
d_1 &= \sqrt{\left\{\frac{u\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)}\right\}^2+\left\{\frac{u^2(Se+e-C)}{(1-Ce)}\right\}^2} \\
d_2 &= \sqrt{\left\{\frac{u\left[(Se-Cs+s-S)+u^2(Se+Cs+s+S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se-Cs-s+S)+u^2(Se+Cs+s+S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)}\right\}^2+\left\{\frac{u^2(Se-e+C)}{(1-Ce)}\right\}^2} \\
d_3 &= \sqrt{\left\{\frac{u\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]}{2(1-Ce)} + \frac{\sqrt{u^2\left[(Se-Cs-s+S)+u^2(Se+Cs-s-S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)}\right\}^2+\left\{\frac{u^2(Ss+e-C)}{(1-Ce)}\right\}^2}
\end{aligned}\right\} \quad (3\text{-}6b)
$$

$$d_4 = \sqrt{\left\{ \frac{u\left[(Se - Cs + s - S) + u^2(Se + Cs + s + S)\right]}{2(1-Ce)} + \frac{\sqrt{u^2\left[(Se - Cs + s - S) + u^2(Se + Cs + s + S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)} \right\}^2 + \left\{ \frac{u^2(Ss - e + C)}{(1-Ce)} \right\}^2}$$

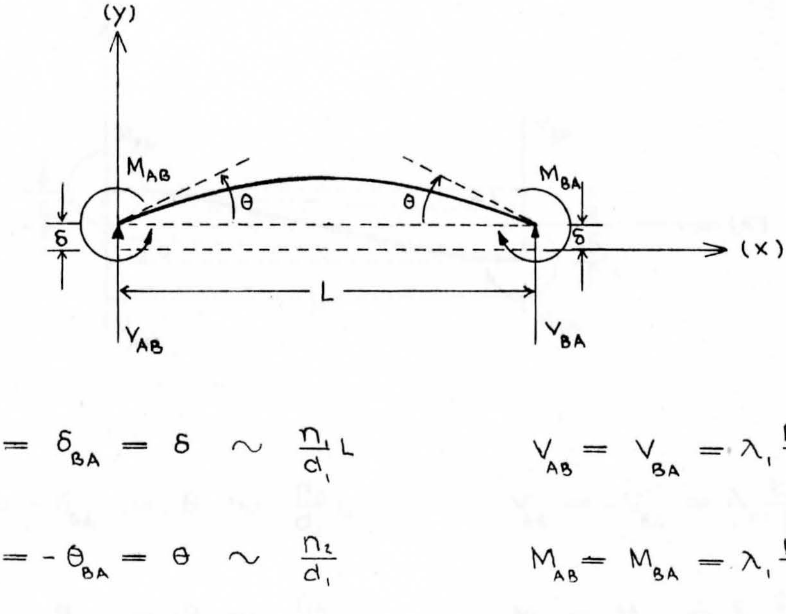
where

$$\frac{n_1}{d_1} = \frac{n_2}{d_3} \;, \qquad \frac{n_2}{d_1} = -\frac{n_5}{d_3}$$

$$\frac{n_3}{d_2} = -\frac{n_4}{d_4} \;, \qquad \frac{n_4}{d_2} = \frac{n_6}{d_4}$$

3.3 Solutions for the Moment, Shear Forces and Variations of Normal Mode Shapes

For the eigenvalue λ_1, the normal mode shape together with moment and shear values are given in Figure (II A).



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{n_1}{d_1} L \qquad V_{AB} = V_{BA} = \lambda_1 \frac{EI}{L^3} \delta$$

$$\theta_{AB} = -\theta_{BA} = \theta \sim \frac{n_2}{d_1} \qquad M_{AB} = M_{BA} = \lambda_1 \frac{EI}{L} \theta$$

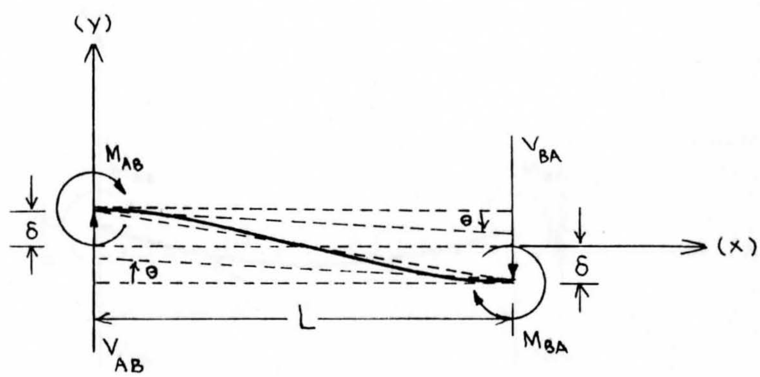
Figure (IIA) Modal Shape of the Vibrating Beam for λ_1

The variation in the mode shape for values of the parameter u where $0 \leqslant u \leqslant 4.7300405$ are shown in Table (IA).

δ \ u	0	2.0	4.0	4.5	4.7300405	5.0
δ_{AB}/L	$1/\sqrt{2}$	0.69856826	0.69510438	0.69293408	0.69128383	0.68833003
θ_{AB}	0	0.10955536	0.12973007	0.14086288	0.14875034	0.16186957
δ_{BA}/L	$1/\sqrt{2}$	0.69856826	0.69510438	0.69293408	0.69128383	0.68833003
θ_{BA}	0	-0.10955536	-0.12973007	-0.14086288	-0.14875034	-0.16186957

<u>Table (IA)</u> Modal Shape Variation for λ_1

For the eigenvalue λ_2, the normal mode shape together with moment and shear values are given in Figure (IIB).



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{n_3}{d_2} L \qquad V_{AB} = -V_{BA} = \lambda_2 \frac{EI}{L^3} \delta$$

$$\theta_{AB} = \theta_{BA} = -\theta \sim \frac{n_4}{d_2} \qquad M_{AB} = M_{BA} = \lambda_2 \frac{EI}{L} \theta$$

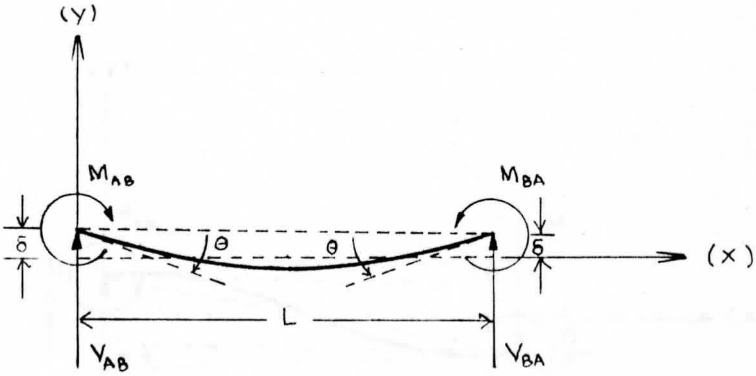
<u>Figure (IIB)</u> Modal shape of the Vibrating Beam for λ_2

The variation in the mode shape for values of the parameter u where $0 \leqslant u \leqslant 7.8532045$ are shown in Table (IB).

δ \ u	0	2.0	4.0	6.0	7.5	7.8532045	8.0
δ_{AB}/L	$1/\sqrt{10}$	0.32847309	0.69964873	0.70533706	0.70263359	0.70145153	0.70076685
θ_{AB}	$-2/\sqrt{10}$	-0.62618321	-0.10242871	0.06897051	0.07941053	0.08925104	0.09447652
δ_{BA}/L	$-1/\sqrt{10}$	-0.32847309	-0.69964873	-0.70533706	-0.70263359	-0.70145153	-0.70076685
θ_{BA}	$-2/\sqrt{10}$	-0.62618321	-0.10242871	0.06897051	0.07941053	0.08925104	0.09447652

Table (IB) Modal Shape Variation for λ_2

For the eigenvalue λ_3, the normal mode shape together with moment and shear values are given in Figure (IIC).



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{n_5}{d_3} L \qquad V_{AB} = V_{BA} = \lambda_3 \frac{EI}{L^3} \delta$$

$$\theta_{AB} = -\theta_{BA} = -\theta \sim \frac{n_2}{d_3} \qquad M_{AB} = -M_{BA} = \lambda_3 \frac{EI}{L} \theta$$

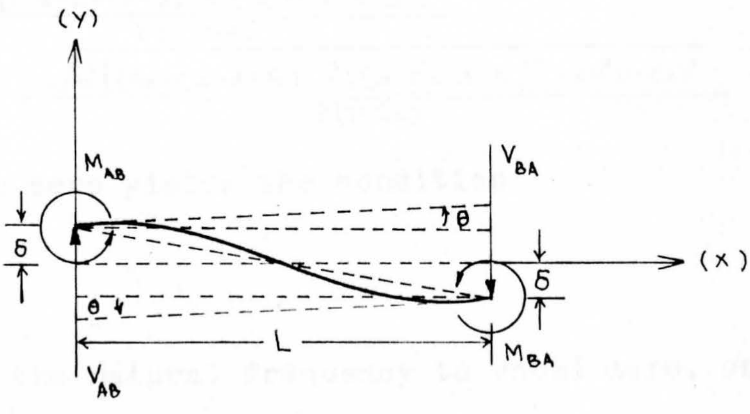
Figure (IIC) Modal Shape of the Vibrating Beam for λ_3

The variation in the mode shape for values of the parameter u where $0 \leq u \leq 4.7300405$ are shown in Table (IC).

δ \ u	0	2.0	4.0	4.5	4.7300405	5.0
δ_{AB}/L	0	-0.45542070	0.11551679	0.13945976	0.14875034	0.15966984
θ_{AB}	$1/\sqrt{2}$	0.54091772	-0.69760724	-0.69321783	-0.69128383	-0.68884362
δ_{BA}/L	0	-0.45542070	0.11551679	0.13945976	0.14875034	0.15966984
θ_{BA}	$-1/\sqrt{2}$	-0.54091772	0.69760724	0.69321783	0.69128383	0.68884362

Table (IC) Modal Shape Variation for λ_3

For the eigenvector λ_4, the normal mode shape together with moment and shear values are given in Figure (IID).



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{n_6}{d_4} L \qquad V_{AB} = -V_{BA} = \lambda_4 \frac{EI}{L^3} \delta$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{n_4}{d_4} L \qquad M_{AB} = M_{BA} = \lambda_4 \frac{EI}{L} \theta$$

Figure (IID) Modal Shape of the Vibrating Beam for λ_4

The variation in the mode shape for value of the parameter u where $0 \leq u \leq 7853{,}2045$ are shown in Table (ID).

δ \ u	0	2.0	4.0	6.0	7.5	7.8532045	8.0
δ_{AB}/L	$2/\sqrt{10}$	0.62618321	0.10242871	0.06897051	0.07941053	0.08925104	0.09447652
θ_{AB}	$1/\sqrt{10}$	0.32847309	0.69964873	-0.70533706	-0.70263359	-0.70145153	-0.70076685
δ_{BA}/L	$-2/\sqrt{10}$	-0.62618321	-0.10242871	-0.06897051	-0.07941053	-0.08925104	-0.09447652
θ_{BA}	$1/\sqrt{10}$	0.32847309	0.69964873	-0.70533706	-0.70263359	-0.70145153	-0.70076685

Table (ID) Modal Shape Variation for λ_4

3.4 Zeros of the Eigenvalues

The first eigenvalue

$$\lambda_1 = \frac{-u\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]}{2(1-Ce)} - \frac{\sqrt{u^2\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]^2 + 4u^4(1-Ce)^2}}{2(1-Ce)}$$

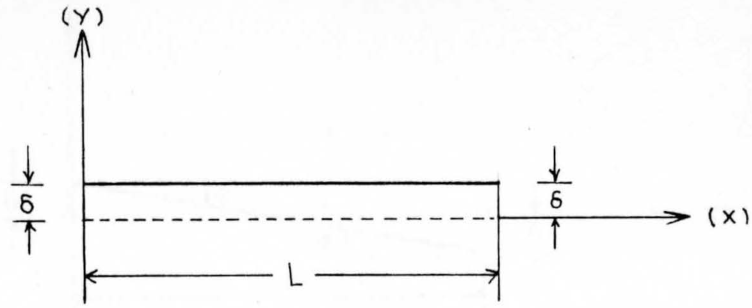
when equated to zero yields the condition

$$u = 0 \tag{3-7a}$$

which requires the natural frequency to equal zero, or

$$\omega = 0 \tag{3-7b}$$

For u = 0 the mode shape, for the condition $\omega = 0$, takes the shape as shown in Figure (IIE); see Table (IA).



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{L}{\sqrt{2}} \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta = 0 \qquad M_{AB} = M_{BA} = 0$$

Figure (IIE) Modal Shape of the Vibrating Beam for the Zero of λ_1

The second eigenvalue

$$\lambda_2 = \frac{-u\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]}{2(1 - Ce)} - \frac{\sqrt{u^2\left[(Se - Cs + s - S) - u^2(Se + Cs + s + S)\right]^2 + 4u^4(1 - Ce)^2}}{2(1 - Ce)}$$

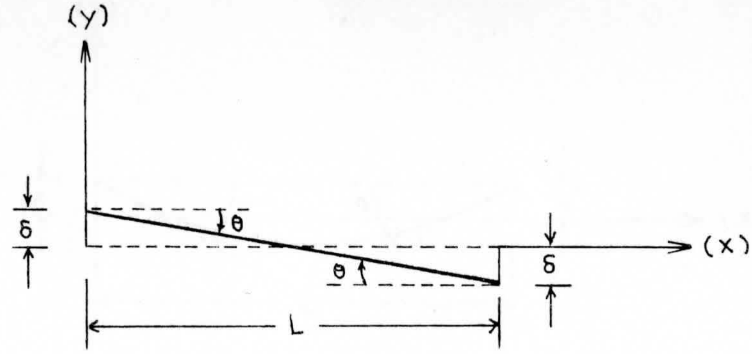
when equated to zero yields the condition

$$u = 0 \tag{3-8a}$$

which requires the natural frequency to equal zero or

$$\omega = 0 \tag{3-8b}$$

For u = 0, the mode shape takes the shape as shown in Figure (IIF); see Table (IB).



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{L}{\sqrt{10}} \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{2}{\sqrt{10}} \qquad M_{AB} = M_{BA} = 0$$

Figure (IIF) Modal Shape of the Vibrating Beam for the Zero of λ_2

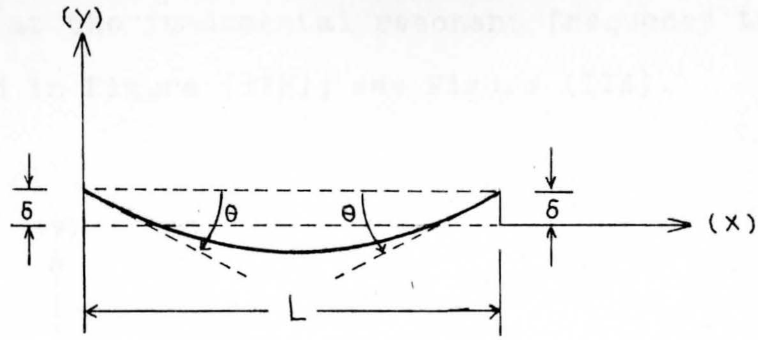
The third eigenvalue

$$\lambda_1 = \frac{-u\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]}{2(1 - Ce)} + \frac{\sqrt{u^2\left[(Se - Cs - s + S) - u^2(Se + Cs - s - S)\right]^2 + 4u^4(1 - Ce)^2}}{2(1 - Ce)}$$

when equated to zero, yields the condition

$$u = 4.7300405;\ 10.9956078;\ 17.2787596\ldots \tag{3-9a}$$

The mode shape at resonant frequency takes the shape shown in Figure (IIG); see Figure (IIJ).



$$\delta_{AB} = \delta_{BA} = \delta \sim 0.14875034L \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = -\theta_{BA} = \theta \sim -0.69128383 \qquad M_{AB} = M_{BA} = 0$$

Figure (IIG) Modal Shapes of the Vibrating Beam for the Third Zero of λ_3

Noting $u = \beta L$, it follows that a natural frequency is obtained by equation (2-6) as

$$\omega = 22.3732831\sqrt{\frac{EI}{\rho AL^4}}\ ;\ 120.9033909\sqrt{\frac{EI}{\rho AL^4}}\ \cdots\cdots \qquad (3\text{-}9b)$$

The fourth eigenvalue

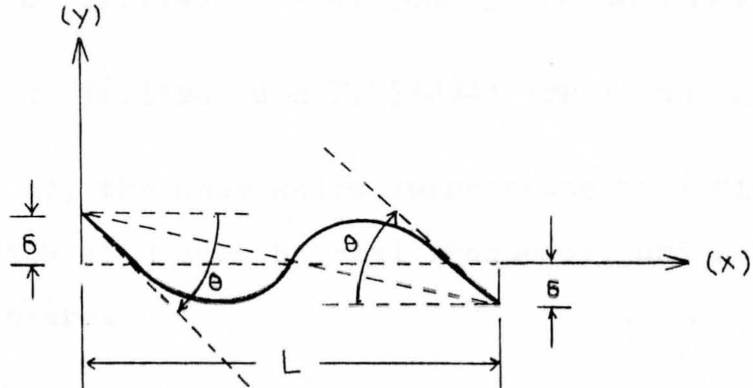
$$\lambda_4 = \frac{-u\left[(Se-Cs+s-S)-u^2(Se+Cs+s+S)\right]}{2(1-Ce)}$$

$$+\ \frac{\sqrt{u^2\left[(Se-Cs+s-S)-u^2(Se+Cs+s+S)\right]^2+4u^4(1-Ce)^2}}{2(1-Ce)}$$

when equated to zero yields the condition

$$u = 7.8532045; 14.1371655; 20.4203523\ \ldots \qquad (3\text{-}10a)$$

The mode shape at the fundamental resonant frequency takes the shape shown in Figure (IIH); see Figure (IIK).



$$\delta_{AB} = -\delta_{BA} = \delta \sim 0.08925104L \quad , \quad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim -0.70145153 \quad , \quad M_{AB} = M_{BA} = 0$$

Figure (IIH) Mode shape of the Vibrating Beam for the Fourth Zero of λ_4

Equation (3-109) yields the values of natural frequency as

$$\omega = 61.6728092\sqrt{\frac{EI}{\rho AL^4}} \; ; \; 199.8594484\sqrt{\frac{EI}{\rho AL^4}} \; \ldots\ldots \qquad (3\text{-}10b)$$

3.5 Interpretation of Result for the Vibrating Beam

The four nonzero eigenvalues define the mode shapes with associated joint moments, shear forces, displacements, and rotations. Equating to zero the four nonzero value λ yields the conditions of natural frequency. The four conditions are

a) $\lambda_1 = 0$ implies $u = 0$ or $\omega = 0$

b) $\lambda_2 = 0$ implies $u = 0$ or $\omega = 0$

c) $\lambda_3 = 0$ implies $u = 4.7300405$ or $\omega = 22.3732831\sqrt{\frac{EI}{\rho AL^4}}$

d) $\lambda_4 = 0$ implies $u = 7.8532045$ or $\omega = 61.6728209\sqrt{\frac{EI}{\rho AL^4}}$

For condition a), the mode shape corresponds to a rigid body translation with zero natural frequency, and zero joint moments and shears.

For condition b), the mode shape corresponds to a rigid body rotation with zero natural frequency, and zero moments and shears.

Condition c) produces even mode shapes at natural frequencies which correspond to the first mode (i.e. n = 1) and higher modes of a free-free beam (See Figure IIJ).

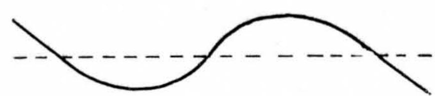
Condition d) produces odd mode shapes at natural frequencies which correspond to the second mode (i.e. n = 2) and higher modes of a free-free beam (See Figure IIK).

$\omega = 4.7300405$ rad/sec

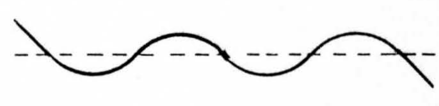
$\omega = 10.9956078$ rad/sec

$\omega = 17.2787596$ rad/sec

<u>Figure (IIJ)</u> Even Mode Shapes of the Vibrating Beam



$\omega = 7.8532045$ rad/sec



$\omega = 14.1371655$ rad/sec

$\omega = 20.4203523$ rad/sec

<u>Figure (IIK)</u> Odd Mode Shapes of the Vibrating Beam

CHAPTER IV

BEAM-COLUMN BENDING PROBLEM

4.1 Equation of Static Equilibrium

The stability stiffness matrix of a general beam-column is obtained from the Equation (2-3), by setting $\Omega = 0$, that is $\beta = 0$. Therefore,

$$\frac{d^4}{dx^4}\underline{\overline{Y}} + \frac{P}{EI}\frac{d^2}{dx^2}\underline{\overline{Y}} = 0 \qquad (4\text{-}1)$$

where the positive sign convention given in Fig. 1 holds. Assuming the solution of the differential Equation (4-1) is

$$\underline{\overline{Y}}(x) = A_m e^{\gamma x} \qquad (4\text{-}2)$$

it follows that

$$\gamma^4 + k^2\gamma^2 = 0 \qquad (4\text{-}3)$$

where

$$k^2 = \frac{P}{EI} \qquad (4\text{-}4)$$

The roots of equation (4-2) become

$$\left.\begin{aligned} \gamma_{1,2} &= 0,0 \\ \gamma_{3,4} &= \pm ik \end{aligned}\right] \qquad (4\text{-}5a)$$

The general solution of the differential equation (4-1) is

$$\overline{Y}(x) = A_1 \cos kx + A_2 \sin kx + A_3 x + A_4 \tag{4-6}$$

where A_1, A_2, A_3 and A_4 are the constants.

4.2 Boundary Conditions

Substituting the boundary conditions given by Equation (2-9a) and (2-9b) into Equation (4-6) yield respectively the matrix forms

$$\begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & K & 1 & 0 \\ c & s & L & 1 \\ -Ks & Kc & 1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \tag{4-7}$$

and

$$\begin{Bmatrix} \dfrac{V_{AB}}{EI} - \dfrac{P\theta_{AB}}{EI} \\ -\dfrac{M_{AB}}{EI} \\ -\dfrac{V_{BA}}{EI} - \dfrac{P\theta_{BA}}{EI} \\ \dfrac{M_{BA}}{EI} \end{Bmatrix} = \begin{bmatrix} 0 & -K^3 & 0 & 0 \\ -K^2 & 0 & 0 & 0 \\ K^3 & -K^3 c & 0 & 0 \\ -K^2 c & -K^2 s & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \tag{4-8}$$

where $s = \sin KL$

$c = \cos KL$

4.3 Stiffness Matrix

Performing the inverse operation on Equation (4-7) gives

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = G \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} \qquad (4\text{-}9)$$

where

$$\left.\begin{aligned}
a_{11} = -a_{13} = a_{32} = a_{34} = a_{43} &= K(1-c) \\
a_{12} = -a_{42} &= s - qc \\
a_{14} = -a_{44} &= q - s \\
-a_{21} = a_{23} &= Ks \\
a_{22} &= -(-1 + c + qs) \\
a_{24} &= -(1-c) \\
a_{31} = -a_{33} &= K^2 s \\
a_{41} &= K(1 - s - qs) \\
G &= \frac{1}{K(2 - 2c - qs)} \\
\text{and} \qquad q &= KL
\end{aligned}\right] \qquad (4\text{-}10)$$

Substituting Equation (4-9) into (4-8), the final form of the stability stiffness matrix relating end forces and moments to displacements and rotation takes the dimensionless form

$$\frac{L}{EI}\begin{Bmatrix} V_{AB}L \\ M_{AB} \\ V_{BA}L \\ M_{BA} \end{Bmatrix} = d_s \begin{bmatrix} q^3 s & & & \text{SYMMETRIC} \\ q^2 - q^2 c & q s - q^2 c & & \\ -q^3 s & q^2 c - q^2 & q^3 s & \\ q^2 - q^2 c & q^2 - q s & q^2 c - q^2 & q s - q^2 c \end{bmatrix} \begin{Bmatrix} \frac{\delta_{AB}}{L} \\ \theta_{AB} \\ \frac{\delta_{BA}}{L} \\ \theta_{BA} \end{Bmatrix} \qquad (4\text{-}11a)$$

or

$$\{f\} = [K_s]\{\delta\} \qquad (4\text{-}11b)$$

where

$$\left.\begin{aligned} q &= KL \\ c &= \cos q \\ s &= \sin q \end{aligned}\right] \qquad (4\text{-}11c)$$

and

$$d_s = \frac{1}{2 - 2c - q s} \qquad (4\text{-}12)$$

The four matrix invariants of $[K_s]$ matrix in Equation (4-11a) are

$$I_1 = \frac{2(q^3 s - q^2 c + q s)}{(2 - 2c - q s)}$$

$$I_2 = \frac{(-4q^5 se + 7q^4 s^2 + 8q^4 e - 8q^4 - 2q^3 se + 2q^3 s)}{(2-2e-qs)^2}$$

$$I_3 = \frac{(-2q^7 s^3 - 8q^6 s^2 e + 8q^6 - 8q^6 e^2 - 16q^5 s + 16q^5 se + 8q^5 s^3)}{(2-2e-qs)^3} \qquad (4\text{-}13)$$

$$I_4 = 0$$

The characteristic equation becomes

$$\lambda^4 - \frac{2(q^3 s - q^2 e + qs)}{(2-2e-qs)}\lambda^3 + \frac{(-4q^5 se + 7q^4 s^2 + 8q^4 e - 8q^4 - 2q^3 se + 2q^3 s)}{(2-2e-qs)^2}\lambda^2$$

$$+ \frac{(2qs - q^2 e - q^2)}{(2-2e-qs)} \cdot \frac{(2q^5 se - 2q^5 s - 4q^4 s^2 + 8q^4 - 8q^4 e)}{(2-2e-qs)^2}\lambda = 0 \qquad (4\text{-}14a)$$

or in quadratic factored form as

$$\lambda\left[\lambda - \frac{(2qs - q^2 e - q^2)}{(2-2e-qs)}\right] \cdot$$

$$\left[\lambda^2 - \frac{(2q^3 s - q^2 e + q^2)}{(2-2e-qs)}\lambda - \frac{(2q^5 se - 2q^5 s - 4q^4 s^2 + 8q^4 - 8q^4 e)}{(2-2e-qs)^2}\right] = 0 \quad (4\text{-}14b)$$

with the four roots determined as

$$\lambda_1 = 0$$

$$\lambda_2 = \frac{(2q^3 s - q^2 e + q^2) - q^2\sqrt{4q^2 s^2 + 4qs(e-1) + 17(e-1)^2}}{2(2-2e-qs)}$$

$$\lambda_3 = \frac{(2qs - q^2 e - q^2)}{(2-2e-qs)} \qquad (4\text{-}14c)$$

$$\lambda_4 = \frac{(2q^3 s - q^2 e + q^2) + q^2\sqrt{4q^2 s^2 + 4qs(e-1) + 17(e-1)^2}}{2(2-2e-qs)}$$

The eigenvalue matrix takes the form

$$[\Delta_s] = \begin{bmatrix} \lambda_1 & & \text{SYMMETRIC} & \\ 0 & \lambda_2 & & \\ 0 & 0 & \lambda_3 & \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{4-14d}$$

4.4 The Eigenvector Matrix

Utilizing equation (1-4), one obtains

$$\Big[[K_s] - \lambda [I]\Big] \{\delta\} = \{0\} \tag{4-15a}$$

Substituting the four roots of λ individually into equation (4-15), the eigenvector matrix is constructed as

$$[U] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\frac{n_1}{d_1} & 0 & \frac{n_3}{d_2} \\ 0 & -\frac{n_2}{d_1} & 1 & \frac{n_2}{d_2} \\ 1 & \frac{n_1}{d_1} & 0 & -\frac{n_3}{d_2} \\ 0 & -\frac{n_2}{d_1} & -1 & \frac{n_2}{d_2} \end{bmatrix} \tag{4-15b}$$

where

$$n_1 = (2q^3 s + q^2 e + q^2) - q^2\sqrt{4q^2 s^2 + 4qs(C-1) + 17(C-1)^2}$$

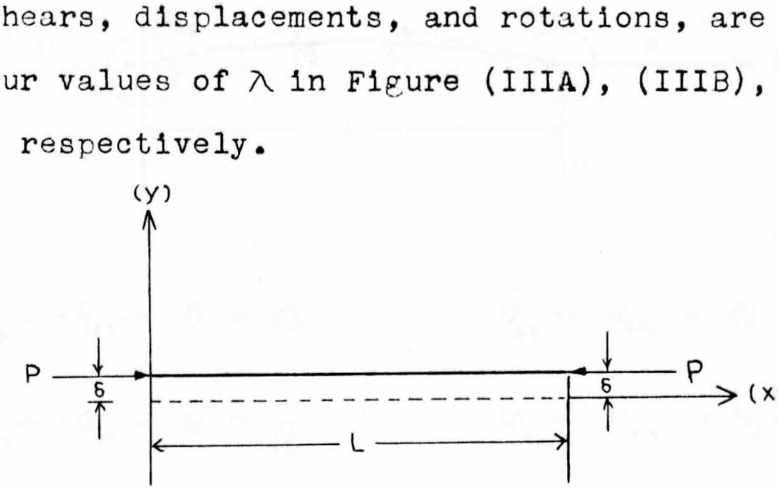
$$
\left.
\begin{aligned}
n_2 &= 4(q^2 - q^2 e) \\
n_3 &= (2q^3 s + q^2 e - q^2) + q^2\sqrt{4q^2 s^2 + 4qs(e-1) + 17(e-1)^2} \\
d_1 &= \sqrt{\left[(2q^3 s + q^2 e - q^2) - q^2\sqrt{4q^2 s^2 + 4qs(e-1) + 17(e-1)^2}\,\right]^2 + 16(q^2 - q^2 e)^2} \\
d_2 &= \sqrt{\left[(2q^3 s + q^2 e - q^2) + q^2\sqrt{4q^2 s^2 + 4qs(e-1) + 17(e-1)^2}\,\right]^2 + 16(q^2 - q^2 e)^2}
\end{aligned}
\right|
\qquad (4\text{-}15c)
$$

Note

$$\frac{n_1}{d_1} = -\frac{n_2}{d_2} \quad , \qquad \frac{n_2}{d_1} = \frac{n_3}{d_2}$$

4.5 Solutions for the Moments, Shear Forces and Vibrations of Normal Mode Shapes

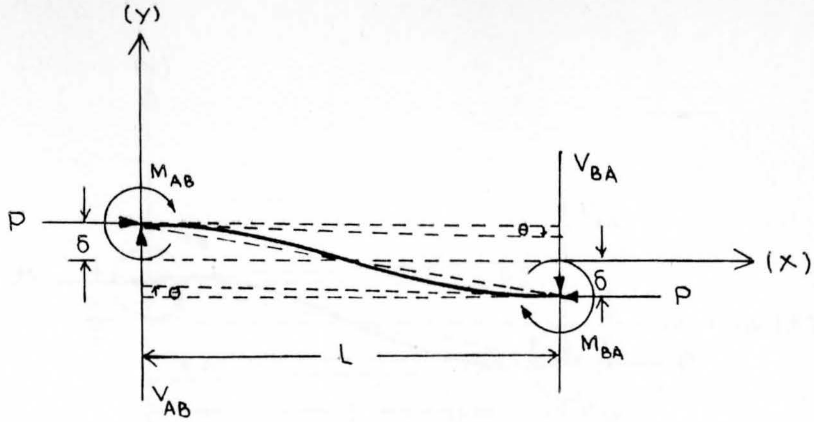
The normal mode shapes, together with the joint moments, shears, displacements, and rotations, are given for the four values of λ in Figure (IIIA), (IIIB), (IIIC) and (IIID) respectively.



$$\delta_{AB} = \delta_{BA} = \bar{\delta} \sim \frac{L}{\sqrt{2}} \qquad\qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta = 0 \qquad\qquad M_{AB} = M_{BA} = 0$$

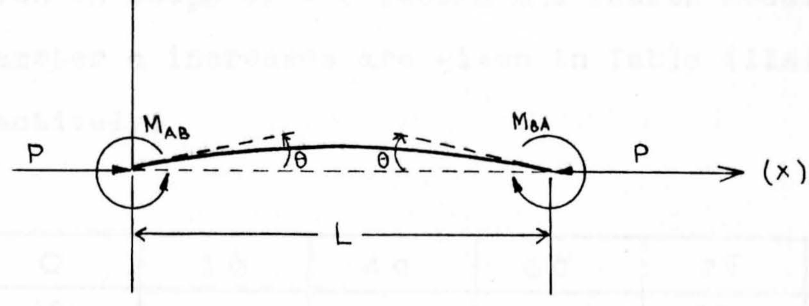
Figure (IIIA) Modal Shape of the Beam-Column for λ_1



$$\delta_{AB} = -\delta_{BA} = \delta \sim -\frac{n_1}{d_1}L \qquad V_{AB} = -V_{BA} = -\lambda_2 \frac{EI}{L^3}\delta$$

$$\theta_{AB} = \theta_{BA} = -\theta \sim -\frac{n_2}{d_1} \qquad M_{AB} = M_{BA} = -\lambda_2 \frac{EI}{L}\theta$$

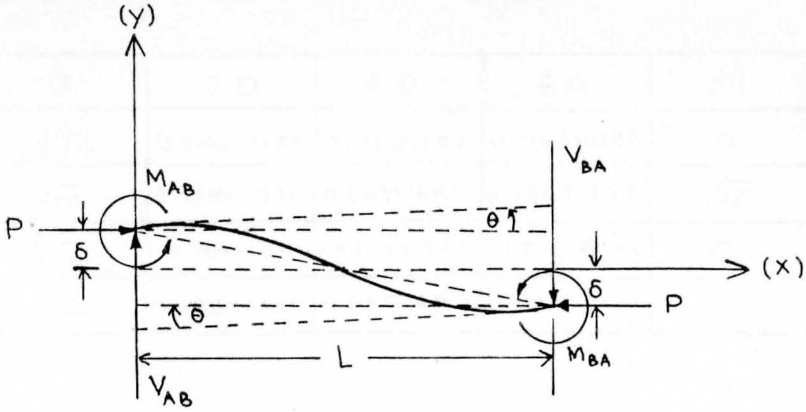
Figure (IIIB) Modal Shape of the Beam-Column for λ_2



$$\delta_{AB} = -\delta_{BA} = \delta = 0 \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = -\theta_{BA} = \theta \sim \frac{1}{\sqrt{2}} \qquad M_{AB} = -M_{BA} = \lambda_3 \frac{EI}{L}\theta$$

Figure (IIIC) Modal Shape of the Beam-Column for λ_3



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{n_3}{d_2} L \qquad V_{AB} = -V_{BA} = \lambda_4 \frac{EI}{L^3}\delta$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{n_2}{d_2} \qquad M_{AB} = M_{BA} = \lambda_4 \frac{EI}{L}\theta$$

Figure (IIID) Modal Shape of the Beam-Column for λ_4

The characteristic shape of first and third modal shapes do not change geometrically as the parameter q increases. The variation in shape of the second and fourth modal shape as the parameter q increases are given in Table (IIA) and (IIB) respectively.

δ \ q	0	2.0	4.0	6.0	2π	6.5
δ_{AB}/L	$1/\sqrt{10}$	0.39842339	0.66311481	0.70691237	$1/\sqrt{2}$	0.01193452
θ_{AB}	$-2/\sqrt{10}$	-0.58417383	-0.24551784	-0.01658826	0	-0.70700624
δ_{BA}/L	$-1/\sqrt{10}$	-0.39842339	-0.66311481	-0.70691237	$-1/\sqrt{2}$	-0.01193452
θ_{BA}	$-2/\sqrt{10}$	-0.58417383	-0.24551784	-0.01658826	0	-0.70700624

Table (IIA) Mode Shape Variation for λ_2

δ \ q	0	2.0	4.0	6.0	2π	6.5
δ_{AB}/L	$2/\sqrt{10}$	0.58417383	0.24551784	0.01658826	0	0.70700624
θ_{AB}	$1/\sqrt{10}$	0.39842339	0.66311481	0.70691237	$1/\sqrt{2}$	0.01193452
δ_{BA}/L	$-2/\sqrt{10}$	-0.58417383	-0.24551784	-0.01658826	0	-0.70700624
θ_{BA}	$1/\sqrt{10}$	0.39842339	0.66311481	0.70691237	$1/\sqrt{2}$	0.01193452

Table (IIB) Mode Shape Variation for λ_4

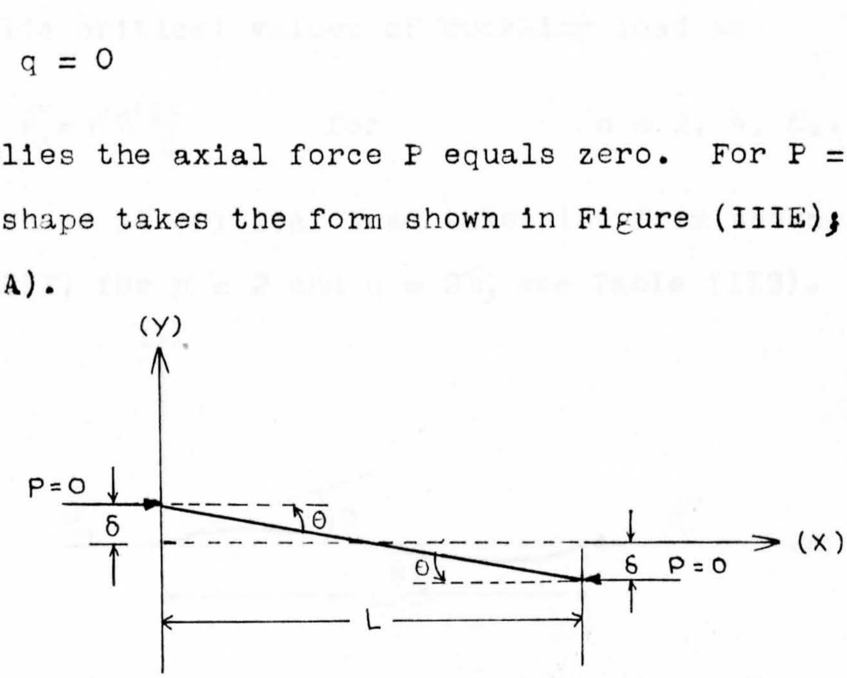
4.6 Zero of the Eigenvalues

The first eigenvalue $\lambda_1 = 0$ corresponds to a rigid body translation as shown in Figure (IIIA).

The second eigenvalue λ_2, given in Equation (4-14C), when equated to zero yields the condition

$$q = 0 \qquad (4\text{-}16)$$

which implies the axial force P equals zero. For P = 0, the mode shape takes the form shown in Figure (IIIE); see Table (IIA).



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{L}{\sqrt{10}} \qquad V_{AB} = -V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim -\frac{2}{\sqrt{10}} \qquad M_{AB} = M_{BA} = 0$$

Figure (IIIE) Modal Shape of the Beam-Column for the Second Zero of λ_2

The third eigenvalue λ_3, when equated to zero yields the condition

$$q = n\pi, \qquad \text{for} \qquad n = 1, 3, 5, \ldots \qquad (4\text{-}17a)$$

noting $q = \sqrt{\frac{P}{EI}}\,L$, it follows that a critical value of axial force is obtained as

$$P_{cr}^{(1)} = n^2\pi^2 \frac{EI}{L^2} \qquad \text{for} \qquad n = 1, 3, 5, \ldots \qquad (4\text{-}17b)$$

For n = 1 and $q = \pi$, the same mode shape occurs as given in Figure (IIIC).

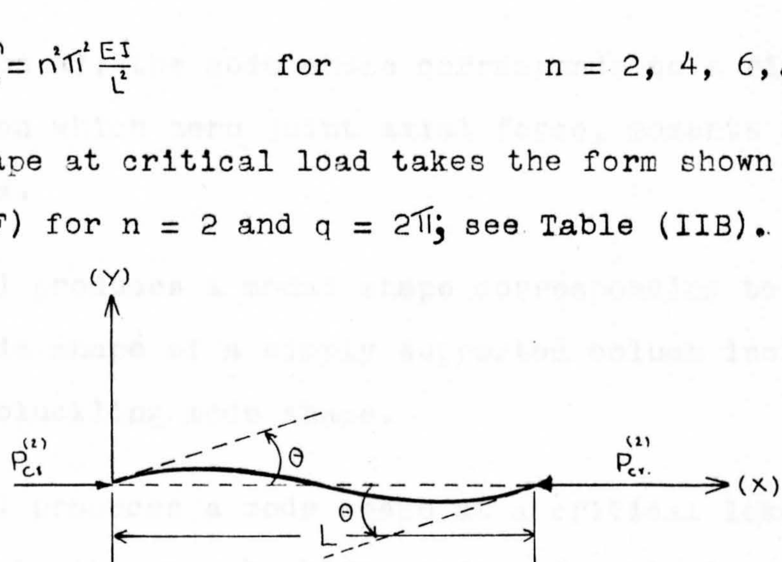
The fourth eigenvalue λ_4 , when equated to zero yields the condition

$$q = n\pi, \qquad \text{for} \qquad n = 2, 4, 6, \ldots \qquad (4\text{-}18a)$$

which yields critical values of buckling load as

$$P_{cr}^{(2)} = n^2\pi^2 \frac{EI}{L^2} \qquad \text{for} \qquad n = 2, 4, 6, \ldots \qquad (4\text{-}18b)$$

The mode shape at critical load takes the form shown in Figure (IIIF) for n = 2 and $q = 2\pi$; see Table (IIB).



$$\delta_{AB} = \delta_{BA} = \delta = 0 \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{1}{\sqrt{2}} \qquad M_{AB} = M_{BA} = 0$$

Figure (IIIF) Modal Shape of the Beam-Column for the Fourth Zero of λ_4

4.7 Interpretation of Results for the Beam-Column

The single zero eigenvalue is obtained for this problem which corresponds to a rigid body translation. The three nonzero eigenvalues define a pure bending mode shape and two additional deformed mode shapes which are associated with joint moments, shears, displacements, and rotations.

Equating to zero the three nonzero values of λ yields the condition of critical buckling load. The three conditions are:

a) $\lambda_1 = 0$ implies $q = 0$ or $P = 0$

b) $\lambda_3 = 0$ implies $q = \pi, n=1$ or $P_{cr}^{(1)} = \pi^2 \frac{EI}{L^2}$

c) $\lambda_4 = 0$ implies $q = 2\pi, n=2$ or $P_{cr}^{(2)} = 4\pi^2 \frac{EI}{L^2}$

For condition a), the mode shape corresponds to a rigid body rotation which zero joint axial force, moments and shear forces.

Condition b) produces a modal shape corresponding to the odd buckling mode shape of a simply supported column including the lowest bluckling mode shape.

Condition c) produces a mode shape at a critical load which corresponds to the even buckling modes of a simply supported beam-column.

CHAPTER V

VIBRATING BEAM-COLUMN PROBLEM

5.1 The Eigenvalues

For the vibrating beam-column problem, the general stiffness matrix is obtained by a matrix $\left[S \right]$ from equation (2-14); it follows that

$$\{f\} = [S]\{\delta\}$$

The four matrix invariants of $\left[S \right]$ matrix are

$$\left.\begin{aligned}
I_1 &= \frac{1}{d}\left[-2(u^2+v^2)\right]\left[Se\, v(1-u^2) - Cs\, u(1+v^2)\right] \\
I_2 &= \frac{1}{d}\left[Ss(u^2+v^2)^2(1-u^2v^2) + 4(1+3Ce)u^3v^3 + 2\left\{2Ce(u^2-v^2)+Ss\,uv\right\}uv(u^2-v^2)\right] \\
I_3 &= \frac{1}{d}\left[2uv(u^2+v^2)\right]\left[Se\, u(u^2-v^4) - Cs\, v(v^2+u^4)\right] \\
I_4 &= \frac{1}{d}\, u^2v^2\left[2(1-Ce)u^3v^3 - Ss(u^2-v^2)\right]
\end{aligned}\right] \qquad (5\text{-}1a)$$

where

$$d = 2uv(1-Ce) + (u^2-v^2)Ss \qquad (5\text{-}1b)$$

The characteristic equation in quadratic factored form becomes

$$
\left.
\begin{aligned}
&\left[\lambda^2 + \frac{(u^2+v^2)\{(Sev - su - Csu + Sv) - (Seu - sv + Csv - Su)uv\}}{d}\lambda \right.\\
&\qquad \left. - \frac{\{uv(u^4+v^4)(1-Ce) - u^2v^2(u^2-v^2)Ss + uv(u^4-v^4)(C-e)\}}{d}\right] = 0\\
&\left[\lambda^2 + \frac{(u^2+v^2)\{(Sev + su - Csu - Sv) - (Seu + sv + Csv + Su)uv\}}{d}\lambda \right.\\
&\qquad \left. - \frac{\{uv(u^4+v^4)(1-Ce) - u^2v^2(u^2-v^2)Ss - uv(u^4-v^4)(C-e)\}}{d}\right] = 0
\end{aligned}
\right]
\qquad (5\text{-}2a)
$$

with the four roots determined as

$$
\left.
\begin{aligned}
\lambda_1 &= \frac{1}{2d}\Bigg[-(u^2+v^2)\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\} \\
&\quad -\sqrt{(u^2+v^2)^2\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\}}\,\Bigg] \\
\lambda_2 &= \frac{1}{2d}\Bigg[-(u^2+v^2)\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\} \\
&\quad +\sqrt{(u^2+v^2)^2\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\}}\,\Bigg] \\
\lambda_3 &= \frac{1}{2d}\Bigg[-(u^2+v^2)\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\} \\
&\quad +\sqrt{(u^2+v^2)^2\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\}}\,\Bigg] \\
\lambda_4 &= \frac{1}{2d}\Bigg[-(u^2+v^2)\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\} \\
&\quad -\sqrt{(u^2+v^2)^2\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\}}\,\Bigg]
\end{aligned}
\right] \quad (5\text{-}2b)
$$

The eigenvalue matrix takes the form

$$\left[\Lambda\right] = \begin{bmatrix} \lambda_1 & & \text{SYMMETRIC} & \\ 0 & \lambda_2 & & \\ 0 & 0 & \lambda_3 & \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \qquad (5\text{-}2c)$$

5.2 <u>The Eigenvector Matrix</u>

Utilizing equation (1-4), one obtains

$$\Big[\left[S\right] - \lambda\left[I\right]\Big]\left\{\delta\right\} = \left\{0\right\} \qquad (5\text{-}3)$$

Substituting the four of λ's individually into equation (5-3), the eigenvector matrix is constructed as

$$\left[U\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} n_1/d_1 & n_3/d_2 & n_5/d_3 & n_6/d_4 \\ n_2/d_1 & n_4/d_2 & n_2/d_3 & n_4/d_4 \\ n_1/d_1 & -n_3/d_2 & n_5/d_3 & -n_6/d_4 \\ -n_2/d_1 & n_4/d_2 & -n_2/d_3 & n_4/d_4 \end{bmatrix} \qquad (5\text{-}4a)$$

where

$$n_1 = \Big[(u^2+v^2)\{(Sev-su-Csu+Sv)+(Seu-sv+Csv-Su)uv\} - \sqrt{(u^2+v^2)^2\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\}}\Big]$$

$$n_2 = 2uv\{-(u^2-v^2)(1-Ce)+2uvSs+(u^2+v^2)(C-e)\}$$

$$n_3 = \Big[(u^2+v^2)\{(Sev+su-Csu-Sv)+(Seu+sv+Csv+Su)uv\} - \sqrt{(u^2+v^2)^2\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\}}\Big]$$

$$n_4 = 2uv\{-(u^2-v^2)(1-Ce)+2uvSs+(u^2+v^2)(C-e)\}$$

$$n_5 = \Big[(u^2+v^2)\{(Sev-su-Csu+Sv)+(Seu-sv+Csv-Su)uv\} + \sqrt{(u^2+v^2)^2\{(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\}}\Big]$$

$$n_6 = \Big[(u^2+v^2)\{(Sev+su-Csu-Sv)+(Seu+sv+Csv+Su)uv\} + \sqrt{(u^2+v^2)^2\{(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\}^2 + 4\{2uv(1-Ce)+(u^2-v^2)Ss\}\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\}}\Big] \quad (5\text{-}4b)$$

$$d_1 = \sqrt{\left[2uv\left\{-(u^2-v^2)(1-Ce)+2uvSs-(u^2+v^2)(C-e)\right\}\right]^2+\left[(u^2+v^2)\left\{(Sev-su-Csu+Sv)+(Seu-sv+Csv-Su)uv\right\}-\sqrt{(u^2+v^2)^2\left[(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\right]^2+4\left\{2uv(1-Ce)+(u^2-v^2)Ss\right\}\left\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\right\}}\right]^2}$$

$$d_2 = \sqrt{\left[2uv\left\{-(u^2-v^2)(1-Ce)+2uvSs-(u^2+v^2)(C-e)\right\}\right]^2+\left[(u^2+v^2)\left\{(Sev+su-Csu-Sv)+(Seu+sv+Csv+Su)uv\right\}-\sqrt{(u^2+v^2)^2\left[(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\right]^2+4\left\{2uv(1-Ce)+(u^2-v^2)Ss\right\}\left\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\right\}}\right]^2}$$

$$d_3 = \sqrt{\left[2uv\left\{-(u^2-v^2)(1-Ce)+2uvSs-(u^2+v^2)(C-e)\right\}\right]^2+\left[(u^2+v^2)\left\{(Sev-su-Csu+Sv)+(Seu-sv+Csv-Su)uv\right\}+\sqrt{(u^2+v^2)^2\left[(Sev-su-Csu+Sv)-(Seu-sv+Csv-Su)uv\right]^2+4\left\{2uv(1-Ce)+(u^2-v^2)Ss\right\}\left\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss+uv(u^4-v^4)(C-e)\right\}}\right]^2}$$

$$d_4 = \sqrt{\left[2uv\left\{-(u^2-v^2)(1-Ce)+2uvSs+(u^2+v^2)(C-e)\right\}\right]^2+\left[(u^2+v^2)\left\{(Sev+su-Csu-Sv)+(Seu+sv+Csv+Su)uv\right\}+\sqrt{(u^2+v^2)^2\left[(Sev+su-Csu-Sv)-(Seu+sv+Csv+Su)uv\right]^2+4\left\{2uv(1-Ce)+(u^2-v^2)Ss\right\}\left\{uv(u^4+v^4)(1-Ce)-u^2v^2(u^2-v^2)Ss-uv(u^4-v^4)(C-e)\right\}}\right]^2}$$

Note

$$\frac{n_1}{d_1} = \frac{n_2}{d_3}, \qquad \frac{n_2}{d_1} = -\frac{n_5}{d_3}$$

$$\frac{n_3}{d_2} = -\frac{n_4}{d_4}, \qquad \frac{n_4}{d_2} = \frac{n_6}{d_4}$$

5.3 Zeros of the Eigenvalues

The first eigenvalue λ_1 when equated to zero, yields the condition

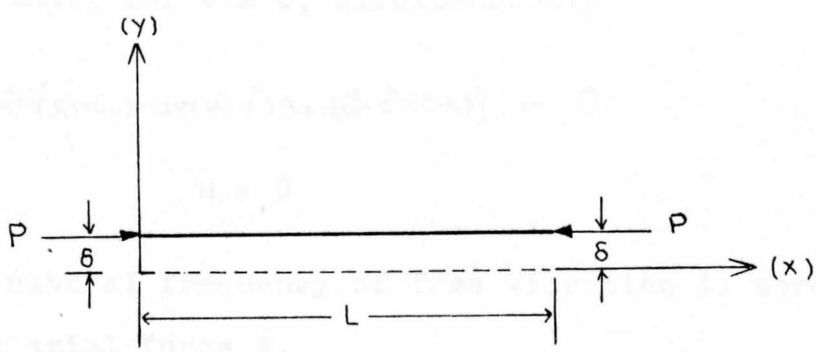
$$uv\left[(u^4+v^4)(1-Cc)-uv(u^2-v^2)Ss+(u^4-v^4)(C-c)\right] = 0 \qquad (5\text{-}5)$$

It follows that, for u = 0, simultaneously

$$\left[(u^4+v^4)(1-Cc)-uv(u^2-v^2)Ss+(u^4-v^4)(C-c)\right] = 0 \qquad (5\text{-}6a)$$

and $$v \neq 0 \qquad (5\text{-}6b)$$

Further, it follows that $\beta^4 = 0$, or the natural frequency of free vibration is zero, that is, $\Omega^2 = 0$.
The mode shape is shown in Figure (IVA) which is the case of rigid body translational motion with arbitrary value of axial force. This shape is similar to that given in static stability problem (see Fig. (IIIA)). The mode shape result is shown in Figure (IVA) for convenience.



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{L}{\sqrt{2}} \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta = 0 \qquad M_{AB} = M_{BA} = 0$$

Figure (IVA) Modal Shape of the Vibrating Beam-Column for the First Zero of λ_1

The variation in the mode shape for values of the parameter v with u = 0 is shown in Table (IIIA). The mode shape does not change as v increases.

v	0^+	π	2π	3π	4π	5π
δ \\ u	0	0	0	0	0	0
δ_{AB}/L	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
θ_{AB}	0	0	0	0	0	0
δ_{BA}/L	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
θ_{BA}	0	0	0	0	0	0

Table (IIIA) Mode Shape Variation for the First Zero of λ_1

The second eigenvalue λ_2, when equated to zero, yields the condition

$$uv\left[(u^4+v^4)(1-Ce)-uv(u^2-v^2)Ss-(u^4-v^4)(C-e)\right] = 0 \tag{5-7}$$

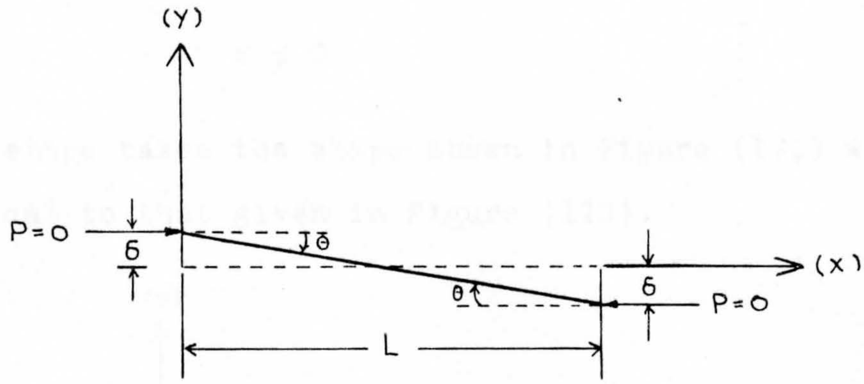
It follows that, for v = 0, simultaneously

$$\left[(u^4+v^4)(1-Ce)-uv(u^2-v^2)Ss-(u^4-v^4)(C-e)\right] = 0 \tag{5-8a}$$

and
$$u = 0 \tag{5-8b}$$

Hence, the natural frequency of free vibration is zero as well as the axial force P.

The mode shape takes the shape shown in Figure (IVB) which is the case of rigid body rotational motion. This shape is for the static stability problem and the beam vibration problem respectively and is identical to that given in Figure (IIF) and Figure (IIIE).



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{L}{\sqrt{10}} \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{2}{\sqrt{10}} \qquad M_{AB} = M_{BA} = 0$$

Figure (IVB) Mode Shape of the Vibrating Beam-Column for the Second Zero of λ_2

The mode shape for the values of the parameters v = 0 and u = 0 is shown in Table (IIIB).

u	v	δ_{AB}/L	θ_{AB}	δ_{BA}/L	θ_{BA}
0	0	$1/\sqrt{10}$	$-2/\sqrt{10}$	$-1/\sqrt{10}$	$-2/\sqrt{10}$

Table (IIIB) Mode Shape Variation for the Second Zero of λ_2

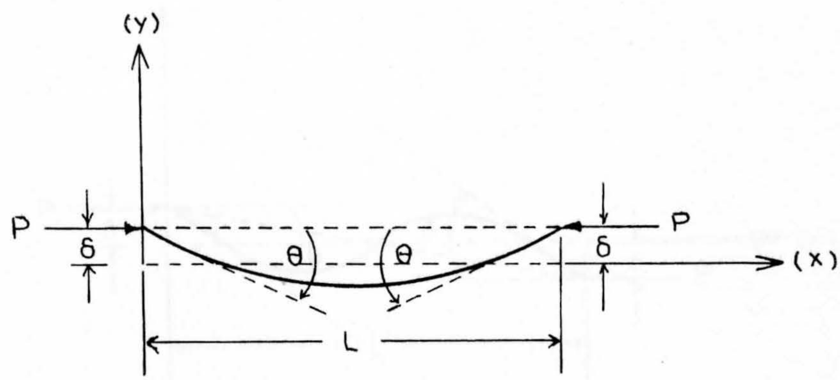
The third eigenvalue λ_3 when equated to zero yields the condition in Equation (5-5).

It follows that,

$$\left[(u^4+v^4)(1-Cc) - uv(u^2-v^2)Ss + (u^4-v^4)(C-c)\right] = 0 \tag{5-9}$$

$$\left.\begin{array}{lll} \text{with} & & u \neq 0 \\ \text{and} & & v \neq 0 \end{array}\right] \tag{5-10a}$$

The mode shape takes the shape shown in Figure (IVC) which is identical to that given in Figure (IIG).



$$\delta_{AB} = \delta_{BA} = \delta \sim \frac{n_5^*}{d_3} L \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = -\theta_{BA} = \theta \sim \frac{n_2^*}{d_3} \qquad M_{AB} = M_{BA} = 0$$

<u>Figure (IVC)</u> Modal Shape of the Vibrating Beam-Column for the Third Zero of λ_3

The variation in the mode shape for values of the parameter u and v are shown in Table (IIIC).

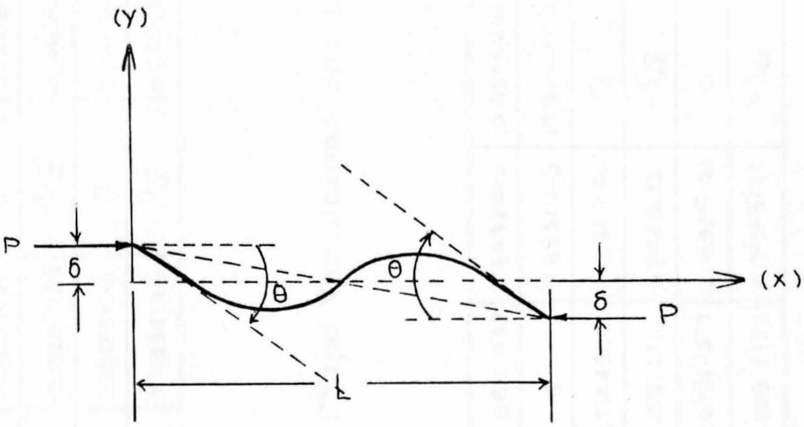
* See Appendix III

The fourth eigenvalue λ_4 when equated to zero yields the condition in Equation (5-7). It follows that

$$\left[(u^4+v^4)(1-Cc) - uv(u^2-v^2)Ss - (u^4-v^4)(C-c)\right] = 0 \tag{5-11}$$

$$\left.\begin{array}{ll} \text{with} & u \neq 0 \\ \text{and} & v \neq 0 \end{array}\right] \tag{5-12a}$$

The mode shape takes the shape shown in Figure (IVD) which is identical to that given in Figure (IIH).



$$\delta_{AB} = -\delta_{BA} = \delta \sim \frac{n_6}{d_4}L^{*} \qquad V_{AB} = V_{BA} = 0$$

$$\theta_{AB} = \theta_{BA} = \theta \sim \frac{n_4}{d_4}^{*} \qquad M_{AB} = M_{BA} = 0$$

Figure (IVD) Modal Shape of the Vibrating Beam-Column for the Fourth Zero of λ_4

The variation in the mode shapes for values of the parameter u and v are shown in Table (IIID).

* See Appendix III

ν	π	4.0080257	4.7300405	7.8532045	8.5187096	3π	9.8330142	10.9956078	14.1371655	14.7116599	5π
δ \ u	0^+	3	4.7300405	7.8532045	4	0^+	6	10.9956078	14.1371655	8	0^+
δ_{AB}/L	0	0.10676242	0.14870374	$1/\sqrt{2}$	0.08207652	0	0.25118475	0.64045949	0.00417104	0.01950135	0
θ_{AB}	$-1/\sqrt{2}$	-0.69900075	-0.69128401	0	-0.70232735	$-1/\sqrt{2}$	-0.66098902	-0.70420052	-0.70709467	-0.70683784	$-1/\sqrt{2}$
δ_{BA}/L	0	0.10676242	0.14870374	$1/\sqrt{2}$	0.08207652	0	0.25118475	0.64045949	0.00417104	0.01950735	0
θ_{BA}	$1/\sqrt{2}$	0.69900075	0.69128401	0	0.70232735	$1/\sqrt{2}$	0.66098902	0.70420052	0.70709467	0.70683784	$1/\sqrt{2}$

Table (IIIC) Mode Shape Variations for the Third Zero of λ_3

ν	4.7300405	5.4425034	2π	6.6885432	7.8532045	10.9956078	11.6174487	4π	13.0204615	14.1371655
δ \ u	4.7300405	3	0^+	4	7.8532045	10.9956078	6	0^+	8	14.1371655
δ_{AB}/L	$1/\sqrt{2}$	0.60173913	0	0.67544370	0.08925106	0	0.70664030	0	0.70413947	0.04988897
θ_{AB}	0	-0.37136273	$-1/\sqrt{2}$	-0.20922732	-0.70145172	$-1/\sqrt{2}$	-0.02568547	$-1/\sqrt{2}$	-0.06471378	-0.70534485
δ_{BA}/L	$-1/\sqrt{2}$	-0.60173913	0	-0.67544370	-0.08925106	0	-0.70664030	0	-0.70413947	-0.04988897
θ_{BA}	0	-0.37136273	$-1/\sqrt{2}$	-0.20922732	-0.70145172	$-1/\sqrt{2}$	-0.02568547	$-1/\sqrt{2}$	-0.06471378	-0.70534485

Table (IIID) Mode Shape Variations for the Fourth Zero of λ_4

5.4 Interpretation of Result for the Vibrating Beam-Column

The four nonzero eigenvalues define the mode shapes with associated joint moments, shear forces, displacement and rotations. Equating to zero the four nonzero values of λ, yields the condition of natural frequency and axial force. The four conditions are

a) $\lambda_1 = 0$ implies $u = 0$ when $v \neq 0$

b) $\lambda_2 = 0$ implies $v = 0$ when $u = 0$

c) $\lambda_3 = 0$ implies $\left[(u^4+v^4)(1-Ce)-uv(u^2-v^2)Ss+(u^4-v^4)(C-e)\right] = 0$
when $u \neq 0$ and $v \neq 0$

d) $\lambda_4 = 0$ implies $\left[(u^4+v^4)(1-Ce)-uv(u^2-v^2)Ss-(u^4-v^4)(C-c)\right] = 0$
when $u \neq 0$ and $v \neq 0$

For condition a), the mode shape corresponds to a rigid body translation with arbitrary value of axial force, zero natural frequency and zero values of joint moments and shear forces.

Condition b), the mode shape corresponds to a rigid body rotation with zero natural frequency and zero values of joint moment and shear force.

Condition c), the mode shape is produced with a natural frequency with corresponds to the odd modes (i.e. n = 1,3,5,7...) of a free-free beam-column.

Condition d), the mode shape is produced with a natural frequency with corresponds to the even modes (i.e. n = 2,4,6...) of a free-free beam-column.

CHAPTER VI

DISCUSSION AND CONCLUSION

6.1 Discussion

In general, four deformed mode shapes are defined in the vibrating beam problem, the beam-column bending problem and the vibrating beam-column problem, with the exception that in the beam-column bending problem one rigid body translational mode shape is present.

The zeros of the eigenvalues of the general stiffness matrix produce two rigid body mode shapes, one a rigid body translational mode shape and the other a rigid body rotational mode shape in each of the three problems. These zero eigenvalues produce the exact values of the natural frequency of free-free vibrating beam, the critical buckling load of a simply supported column, and the resonant frequencies of a vibrating beam-column respectively.

For the vibrating beam-column problem, the relationships between the parameter u and v in term of the axial force and natural frequency of free vibration are shown in Figures (VA), (VB), and (VC). Figures (VA) and (VB) correspond to the zeros of the eigenvalues of λ_1, λ_3 and λ_2, λ_4 respectively. Figure (VC) is a superposition of the latter two figures.

The special case of the zeros of the eigenvalues for the three problems, yields the condition of natural frequency, critical buckling load in terms of parameter u and v. The conditions are (see Figure (VC)),

1) $v \neq 0$ when $u = 0$ (i.e. the V axis) correspond to the rigid body translational modes and arbitrary axial compressive force.

2) $v = 0$ when $u = 0$ (i.e. the origin of coordinates) corresponds to the rigid body rotational mode with zero natural frequency and zero axial force.

3) $v = 0$ when $u \neq 0$ (i.e. the u axis) corresponds to the static column with zero natural frequency and an increasing axial tension force.

4) $u = v$ (i.e. 45° line) corresponds to the natural frequencies of the vibrating beam problem only.

The shaded areas in Figure (VC) corresponds to combinations of the parameters u and v where stable oscillation occur for the vibrating beam-column. The unshaded areas correspond to combinations of u and v where unstable oscillations of the vibrating beam-column occur; stable zones above and to the left of the 45° line define conditions of axial compression force. Stable zone below and to the right of the 45° line indicate the condition of axial tensile load.

6.2 Conclusion

The stiffness matrices in each of the three problems considered are (4x4) symmetric matrices possessing four eigenvalues. Only in the case of the static beam-column bending problem does there exist a zero eigenvalue. This special case produces a zero determinant of the stiffness matrix. Hence its inverse does not exist (i.e. the matrix is positive indefinite). This eigenvalue corresponds to a rigid body translation of beam column. In the other two problems, zero eigenvalues do not occur for the stiffness matrix which implies that their inverses exist when the structural member is subjected to nonzero end shear forces and end moments.

Equating to zero all nonzero eigenvalues leads to a condition of zero determinate in all three problems. All these resulting systems exist under a condition of zero end shear and zero end moment. Hence, the conditions of natural frequency of free vibration and critical buckling load are produced.

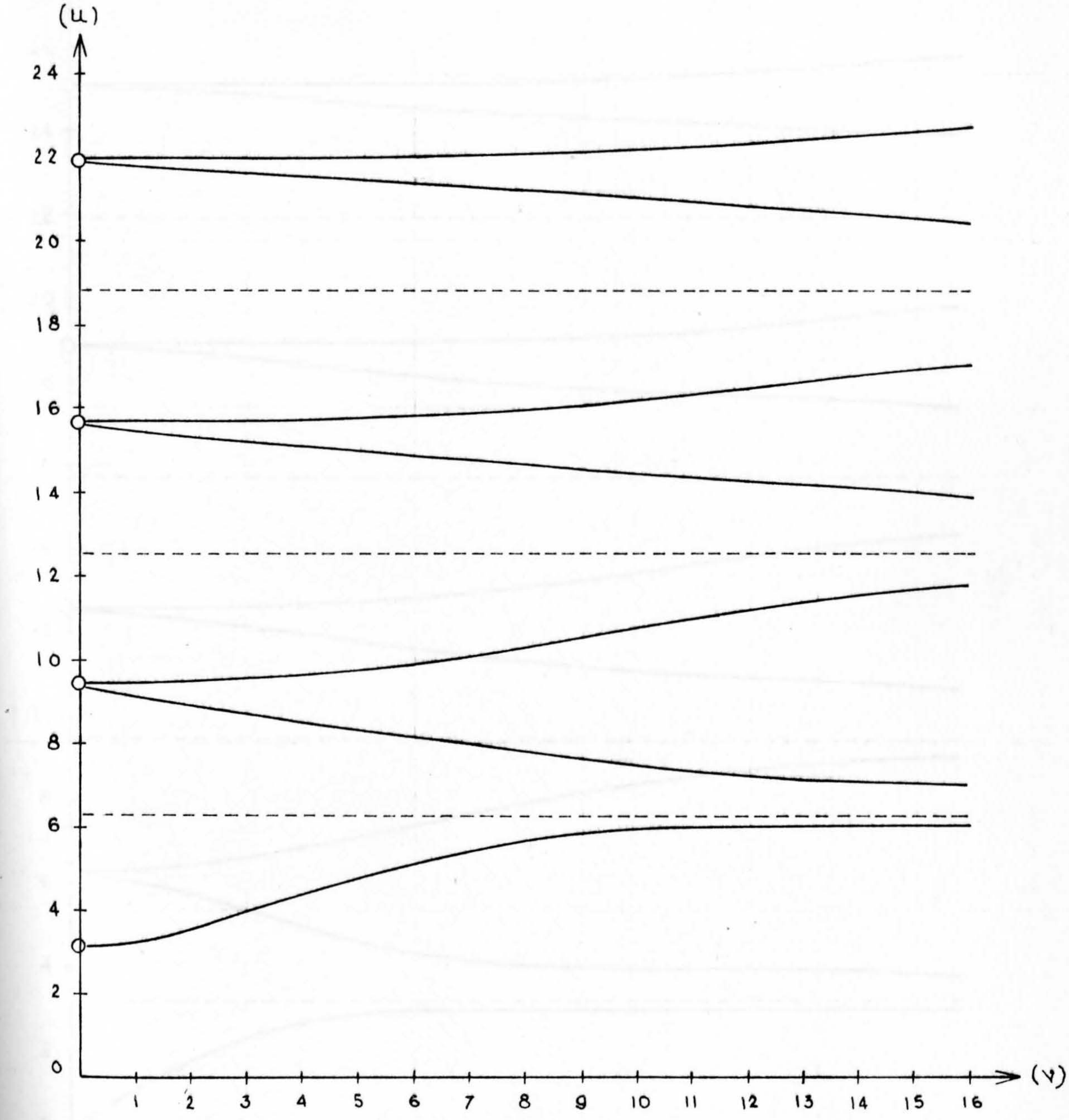


Figure (VA) Plot of u vs. v for the Zeros of the Eigenvalue- λ_1, λ_3

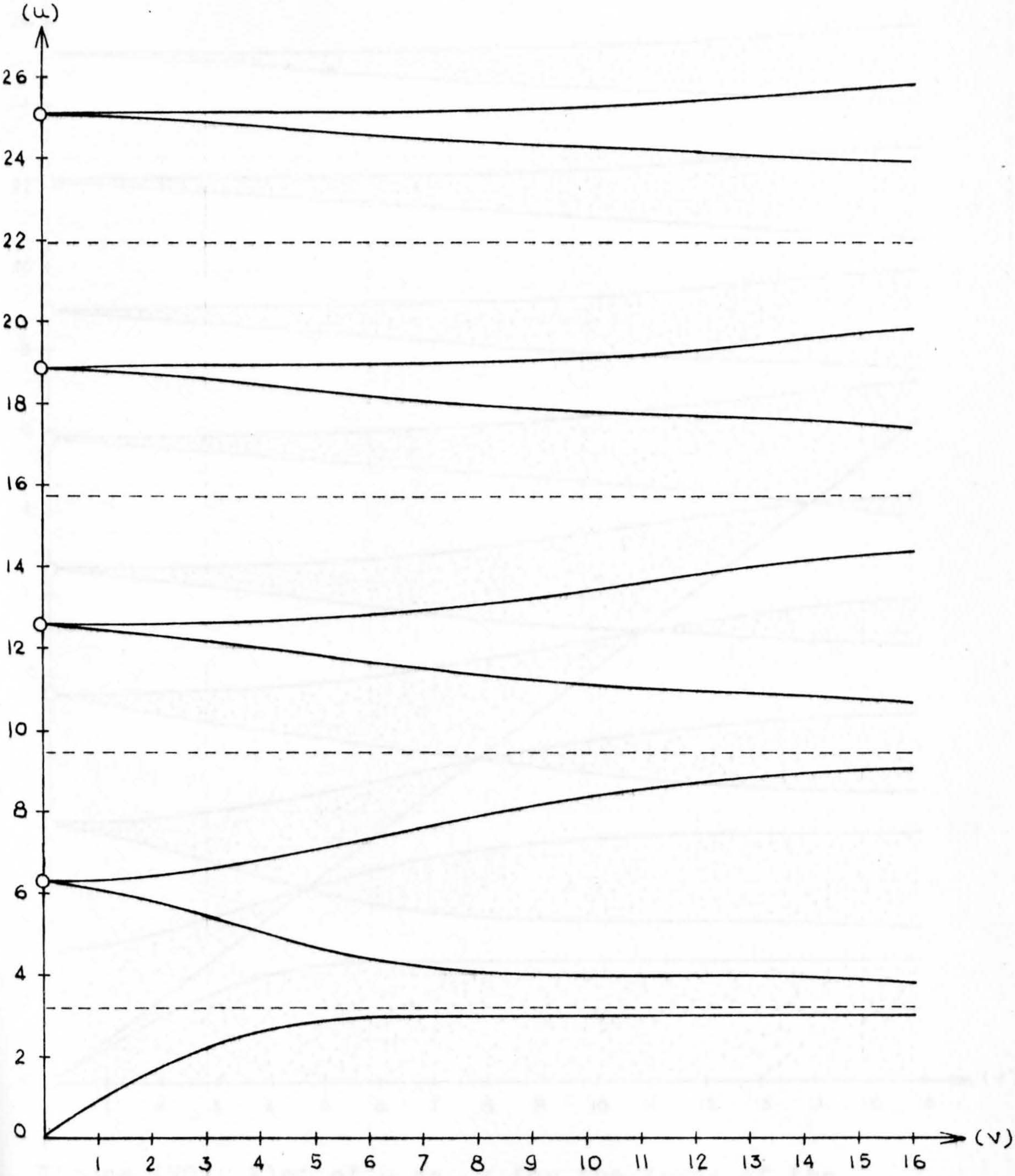


Figure (VB) Plot of u vs. v for the Zeros of the Eigenvalue- λ_2 , λ_4

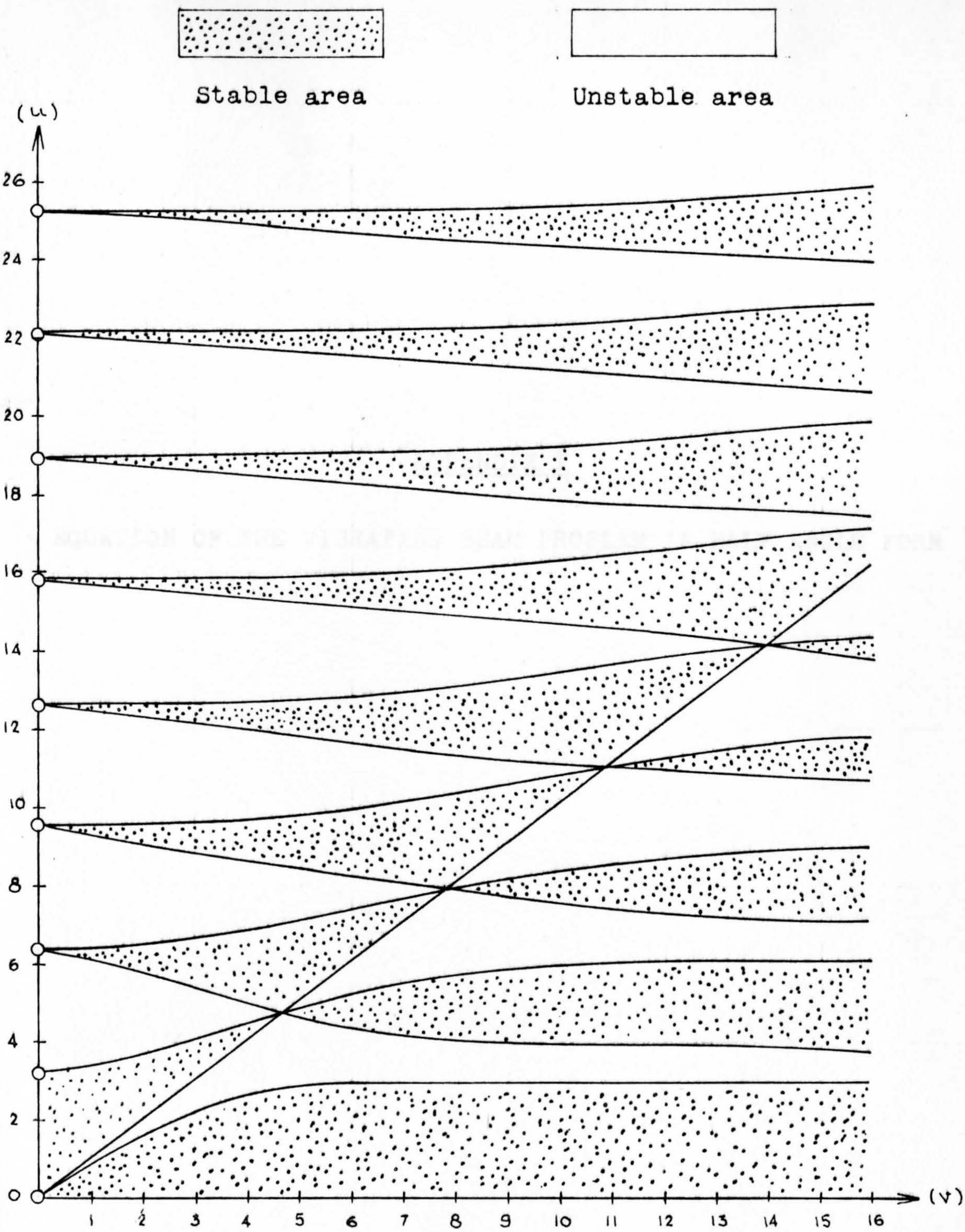


Figure (VC) Plot of u vs. v for the Zeros of the Eigenvalue- $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

APPENDIX I

EQUATION OF THE VIBRATING BEAM PROBLEM IN HALF ANGLE FORM

The four roots in Equation (3-5c) are written in half angle form as

$$\lambda_1 = \frac{2u(\hat{C}\hat{c} - u^2\hat{S}\hat{s}) - \sqrt{4u^2(\hat{C}\hat{c} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{c})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{c})}$$

$$\lambda_2 = \frac{2u(\hat{S}\hat{s} + u^2\hat{C}\hat{c}) - \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{c}) + u^4(\hat{C}\hat{s} - \hat{S}\hat{c})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{c})} \qquad \text{(I-1)}$$

$$\lambda_3 = \frac{2u(\hat{C}\hat{c} - u^2\hat{S}\hat{s}) + \sqrt{4u^2(\hat{C}\hat{c} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{c})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{c})}$$

$$\lambda_4 = \frac{2u(\hat{S}\hat{s} + u^2\hat{C}\hat{c}) + \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{c})^2 + u^4(\hat{C}\hat{s} - \hat{S}\hat{c})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{c})}$$

where

$$\hat{C} = \cosh\frac{u}{2}$$

$$\hat{S} = \sinh\frac{u}{2} \qquad \text{(I-2)}$$

$$\hat{c} = \cos\frac{u}{2}$$

$$\hat{s} = \sin\frac{u}{2}$$

The components of eigenvector matrix in Equation (3-6b) are written in half angle form as

$$n_1 = \frac{-2u(\hat{C}\hat{c} + u^2\hat{S}\hat{s}) - \sqrt{4u^2(\hat{C}\hat{c} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{c})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{c})}$$

$$n_2 = \frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{c} + 2\hat{c}^2 - 2\hat{C}^2)}{(\hat{C}\hat{s} + \hat{S}\hat{c})(\hat{C}\hat{s} - \hat{S}\hat{c})}$$

$$n_3 = \frac{-2u(\hat{S}\hat{s} - u^2\hat{C}\hat{e}) - \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{e})^2 + u^4(\hat{C}\hat{s} - \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{e})}$$

$$n_4 = \frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{e} - 2\hat{e}^2 + 2\hat{C}^2)}{(\hat{C}\hat{s} + \hat{S}\hat{e})(\hat{C}\hat{s} - \hat{S}\hat{e})}$$

$$n_5 = \frac{-2u(\hat{C}\hat{e} + u^2\hat{S}\hat{s}) + \sqrt{4u^2(\hat{C}\hat{e} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{e})}$$

$$n_6 = \frac{-2u(\hat{S}\hat{s} - u^2\hat{C}\hat{e}) + \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{e})^2 + u^4(\hat{C}\hat{s} - \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{e})}$$

$$d_1 = \sqrt{\left[\frac{-2u(\hat{C}\hat{e} + u^2\hat{S}\hat{s}) - \sqrt{4u^2(\hat{C}\hat{e} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{e})}\right]^2 + \left[\frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{e} + 2\hat{e}^2 - 2\hat{C}^2)}{(\hat{C}^2\hat{s}^2 - \hat{S}^2\hat{e}^2)}\right]^2} \qquad \text{(I-3)}$$

$$d_2 = \sqrt{\left[\frac{-2u(\hat{S}\hat{s} - u^2\hat{C}\hat{e}) - \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{e})^2 + u^4(\hat{C}\hat{s} - \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{e})}\right]^2 + \left[\frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{e} - 2\hat{e}^2 + 2\hat{C}^2)}{(\hat{C}^2\hat{s}^2 - \hat{S}^2\hat{e}^2)}\right]^2}$$

$$d_3 = \sqrt{\left[\frac{-2u(\hat{C}\hat{e} + u^2\hat{S}\hat{s}) + \sqrt{4u^2(\hat{C}\hat{e} - u^2\hat{S}\hat{s})^2 + u^4(\hat{C}\hat{s} + \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} + \hat{S}\hat{e})}\right]^2 + \left[\frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{e} + 2\hat{e}^2 - 2\hat{C}^2)}{(\hat{C}^2\hat{s}^2 - \hat{S}^2\hat{e}^2)}\right]^2}$$

$$d_4 = \sqrt{\left[\frac{-2u(\hat{S}\hat{s} - u^2\hat{C}\hat{e}) + \sqrt{4u^2(\hat{S}\hat{s} + u^2\hat{C}\hat{e})^2 + u^4(\hat{C}\hat{s} - \hat{S}\hat{e})^2}}{(\hat{C}\hat{s} - \hat{S}\hat{e})}\right]^2 + \left[\frac{u^2(4\hat{S}\hat{C}\hat{s}\hat{e} - 2\hat{e}^2 + 2\hat{C}^2)}{(\hat{C}^2\hat{s}^2 - \hat{S}^2\hat{e}^2)}\right]^2}$$

APPENDIX II

EQUATION OF THE VIBRATING BEAM-COLUMN PROBLEM IN HALF ANGLE FORM

The four roots in Equation (5-2b) are written in half angle form as

$$
\left.
\begin{aligned}
\lambda_1 &= \frac{1}{\hat{d}}\Big[-2(u^2+v^2)\{\hat{S}\hat{C}v(\hat{c}^2+\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2+\hat{S}^2v^2)\} \\
&\quad -\sqrt{4(u^2+v^2)^2\{\hat{S}\hat{C}v(\hat{c}^2+\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2+\hat{S}^2v^2)\}^2} \\
&\quad \overline{+2uv\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\}\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}+2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\}}\Big] \\
\lambda_2 &= \frac{1}{\hat{d}}\Big[2(u^2+v^2)\{\hat{S}\hat{C}v(\hat{s}^2+\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2+\hat{C}^2v^2)\} \\
&\quad +\sqrt{4(u^2+v^2)^2\{\hat{S}\hat{C}v(\hat{s}^2+\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2+\hat{C}^2v^2)\}^2} \\
&\quad \overline{+2uv\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\}\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}-2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\}}\Big] \\
\lambda_3 &= \frac{1}{\hat{d}}\Big[-2(u^2+v^2)\{\hat{S}\hat{C}v(\hat{c}^2+\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2+\hat{S}^2v^2)\} \\
&\quad +\sqrt{4(u^2+v^2)^2\{\hat{S}\hat{C}v(\hat{c}^2+\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2+\hat{S}^2v^2)\}^2} \\
&\quad \overline{+2uv\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\}\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}+2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\}}\Big] \\
\lambda_4 &= \frac{1}{\hat{d}}\Big[2(u^2+v^2)\{\hat{S}\hat{C}v(\hat{s}^2+\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2+\hat{C}^2v^2)\} \\
&\quad -\sqrt{4(u^2+v^2)\{\hat{S}\hat{C}v(\hat{s}^2+\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2+\hat{C}^2v^2)\}^2} \\
&\quad \overline{+2uv\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\}\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}-2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\}}\Big]
\end{aligned}
\right\} \quad \text{(II-1)}
$$

where

$$
\left.
\begin{aligned}
\hat{C} &= \cosh\frac{u}{2} \\
\hat{S} &= \sinh\frac{u}{2} \\
\hat{c} &= \cos\frac{v}{2} \\
\hat{s} &= \sin\frac{v}{2}
\end{aligned}
\right\} \quad \text{(II-2)}
$$

and

$$\hat{d} = 2uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+4(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}$$

The components of eigenvector matrix in Equation (5-4a) are written in half angle form as

$$
\left.
\begin{aligned}
n_1 &= 4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{c}^2-\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2-\hat{S}^2v^2)\right\}-Z_1\\
n_2 &= 2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}-2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\\
n_3 &= -4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{s}^2-\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2-\hat{C}^2v^2)\right\}-Z_2\\
n_4 &= 2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}+2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\\
n_5 &= 4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{c}^2-\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2-\hat{S}^2v^2)\right\}+Z_1\\
n_6 &= -4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{s}^2-\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2-\hat{C}^2v^2)\right\}+Z_2\\
d_1 &= \sqrt{\begin{aligned}&\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}-2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\right]^2\\&+\left[4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{c}^2-\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2-\hat{S}^2v^2)\right\}-Z_1\right]^2\end{aligned}}\\
d_2 &= \sqrt{\begin{aligned}&\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}+2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\right]^2\\&+\left[4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{s}^2-\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2-\hat{C}^2v^2)\right\}-Z_2\right]^2\end{aligned}}\\
d_3 &= \sqrt{\begin{aligned}&\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}-2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\right]^2\\&+\left[4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{c}^2-\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2-\hat{S}^2v^2)\right\}+Z_1\right]^2\end{aligned}}\\
d_4 &= \sqrt{\begin{aligned}&\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}+2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right)\right]^2\\&+\left[4(u^2+v^2)\left\{\hat{S}\hat{C}v(\hat{s}^2-\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2-\hat{C}^2v^2)\right\}+Z_2\right]^2\end{aligned}}
\end{aligned}
\right] \quad \text{(II-3)}
$$

67

where

$$
\left.
\begin{aligned}
Z_1 &= 2\sqrt{\begin{aligned}&4(u^2+v^2)^2\left\{\hat{S}\hat{C}v(\hat{c}^2+\hat{s}^2u^2)-\hat{s}\hat{c}u(\hat{C}^2+\hat{S}^2v^2)\right\}^2\\ &+2uv\left\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\right\}\left\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}+2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\right\}\end{aligned}}\\
Z_2 &= 2\sqrt{\begin{aligned}&4(u^2+v^2)^2\left\{\hat{S}\hat{C}v(\hat{s}^2+\hat{c}^2u^2)+\hat{s}\hat{c}u(\hat{S}^2+\hat{C}^2v^2)\right\}^2\\ &+2uv\left\{uv(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+2(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}\right\}\left\{(u^4+v^4)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)-4uv(u^2-v^2)\hat{S}\hat{C}\hat{s}\hat{c}-2(u^4-v^4)(\hat{C}^2-\hat{c}^2)\right\}\end{aligned}}
\end{aligned}
\right]
\tag{II-4}
$$

APPENDIX III

MODE SHAPE CHARACTERISTIC FOR THE VIBRATING BEAM-COLUMN IN HALF ANGLE FORM

$$
\left.
\begin{aligned}
n_2 &= 2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}-2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\\
n_4 &= 2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}+2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\\
n_5 &= 8(u^2+v^2)(\hat{S}\hat{C}\hat{c}^2v-\hat{C}^2\hat{s}\hat{c}u)\\
n_6 &= -8(u^2+v^2)(\hat{S}\hat{C}\hat{s}^2v+\hat{S}^2\hat{s}\hat{c}u)\\
d_3 &= \sqrt{\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}-2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\right]^2+\left[8(u^2+v^2)(\hat{S}\hat{C}\hat{c}^2v-\hat{C}^2\hat{s}\hat{c}u)\right]^2}\\
d_4 &= \sqrt{\left[2uv\left\{-(u^2-v^2)(\hat{C}^2\hat{s}^2-\hat{S}^2\hat{c}^2)+8uv\hat{S}\hat{C}\hat{s}\hat{c}+2(u^2+v^2)(\hat{C}^2-\hat{c}^2)\right\}\right]^2+\left[8(u^2+v^2)(\hat{S}\hat{C}\hat{s}^2v+\hat{S}^2\hat{s}\hat{c}u)\right]^2}
\end{aligned}
\right] \quad \text{(III-1)}
$$

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