OPTIMUM DESIGN OF STATICALLY INDETERMINATE COMPOSITE STRUCTURES

by

Frank F. Lam

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science in Engineering

in the

Civil Engineering

Program

Advisor　　　　　　　　　　　　　　　　　　Date

Dean of the Graduate School　　　　　　　　　　Date

YOUNGSTOWN STATE UNIVERSITY

June, 1976

ABSTRACT

OPTIMUM DESIGN OF STATICALLY INDETERMINATE
COMPOSITE STRUCTURES

Frank F. Lam
Master of Science in Engineering
Youngstown State University, 1976

This study was primarily concerned with a systematic approach for determining an optimum design for an indeterminate composite structure. In this case, an optimum design was taken to mean a minimum in the total structural weight while conforming to the AISC design specifications. Matrix techniques were adopted for solving the reactions, moments and stresses in the indeterminate structure. The cutting plane method was used for converting the resultant nonlinear problem into a linear programming problem. The simplex method, an effecient and rapid algorithm, was then utilized to obtain the optimum solution. Finally, computer programs were presented along with numerous optimum design examples.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. Jack D. Bakos and Dr. Michael K. Householder for their invaluble assistance in the formulation and development of this thesis. It is also a great pleasure to thank Dr. John N. Cernica and Dr. Paul X. Bellini for their assistance in various ways. Thanks also goes to the Youngstown State University Computer Center for their contributions to make this thesis possible.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

THE SYSTEM APPROACH

1-1. Introduction

In order to understand the nature of a problem, the engineer always approached the question with a model or hypotheses. A model can be an object, an event, a process, an operation, a system or a mathematical expression (or mathematical model) that represents the reality of the engineering concept. Based on the problem model, the engineer must develop an analysis and design procedure that will enable him to define the problem and select the best solution. This is the fundamental concept of the system approach.

A complex problem can be handled with success when the system approach is effectively applied. Cost effectiveness is one of the devices for measuring success. Cost effectiveness was applied to this study with the end result to obtained a minimum weight design for a statically indeterminate composite structure.

1-2. Formulating the problem and model

Beginning with the basic concept, i.e., to develop a minimum weight design for a simple system, a simple cantilever beam will be initially considered. A concentrated load P will be placed at the span center as illustrated in Fig. 1-1. The maximum moment is $-PL/2$ and will occur at the fixed end.

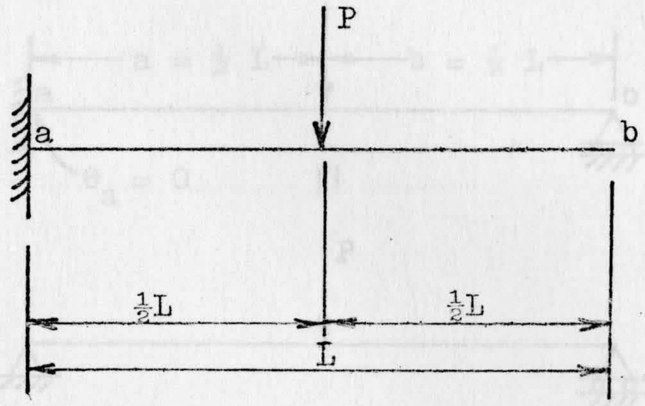


Figure 1-1 Cantilever beam

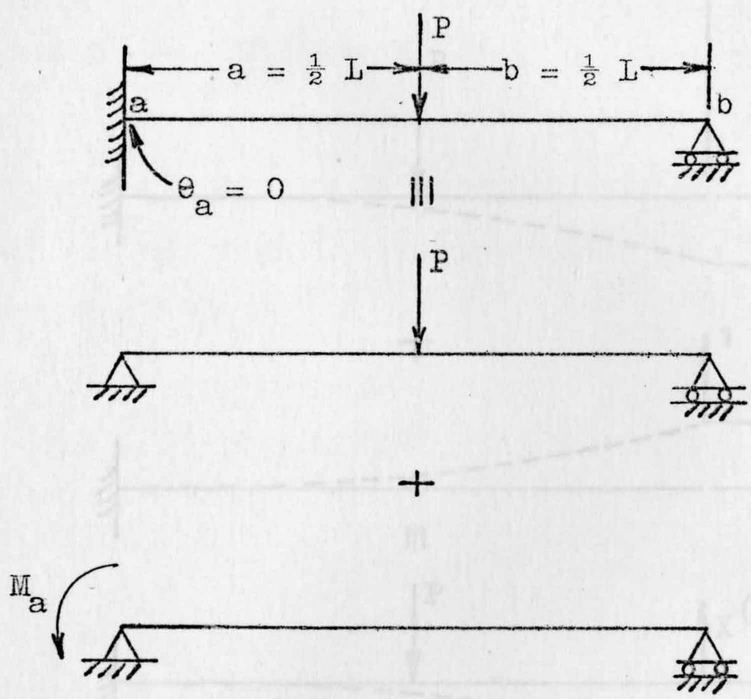
Secondly, consider a simple roller support placed at the free end b of the previous structure model (see Fig. 1-2). This model becomes a statically indeterminate to the first degree, that is to say, there is one redundant reaction. The beam in Fig. 1-2 can be rendered statically determinate by replacing the fixed support with a hinged support. In addition to the original concentrated loading, a redundant moment M_a is then applied to the primary structure, also illustrated in Fig. 1-2. The unknown moment M_a can be solved by applying Eq. 1-1.[4]

$$M = -PL\beta(1-\beta^2)/2 \qquad (1\text{-}1)$$

where

$\beta = b/L$

For b is $\frac{1}{2}$ L and β is $\frac{1}{2}$, the moment M_a at the fixed end due to the load P is -3PL/16.



(M_a such that $\theta_a = 0$)

Figure 1-2 Statically indeterminate structure

Finally, consider a circular tie rod providing the vertical support at point b with the tie rod hinged at point c, as illustrated in Fig. 1-3. By the statics, the moment at the fixed end is PL/2 - RL. Note that R can be solved by the principle of superposition, which denoted by Eq. 1-2 is

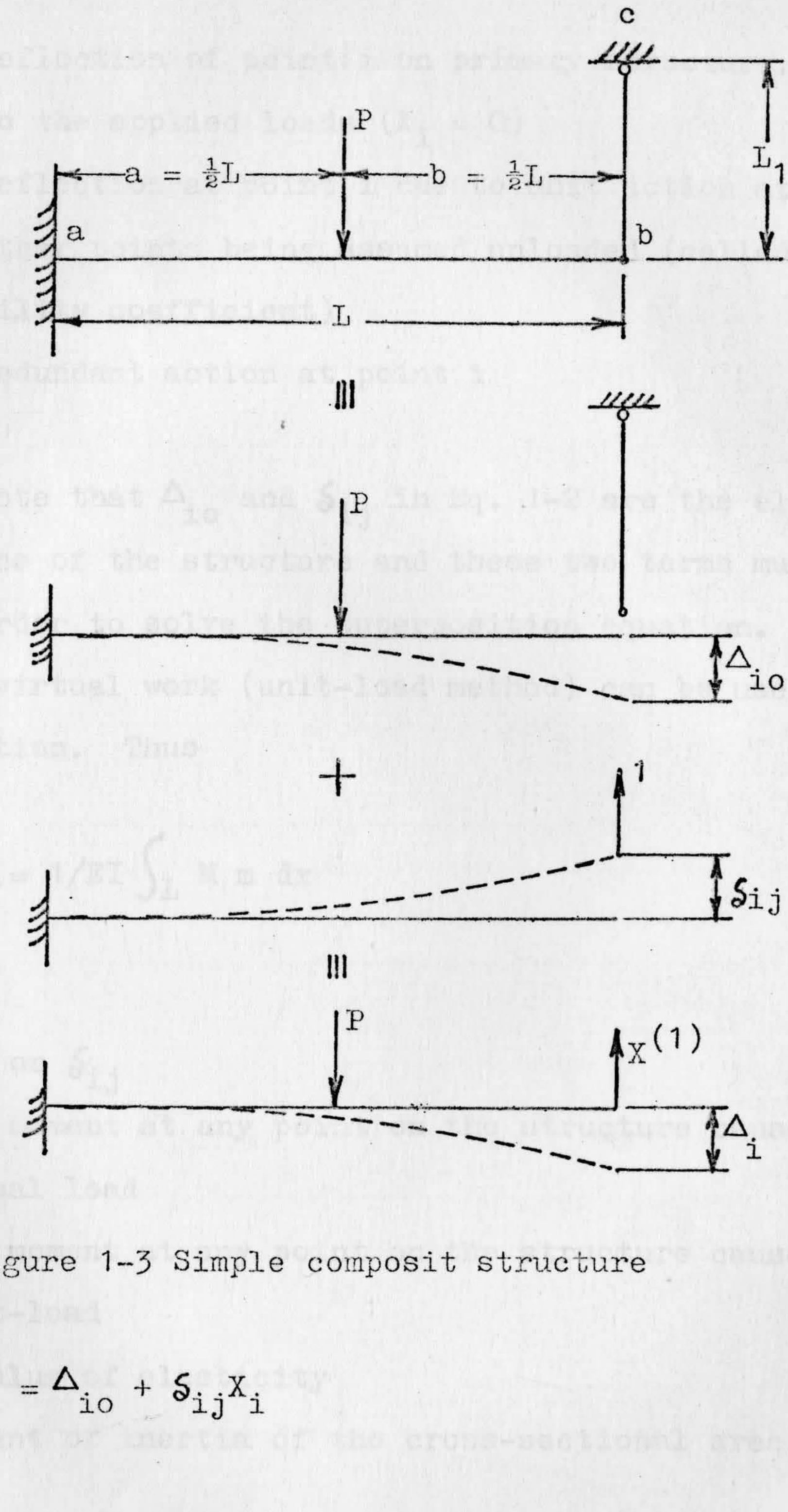


Figure 1-3 Simple composit structure

$$\Delta_i = \Delta_{io} + \delta_{ij} X_i \qquad (1\text{-}2)$$

where

Δ_i = final deflection in direction of release at point i on primary structure

Δ_{io} = deflection of point i on primary structure, due only to the applied loads ($X_i = 0$)

δ_{ij} = deflection at point i due to unit action at j, all other points being assumed unloaded (called flexibility coefficient)

X_i = redundant action at point i

Note that Δ_{io} and δ_{ij} in Eq. 1-2 are the elastic deformations of the structure and these two terms must be found in order to solve the superposition equation. The method of virtual work (unit-load method) can be used for this intention. Thus

$$\Delta = 1/EI \int_L M\, m\, dx \qquad (1\text{-}3)$$

where

$\Delta = \Delta_{io}$ or δ_{ij}

M = the moment at any point on the structure caused by the actual load

m = the moment at any point on the structure caused by the unit-load

E = modulus of elasticity

I = moment of inertia of the cross-sectional area

The values of the product integral $\int_L M\, m\, dx$ can be obtained graphically by matching the appropriate moment diagrams using the table in Appendix C. In this case, Δ_{io} is $-5PL^3/48EI$ and δ_{ij} is $L^3/3EI$. The tie rod support at b is elastic and will yield at k, per unit force. The final deflection Δ_i in the direction of release at point b on primary structure caused by the concentrated load P plus redundant X_i is not 0 but is equal to $-kX_i$ as illustrated in Fig. 1-4. By substitution, X_i is $15PAL^3/48(AL^3 + 3IL_1)$ which is identical to R.

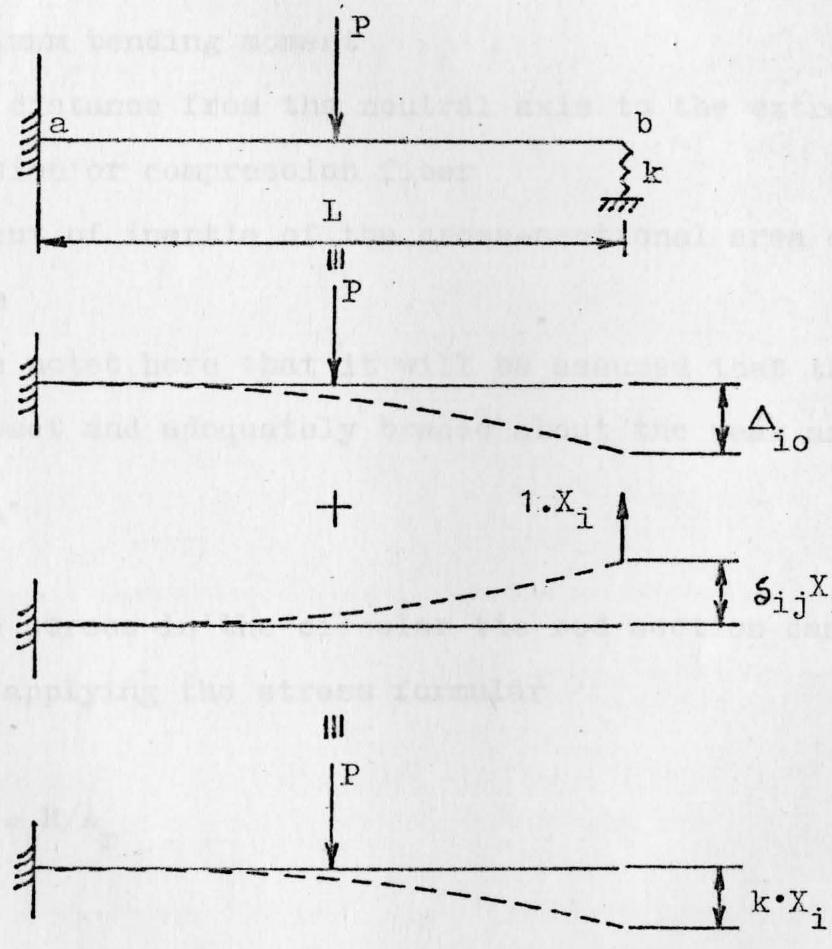


Figure 1-4 Superposition Method

The beam section for design of all models can be solved by applying the flexure formular

$$\sigma_b = M c / I \tag{1-4}$$

Where

σ_b^* = maximum allowable flexural stress in the beam which will be taken as 0.60 F_y (F_y is the yield stress for the beam material)

M = maximum bending moment

c = the distance from the neutral axis to the extreme tension or compression fiber

I = moment of inertia of the cross-sectional area of the beam

* It must be noted here that it will be assumed that the beam is both compact and adequately braced about the weak axis, i.e., $L_b < L_u$.

The stress in the circular tie rod section can be obtained by applying the stress formular

$$\sigma_r = R/A_r \tag{1-5}$$

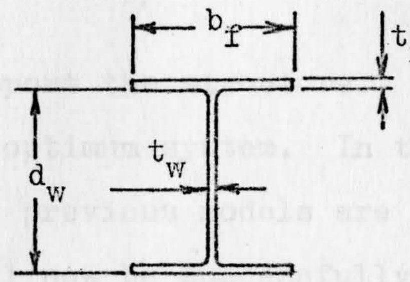
where

σ_r = maximum allowable normal stress which is 0.60 F_y (F_y is the yield stress for the tie rod material)

R = axial load in the tie rod

A_r = the cross-sectional area of the tie rod

The beam section for all models will be H-shaped and the modulus of elasticity E will be taken the same for both the beam and the tie rod. Figure 1-5 shows typical sections for both the beam and the tie rod.



A_r = Area

TIE ROD SECTION

$$c = \tfrac{1}{2}d_w + t_f$$

$$I = \frac{t_w d_w^3}{12} + 2b_f t_f \left(\frac{d_w + t_f}{2}\right)^2$$

$$A_b = 2b_f t_f + d_w t_w$$

$$\text{Vol.} = AL$$

I - BEAM SECTION

Figure 1-5 Typical sections

1-3. Summary

The results of the various system are tabulated in Table 1-1. In the first model, the fixed-end has the largest moment M and thus requires the largest I to resist the distress. In the second model, the simple support requires the the largest tie rod area (such as column or large pier) to

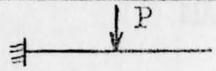
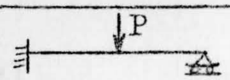
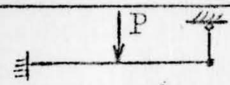
	Model	M	A_r	I_b
1		$- P L / 2$	0	Largest
2		$- 3 P L / 16$	∞	Smallest
3		$P L / 2 - R L$	A	I

Table 1-1 The system results

support the structure. These two models are, therefore, not an optimum system. In the last model the advantages of the two previous models are utilized. The systematic approach will now be successfully developed to obtain a minimum weight design for this system.

CHAPTER II

MATRIX TECHNIQUES

2-1. General

Eq. 1-2 can be written in matrix form as

$$\overline{D} = \overline{S} + \overline{F}\,\overline{X} \tag{2-1}$$

where

$\overline{D}$ = column matrix of the final deflections or the original structure in direction of releases

$\overline{S}$ = column matrix representing the deflections due to the loads on release structure in the direction of the releases

$\overline{F}$ = flexibility matrix of the structure, i.e., the total deflections in the direction of the releases caused by only unit actions in the direction of the releases

$\overline{X}$ = column matrix of redundant reactions

A third order matrix has the general form of

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} S_a \\ S_b \\ S_c \end{bmatrix} + \begin{bmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ba} & f_{bb} & f_{bc} \\ f_{ca} & f_{cb} & f_{cc} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} \tag{2-2}$$

2-2. Applications of matrix techniques

Consider a statically indeterminate composite structure with three redundants, namely M_a, R_b and R_c, as illustrated in Fig. 2-1. The column matrix of redundant reactions $\overline{X}$ is obtained

$$\begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} = \begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix} \tag{2-3}$$

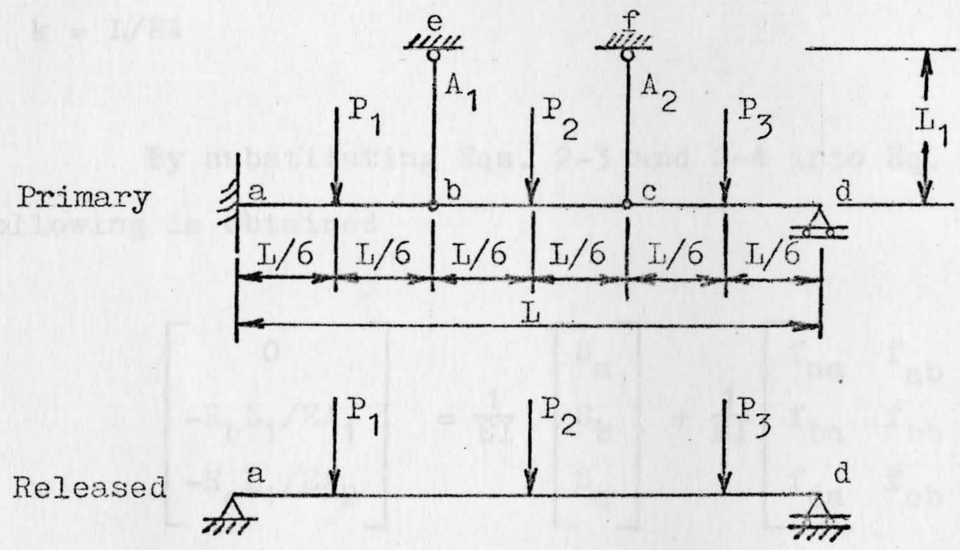


Figure 2-1 Third degree composite structure

The final displacement, i.e., at point a is zero as evidenced by the fixed-end support. Point b and point c are elastic supports and ultimately yield at k_b and k_c, per unit force, respectively. The final deflection at point b in the direction of the release is equal to $-k_bR_b$.* Using the same approach, the final deflection at point c is equal to $-k_cR_c$. The column matrix $\bar{D}$ which represents the final displacements in the direction of the releases is

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} 0 \\ -k_bR_b \\ -k_cR_c \end{bmatrix} = \begin{bmatrix} 0 \\ -R_bL_1/EA_1 \\ -R_cL_1/EA_2 \end{bmatrix}$$

where

$k = L/EA$

By substituting Eqs. 2-3 and 2-4 into Eq. 2-2 the following is obtained

$$\begin{bmatrix} 0 \\ -R_bL_1/EA_1 \\ -R_cL_1/EA_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} S_a \\ S_b \\ S_c \end{bmatrix} + \frac{1}{EI}\begin{bmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ba} & f_{bb} & f_{bc} \\ f_{ca} & f_{cb} & f_{cc} \end{bmatrix}\begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix}$$

multiplied EI for both side and obtained

$$\begin{bmatrix} 0 \\ -R_bL_1I/A_1 \\ -R_cL_1I/A_2 \end{bmatrix} = \begin{bmatrix} S_a \\ S_b \\ S_c \end{bmatrix} + \begin{bmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ba} & f_{bb} & f_{bc} \\ f_{ca} & f_{cb} & f_{cc} \end{bmatrix}\begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix}$$

* The negative sign indicates a displacement in a direction opposite to the direction of the unit actions (see Fig. 2-2).

or

$$\begin{bmatrix} -S_a \\ -S_b \\ -S_c \end{bmatrix} = \begin{bmatrix} 0 \\ R_b L_1 I/A_1 \\ R_c L_1 I/A_2 \end{bmatrix} + \begin{bmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ba} & f_{bb} & f_{bc} \\ f_{ca} & f_{cb} & f_{cc} \end{bmatrix} \begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix} \tag{2-5}$$

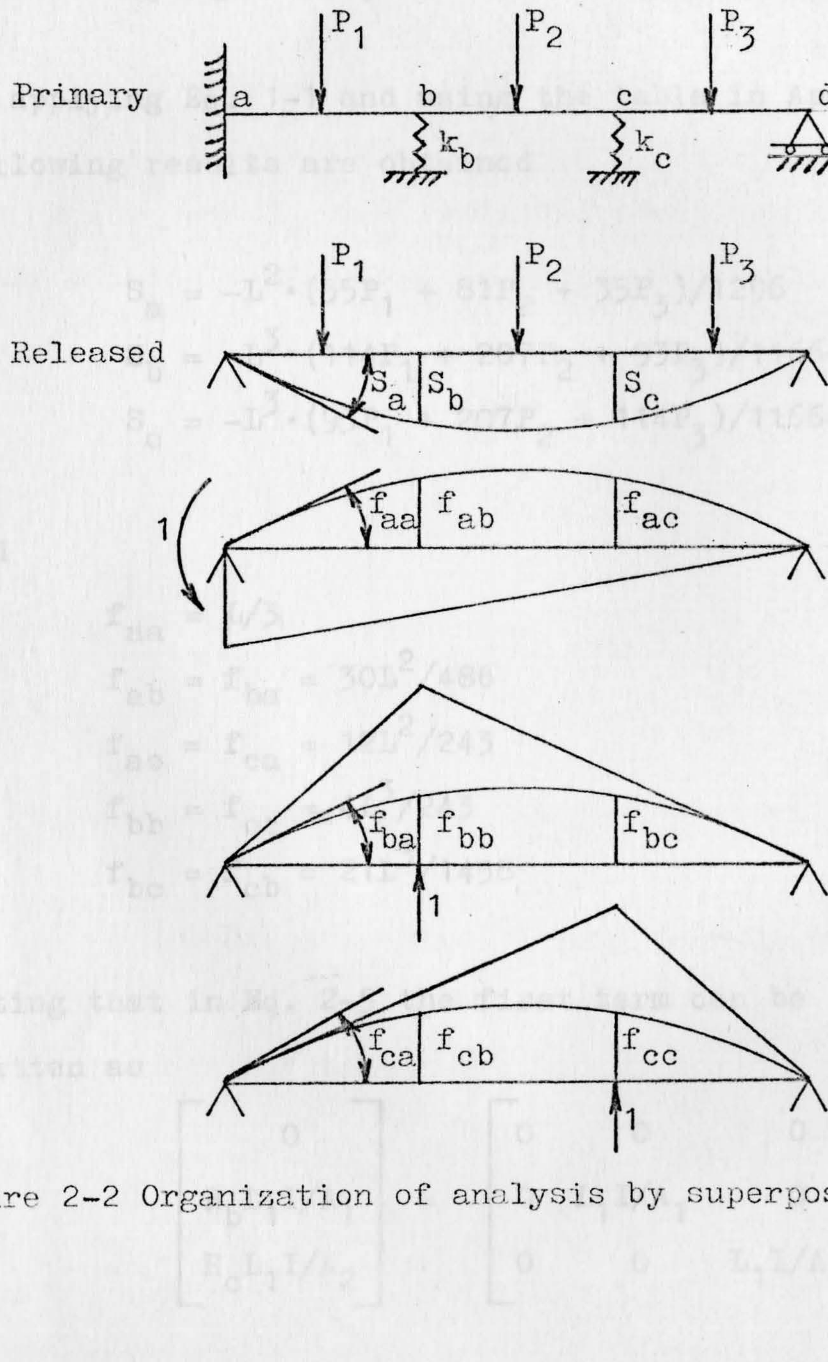


Figure 2-2 Organization of analysis by superposition equations

As illustrated in Fig. 2-2, the principle of superposition, discussed earlier in Chapter I, may be now used to solve the third degree indeterminate structure. In which

$$S_i = \int_L M\, m\, dx \tag{2-6}$$

$$f_{ij} = \int_L m_i\, m_j\, dx \tag{2-7}$$

By applying Eq. 1-1 and using the table in Appendix C, the following results are obtained

$$\begin{aligned} S_a &= -L^2 \cdot (55P_1 + 81P_2 + 35P_3)/1296 \\ S_b &= -L^3 \cdot (114P_1 + 207P_2 + 93P_3)/11664 \\ S_c &= -L^3 \cdot (93P_1 + 207P_2 + 114P_3)/11664 \end{aligned} \tag{2-6a}$$

and

$$\begin{aligned} f_{aa} &= L/3 \\ f_{ab} &= f_{ba} = 30L^2/486 \\ f_{ac} &= f_{ca} = 12L^2/243 \\ f_{bb} &= f_{cc} = 4L^3/243 \\ f_{bc} &= f_{cb} = 21L^3/1458 \end{aligned} \tag{2-7a}$$

Noting that in Eq. 2-5 the first term can be conveniently written as

$$\begin{bmatrix} 0 \\ R_b L_1 I/A_1 \\ R_c L_1 I/A_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L_1 I/A_1 & 0 \\ 0 & 0 & L_1 I/A_2 \end{bmatrix} \begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix} \tag{2-5a}$$

Substituting Eq. 2-5a, Eq. 2-6a and Eq. 2-7a into Eq. 2-5 and factoring out the common terms the following is obtained

$$\begin{bmatrix} M_a \\ R_b \\ R_c \end{bmatrix} = \begin{bmatrix} \frac{L}{3} & \frac{30L^2}{486} & \frac{12L^2}{243} \\ \frac{30L^2}{486} & \frac{4L^3}{243}+\frac{L_1 I}{A_1} & \frac{21L^3}{1458} \\ \frac{12L^2}{243} & \frac{21L^3}{1458} & \frac{4L^3}{243}+\frac{L_1 I}{A_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{L^2(55P_1+81P_2+35P_3)}{1296} \\ \frac{L^3(114P_1+207P_2+93P_3)}{11664} \\ \frac{L^3(93P_1+207P_2+114P_3)}{11664} \end{bmatrix}$$

As illustrated in Fig. 2-3 and using now known redundants M_a, R_b and R_c, the remaining reactions R_a and R_d can be easily solved by applying the simple static techniques.

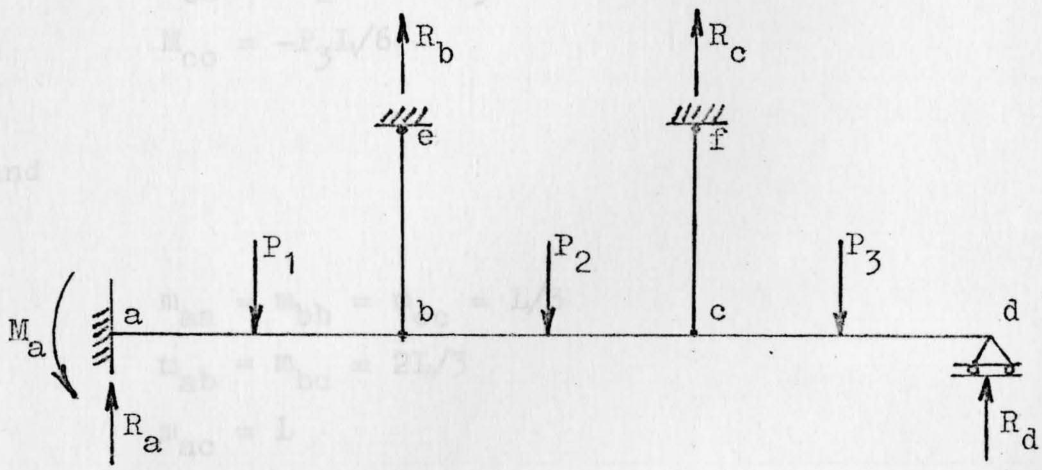


Figure 2-3 Reactions of the composite structure

In order to design the cross-sectional area of the beam, the maximum moment must be obtained. Eq. 2-8 expresses the matrix form for this intention

$$\overline{M}_i = \overline{M}_{io} + \overline{m}_{ij}\overline{R}_i \qquad (2\text{-}8)$$

where

$\bar{M}_i$ = the final moment at point i

$\bar{M}_{io}$ = the moment at point i due only to the applied loads

$\bar{m}_{ij}$ = moment at point i due to reaction at j, all other points being assumed unload

$\bar{R}_i$ = reaction force at i

In this case, i is equal to points a, b and c. For which

$$M_{ao} = -P_1L/6 - P_2L/2 - 5P_3L/6$$
$$M_{bo} = -P_2L/6 - P_3L/2$$
$$M_{co} = -P_3L/6 \qquad (2\text{-}9)$$

and

$$m_{aa} = m_{bb} = m_{cc} = L/3$$
$$m_{ab} = m_{bc} = 2L/3$$
$$m_{ac} = L$$
$$m_{ba} = m_{ca} = m_{cb} = 0 \qquad (2\text{-}10)$$

Substituting Eq. 2-9 and Eq. 2-10 into Eq. 2-8

$$\begin{bmatrix} M_a \\ M_b \\ M_c \end{bmatrix} = \begin{bmatrix} \frac{-P_1L}{6} - \frac{P_2L}{2} - \frac{5P_3L}{6} \\ \frac{-P_2L}{6} - \frac{P_3L}{2} \\ \frac{-P_3L}{6} \end{bmatrix} + \begin{bmatrix} \frac{L}{3} & \frac{2L}{3} & L \\ 0 & \frac{L}{3} & \frac{2L}{3} \\ 0 & 0 & \frac{L}{3} \end{bmatrix} \begin{bmatrix} R_b \\ R_c \\ R_d \end{bmatrix} \qquad (2\text{-}11)$$

By using basic matrix operations, M_a, M_b and M_c in Eq. 2-11 can be easily obtained and the largest of these moments can be used in determining the required moment of inertia for the beam or recalling

$$\sigma_b = 0.60\ F_y = \frac{M_{max.}\ c}{I_b}$$

$$I_{(req'd)} = \frac{(M_a \text{ or } M_b \text{ or } M_c)\cdot c}{0.60\ F_y}$$

For the tie rods

$$\sigma_r = 0.60\ F_y = \frac{R_b}{A_1}\ ;\quad A_1 = \frac{R_b}{0.60\ F_y}$$

and

$$\sigma_r = 0.60\ F_y = \frac{R_c}{A_2}\ ;\quad A_2 = \frac{R_c}{0.60\ F_y}$$

CHAPTER III

CUTTING PLANE METHOD

3-1. General

In general, nonlinear programming problems are quite difficult to solve. There has been relatively little previous work devoted to finding general computational techniques for handling them. However, useful techniques has been developed for certain problems. The cutting plane method applied by F. Moses[10] is very helpful in this matter.

A close examination of the cutting plane method reveals that it actually utilizes piecewise linearization of nonlinear terms through first-order Taylor series transformations. The nonlinear programming problem is then effectively converted into a linear programming problem which is easily solved by the simplex algorithms.

For every optimization problem, linear or nonlinear, two considerations must be formed in order to organize the model such as that discussed in Chapter I. The first consideration is the objective function which describes the design parameters of a given function. An objective function can be a cost function, a weight function (which can be expressed in terms of dimensions), etc., Secondly, consideration must be given to the constraints which controlled the design of the model such as those imposed by the AISC Specifications.

A nonlinear programming problem may be represented by the following general form

$$\text{Minimize } f(x_i) \qquad i = 1 \text{ to } n$$
$$\text{Subject to } g_j(x_i) \leqq 0 \qquad j = 1 \text{ to } m$$
$$L_i \leqq x_i \leqq U_i \tag{3-1}$$

where

$f(x_i)$ = the objective function

$g_j(x_i)$ = the constraints which controlled the design parameters of the problem

x_i = the design parameters which must be positive quantities

L_i = the lower limits of the design parameters

U_i = the upper limits of the design parameters

The linear transformation of the first-order Taylor series for the objective function and constraints are

$$\text{Minimize } \left[f(x_i^o) + \sum_{i=1}^{n} \frac{\partial f(x_i^o)}{\partial x_i} \cdot (x_i - x_i^o) \right] \tag{3-1a}$$

$$\text{Subject to } \left[\sum_{i=1}^{n} \frac{\partial g_j(x_i^o)}{\partial x_i} \cdot (x_i - x_i^o) \right] \leqq 0 \tag{3-1b}$$

The expansion x_i^o in the Taylor series is the solution to the linear programming problem which obtained in the previous iteration. The expansion in the first iteration is the trial design set by the designer.

3-2. Applications of the Cutting Plane Method

The solution of a third degree indeterminate composite structure were discussed in Chapter II, see Fig. 2-3. It will be assumed that the structure has sufficient lateral support and the density of the rolled steel can be taken as 490 lbs/ 1728 in^3. The objective function to be minimized in this problem is the weight function (which expressed in terms of the dimensions of the H-shaped beam and the cross-sectional area of the tie rods) may be represented by the following mathematical expression (mathematical model)

$$f(x_i) = wt = \left[(2b_f t_f + d_w t_w)\cdot L + (A_1 + A_2)\cdot L_1\right] \cdot \frac{490}{1728} \quad (3\text{-}2)$$

where x_i is equal to six parameters as following

$x_1 = b_f$

$x_2 = t_f$

$x_3 = d_w$

$x_4 = t_w$

$x_5 = A_1$

$x_6 = A_2$

All of this six parameters must be non-negative variables. Eq. 3-2 is nonlinear since the expression contains terms consisting of products of the variables, i.e., $x_1 \cdot x_2$, $x_3 \cdot x_4$, etc.. Applying the linearization shown in Eq. 3-1a the following linear equation is obtained

$$f(x_i) = 2t_f^o b_f L + 2b_f^o t_f L + t_w^o d_w L + d_w^o t_w L + A_1 L_1 + A_2 L_1 \quad (3\text{-}2a)$$

Eq. 3-2a is applied only in the solution for the simplex algorithm. The actual weight of the structure is computed from Eq. 3-2.

The constraints which controlled the design variables of the problem are

$$\frac{M \cdot c}{I} \leq \sigma_b \quad \text{where } c = \frac{d_w}{2} + t_f \quad (3\text{-}3)$$

$$I = \frac{t_w d_w^{\,3}}{12} + 2b_f t_f \left(\frac{d_w + t_f}{2}\right)^2$$

$$\frac{R_b}{A_1} \leq \sigma_r \quad (3\text{-}4)$$

$$\frac{R_c}{A_2} \leq \sigma_r \quad (3\text{-}5)$$

From AISC Specification Section 1.5.1.4.1-b, the width-thickness ratio for an unstiffened element such as a projecting compression flange

$$b_f \leq 2t_f \cdot (52.2/\sqrt{F_y}) \quad (3\text{-}6)$$

From AISC Specification Section 1.10.2, the minimum thickness of web

$$d_w \leq \frac{14\times10^3 \cdot t_w}{\sqrt{F_y(16.5+F_y)}} \tag{3-7}$$

From the AISC handbook smallest thickness tabulated for either a flange or web or a rolled section is 0.25 inches. Therefore,

$$t_f \geq 0.25 \tag{3-8}$$

$$t_w \geq 0.25 \tag{3-9}$$

Although not required by the specifications, but it has been common for the good engineering practice to take

$$b_f \geq 10\ t_f \tag{3-10}$$

also

$$d_w \geq d_L \tag{3-11}$$

where d_L is a lower limit on d_w which is provided by the designer to say, meet architectural considerations.

As the expression in Eq. 3-1, let

$$g_1 = \frac{M\cdot(\frac{d_w}{2} + t_f)}{\frac{t_w d_w^3}{12} + 2b_f t_f\cdot(\frac{d_w + t_f}{2})^2} - \sigma_b \leq 0 \tag{3-12}$$

$$g_2 = (R_b/A_1) - \sigma_r \leq 0 \tag{3-13}$$

$$g_3 = (R_c/A_2) - \sigma_r \leq 0 \tag{3-14}$$

$$g_4 = b_f - 2t_f\cdot(52.2/\sqrt{F_y}) \leq 0 \tag{3-15}$$

$$g_5 = d_w - \frac{14x10^3 \cdot t_w}{\sqrt{F_y(16.5+F_y)}} \leqq 0 \qquad (3\text{-}16)$$

$$g_6 = -t_f + 0.25 \leqq 0 \qquad (3\text{-}17)$$

$$g_7 = -t_w + 0.25 \leqq 0 \qquad (3\text{-}18)$$

$$g_8 = -b_f + 10t_f \leqq 0 \qquad (3\text{-}19)$$

$$g_9 = -d_w + d_L \leqq 0 \qquad (3\text{-}20)$$

Note that in Eq. 3-17 to Eq. 3-20, some design parameters are negative which seem to violate the requirements stated in Eq. 3-1. However, for the time being the linear transformation will be continued and an explaination for the negative values will be provided later.

Apply the linearization shown in Eq. 3-1b, the following linear equations for the constraints are obtained

$$g_1'(b_f^o) = \frac{-M \cdot c^o \cdot \left[2t_f^o\left(\frac{d_w^o + t_f^o}{2}\right)^2\right]}{(I^o)^2} \cdot (b_f - b_f^o)$$

$$g_1'(t_f^o) = \frac{M \cdot \left\{\frac{1}{I^o} - c^o \cdot \left[2b_f^o\left(\frac{d_w^o + t_f^o}{2}\right)^2 + 2b_f^o t_f^o\left(\frac{d_w^o + t_f^o}{2}\right)\right]\right\}}{(I^o)^2} \cdot (d_w - d_w^o)$$

$$g_1'(d_w^o) = \frac{M \cdot \left\{\frac{1}{2I^o} - c^o \cdot \left[2b_f^o t_f^o\left(\frac{d_w^o + t_f^o}{2}\right) + \frac{t_w^o \, d_w^{o\,2}}{4}\right]\right\}}{(I^o)^2} \cdot (d_w - d_w^o)$$

$$g_1'(t_w^o) = \frac{-M \cdot c^o \cdot \left(\frac{d_w^{o\,3}}{12}\right)}{(I^o)^2} \cdot (t_w - t_w^o)$$

Therefore,

$$g_1' = g_1'(b_f^o) + g_1'(t_f^o) + g_1'(d_w^o) + g_1'(t_w^o) = 0$$

and where

$$c^o = \frac{d_w^o}{2} + t_f^o$$

$$I^o = 2b_f^o t_f^o \cdot \left(\frac{d_w^o + t_f^o}{2}\right)^2 + \frac{t_w^o \, d_w^{o\,3}}{12}$$

Taking the next constraint functions g_2 through g_9 subject to Eq. 3-1b

$$g_2' = (-R_b/A_1^{o2}) \cdot (A_1 - A_1^o) = 0$$

$$g_3' = (-R_c/A_2^{o2}) \cdot (A_2 - A_2^o) = 0$$

$$g_4'(b_f^o) = 1 \cdot (b_f - b_f^o)$$

$$g_4'(t_f^o) = (-104.4/\sqrt{F_y}) \cdot (t_f - t_f^o)$$

$$g_4' = g_4'(b_f^o) + g_4'(t_f^o) = 0$$

$$g_5'(d_w^o) = 1 \cdot (d_w - d_w^o)$$

$$g_5'(t_w^o) = \frac{-14x10^3}{\sqrt{F_y(16.5 + F_y)}} \cdot (t_w - t_w^o)$$

$$g_5' = g_5'(d_w^o) + g_5'(t_w^o) = 0$$

$$g_6' = -1 \cdot (t_f - t_f^o) = 0$$

$$g_7' = -1 \cdot (t_w - t_w^o) = 0$$

$$g'_8(b_f^o) = -1\cdot(b_f - b_f^o)$$

$$g'_8(t_f^o) = 10\cdot(t_f - t_f^o)$$

$$g'_8 = g'_8(b_f^o) + g'_8(t_f^o) = 0$$

$$g'_9 = -1\cdot(d_w - d_w^o) = 0 \qquad (3\text{-}21)$$

3-3. Summary

After transforming the nonlinear problem into the linear programming problem (see linear Eqs. 3-2a and 3-21), the optimization is ready for the simplex algorithms. Each iteration of the simplex method contains the linear expressions of the objective function and constraints, i.e., Eq. 3-1a and Eq. 3-1b. The following steps will be followed in the solution(7):

1. The expansion x_i^o, in the Taylor series, in the first iterations will be the trial selections set by the designer which, in general, satisfy the boundary conditions $L_i \leqq x_i \leqq U_i$ (see Eqs. 3-3 to 3-11).
2. The linear programming problem is then solved by the simplex method using the initial assumed values, x_i^o, to determine a new set of design parameters, x_i. The solution x_i will satisfy Eq. 3-1a but may not satisfy, in general, the constraints equation Eq. 3-1b.
3. If any of the x_i values violate the constraint equations,

the variables will be uniformly adjusted until all violations are eliminated.

4. Steps 2 and 3 will be repeated until a specified accuracy is achieved. The approximate "optimal" solution will be obtained when successive designs approach a constant value of the objective function.

CHAPTER IV

THE SIMPLEX METHOD

4-1. General

Linear programming problems are characterized by a linear objective function subject to a set of linear constraints. An algorithm is nothing more than a set of computations which can be performed repeatedly following a cyclic pattern. Because of its repetitive nature it can be readily programmed and proceeding using electronic computation. The simplex algorithm provides the most general and an effecient method for solving the linear programming problems.

A linear programming problem may be represented by the following general form

$$f(x_i) = \sum_{i=1}^{n} c_i x_i \qquad i = 1 \text{ to } n$$

subject to

$$a_{ji} x_i \leqq b_j \qquad j = 1 \text{ to } m$$

and $$L_i \leqq x_i \leqq U_i \qquad (4\text{-}1)$$

and where (with aspect to the problem at hand)

$f(x_i)$ = the linear objective function

$c_i = (c_1, c_2, \ldots, c_n)$ is the weight coefficients of the design parameter and is a row vector

$x_i = (x_1, x_2, \ldots, x_n)$ is the structural design parameters and is a column vector

$a_{ji} = (a_{11}, a_{12}, \ldots, a_{1n}; a_{21}, a_{22}, \ldots, a_{2n}; \ldots\ldots\ldots; a_{m1}, a_{m2}, \ldots, a_{mn})$ is the structural form coefficients; and is a rectangular matrix

$b_j = (b_1, b_2, \ldots, b_m)$ is the stipulations and is a column vector

L_i and U_i are the lower and upper limits of the design parameters as specified in Chapter III.

In most cases, the number of constraints m are not equal to the number of design variables n. The structural form (or proportions) coefficients a_{ji} are, therefore, generally in rectangular form. The values of x_i and b_j are conventionally non-negative and c_i and a_{ji} are unrestricted in sign.

The method of the artificial basis(3) is a satisfactory way to start the simplex process. This procedure is known as an advanced start(8), and is designed to reduce the number of iterations required. It also determines whether or not the problem has any feasible solutions. As the name implies, the artifical basis does not possess any physical meaning and must be eliminated as soon as possible. This can be accomplished by introducing artifical variables into the objective function with large positive coefficients (minimization case). The general form is as follows

$$f(x_i) = \sum_{i=1}^{n} c_i n_i + M \sum_{j=1}^{m} x_{n+j} \qquad (4\text{-}2)$$

subject to

$$a_{ji} x_i + 1 \cdot x_{n+j} \leqq b_j \qquad (4\text{-}3)$$

where M is large positive coefficients and $1 \cdot x_{n+j}$ is the unit vector basis or an artificial basis.

There are a few general rules for operating the simplex algorithm

1. The formulation expressed as the objective function in Eq. 4-1 is set up to maximize rather than minimize. Minimizing the linear function can be accomplished by minimizing its negative or

$$\text{minimize } f(x_i) = \text{minimize} \left[- \sum_{i=1}^{n} c_i x_i \right] \qquad (4\text{-}4)$$

2. The artificial basis method is used to create an initial solution. This basis must then be given large negative coefficients in the objective function as follows

$$\text{minimize } f(x_i) = - \sum_{i=1}^{n} c_i x_i - M \sum_{j=1}^{m} x_{n+j} \qquad (4\text{-}5)$$

3. Determine the coefficients of the objective function and choose the variables which yields the largest value as the pivot column in the simplex tableau.
4. Compute the ratios of the stipulations (b_j) in the right-hand column of the simplex tableau to the corresponding coefficients in the pivot column and choose the smallest positive, nonzero ratio as the pivot row.

The simplex computational procedure is then followed (utilizing tableaus) until the coefficients in the objective function are all reduced to zero or negative values, which indicates that an optimal solution has been obtained.

4-2. Application of simplex method

As indicated in Chapter III, Eq. 3-17 to Eq. 3-20 are in violation of the requirements specified by Eq. 3-1, i.e., the following design parameters have the negative values

$$-t_f = -0.25$$
$$-t_w = -0.25$$
$$-b_f = -10 \cdot t_f$$
$$-d_w = -d_L$$

This situation can be resolved by adding a positive constant which is large enough to form a non-negative value on the right-hand side (b_j values). This procedure takes the following form

$$TF - t_f = TF - 0.25 = \text{positive value}$$

$$TW - t_w = TW - 0.25 = \text{positive value}$$

$$BF - b_f = BF - 10t_f = \text{positive value}$$

$$DW - d_w = DW - d_L = \text{positive value}$$

Where TF, TW, BF, DW are positive constants, more generally named C_i. Recalled that t_f, t_w, b_f, d_w are x_i varialbes. Letting the left-hand side be XX_i the following general form is obtained

$$XX_i = C_i - x_i \qquad (4\text{-}6)$$

In order to obtain consistent solution for the whole problem, XX_i has to be transformed in all the design parameters used in the objective function and in the constraint equations. The linear programming problem then yields the new variable XX_i. After obtaining the optimal solution from the simplex algorithm the XX_i values can be deducted from C_i to get x_i, the final desired optimal solution.

It is interesting the note from Eq. 4-6 that because of the transformation to positive values, the b_j values on the right-hand side of the constraint equations affords the designer the convenient capability of setting the desired upper limit values on any designed parameter.

Applying Eq. 4-6 and Eq. 3-1a to objective function Eq. 3-2, reveals that the deviative of the positive values C_i are zero and hence, the term $-x_i$ is again introduced as a negative result for Eq. 3-2a. In equation form this becomes

$$f(x_i) = -2Lt_f^o\, b_f - 2Lb_f^o\, t_f - Lt_w^o\, d_w - Ld_w^o\, t_w - L_1\, A_1 - L_1\, A2 \tag{4-7}$$

By the same procedure, applying Eq. 4-6 and Eq. 3-1b to the constraints equations, Eq. 3-12 to Eq. 3-20, the same negative result is obtained for Eq. 3-21. Notice that Eq. 3-21 contains the following structural form coefficients a_{ji}

$$a_{11} = \frac{M \cdot c^o \cdot \left[2t_f^o\left(\frac{d_w^o + t_f^o}{2}\right)\right]}{(I^o)^2}$$

$$a_{12} = \frac{-M\left\{\frac{1}{I^o} - c^o \cdot \left[2b_f^o\left(\frac{d_w^o + t_f^o}{2}\right)^2 + 2b_f^o t_f^o\left(\frac{d_w^o + t_f^o}{2}\right)\right]\right\}}{(I^o)^2}$$

$$a_{13} = \frac{-M\left\{\frac{1}{2I^o} - c^o \cdot \left[2b_f^o t_f^o\left(\frac{d_w^o + t_f^o}{2}\right) + \frac{t_w^o {d_w^o}^2}{4}\right]\right\}}{(I^o)^2}$$

$$a_{14} = \frac{M \cdot c^o\left(\frac{{d_w^o}^3}{12}\right)}{(I^o)^2}$$

$$a_{15} = 0$$

$$a_{16} = 0$$

$$a_{21} = 0$$

$a_{22} = 0$

$a_{23} = 0$

$a_{24} = 0$

$a_{25} = R_b/A_1^{o\,2}$

$a_{26} = 0$

$a_{31} = 0$

$a_{32} = 0$

$a_{33} = 0$

$a_{34} = 0$

$a_{35} = 0$

$a_{36} = R_c/A_2^{o\,2}$

$a_{41} = -1$

$a_{42} = 104.4/\sqrt{F_y}$

$a_{43} = 0$

$a_{44} = 0$

$a_{45} = 0$

$a_{46} = 0$

$a_{51} = 0$

$a_{52} = 0$

$a_{53} = -1$

$a_{54} = 14x10^3/\sqrt{F_y(16.5 + F_y)}$

$a_{55} = 0$

$a_{56} = 0$

$$a_{61} = 0$$
$$a_{62} = 1$$
$$a_{63} = 0$$
$$a_{64} = 0$$
$$a_{65} = 0$$
$$a_{66} = 0$$

$$a_{71} = 0$$
$$a_{72} = 0$$
$$a_{73} = 0$$
$$a_{74} = 1$$
$$a_{75} = 0$$
$$a_{76} = 0$$

$$a_{81} = 1$$
$$a_{82} = -10$$
$$a_{83} = 0$$
$$a_{84} = 0$$
$$a_{85} = 0$$
$$a_{86} = 0$$

$$a_{91} = 0$$
$$a_{92} = 0$$
$$a_{93} = 1$$
$$a_{94} = 0$$
$$a_{95} = 0$$
$$a_{96} = 0 \qquad (4\text{-}8)$$

The stipulations b_j appearing in the linear constraint equations are obtained from Eq. 3-3 to Eq. 3-11 plus the remainders from Eq. 3-21. In equation form

$$b_1 = \sigma_b - \frac{M \cdot c^o}{I^o} + a_{11}(b_f^o - BF) + a_{12}(t_f^o - TF) + a_{13}(d_w^o - DW) + a_{14}(t_w^o - TW)$$

$$b_2 = \sigma_r - \frac{R_b}{A_1^o} + a_{25}(A_1^o - A1)$$

$$b_3 = \sigma_r - \frac{R_c}{A_2^o} + a_{36}(A_2 - A2)$$

$$b_4 = a_{41}(-BF) + a_{42}(-TF)$$

$$b_5 = a_{53}(-DW) + a_{54}(-TW)$$

$$b_6 = TF - 0.25$$

$$b_7 = TW - 0.25$$

$$b_8 = BF - 10 \cdot TF$$

$$b_9 = DW - d_L \qquad (4-9)$$

After all this necessity linearization, the simplex algorith can be now applied. Eq. 4-5 will be used to obtain the objective function and will be listed in the last row of Tableau 4-1. The general form for iteration of the simplex algorithm is presented in Tableau 4-1. The computer program will be utilized to solve the problem.

BF	TF	DW	TW	A1	A2	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	
a_{11}	a_{12}	a_{13}	a_{14}	0	0	1	0	0	0	0	0	0	0	0	b_1
0	0	0	0	a_{25}	0	0	1	0	0	0	0	0	0	0	b_2
0	0	0	0	0	a_{36}	0	0	1	0	0	0	0	0	0	b_3
a_{41}	a_{42}	0	0	0	0	0	0	0	1	0	0	0	0	0	b_4
0	0	a_{53}	a_{54}	0	0	0	0	0	0	1	0	0	0	0	b_5
0	a_{62}	0	0	0	0	0	0	0	0	0	1	0	0	0	b_6
0	0	0	a_{74}	0	0	0	0	0	0	0	0	1	0	0	b_7
a_{81}	a_{82}	0	0	0	0	0	0	0	0	0	0	0	1	0	b_8
0	0	a_{93}	0	0	0	0	0	0	0	0	0	0	0	1	b_9
$2Lt^{o}_{f}$	$2Lb^{o}_{f}$	Lt^{o}_{w}	Ld^{o}_{w}	L_1	L_1	-M	-M	-M	-M	-M	-M	-M	-M	-M	

Tableau 4-1 General form for simplex method

(Notice that x_7 to x_{15} are the artificial bases from Eq. 4-3)

CHAPTER V

INTERACTIVE COMPUTER PROGRAM

5-1. Program description

The program presented herein has been written in the BASIC language. A knowledge of the BASIC language is, therefore, useful, but not necessary to design using the minimum weight criteria for statically indeterminate composite structures. The user can simply interact with the computer by supplying the input data at the terminal as "called for" by the computer.

The main program handles all input and output. The "INDETERMINATE ANALYSIS" subroutine program carries out an indeterminate analysis using the flexibility method for the maximum beam moment, reactions and stresses. The "COEFFICIENT" subroutine program generates the coefficients for the linear programming problem. Finally, the "SIMPLEX ALGORITHM" subroutine program is the simplex routine calculation which was used to solve the linear programming as formulated in Eq. 4-1.

The dimensions for the H-shaped beam are in inches and the tie-rod areas are in inches square. The concentrated loads on the structure are in kips and E and F_y are in kips per square inch. The design weight of the whole structure is in pounds.

The designer interacts with the computer by inputting the following input-data: first, the load magnitudes for P_1, P_2 and P_3; the length of the span L; the length of the tie rods L_1 and the modulus of elasticity E of the steel. Secondly, the yield stress of steel for the beam and tie rods must be inputted. Finally entered are the upper limit dimensions of the H-shaped beam, namely, BF, TF, DW and TW; the upper limit of the tie-rod areas A_1 and A_2 and the lower limit of the depth of the beam d_L.

A flow diagram has been constructed to better illustrate the significant feature of the iteration routine of the computer program (see Fig. 5-1).

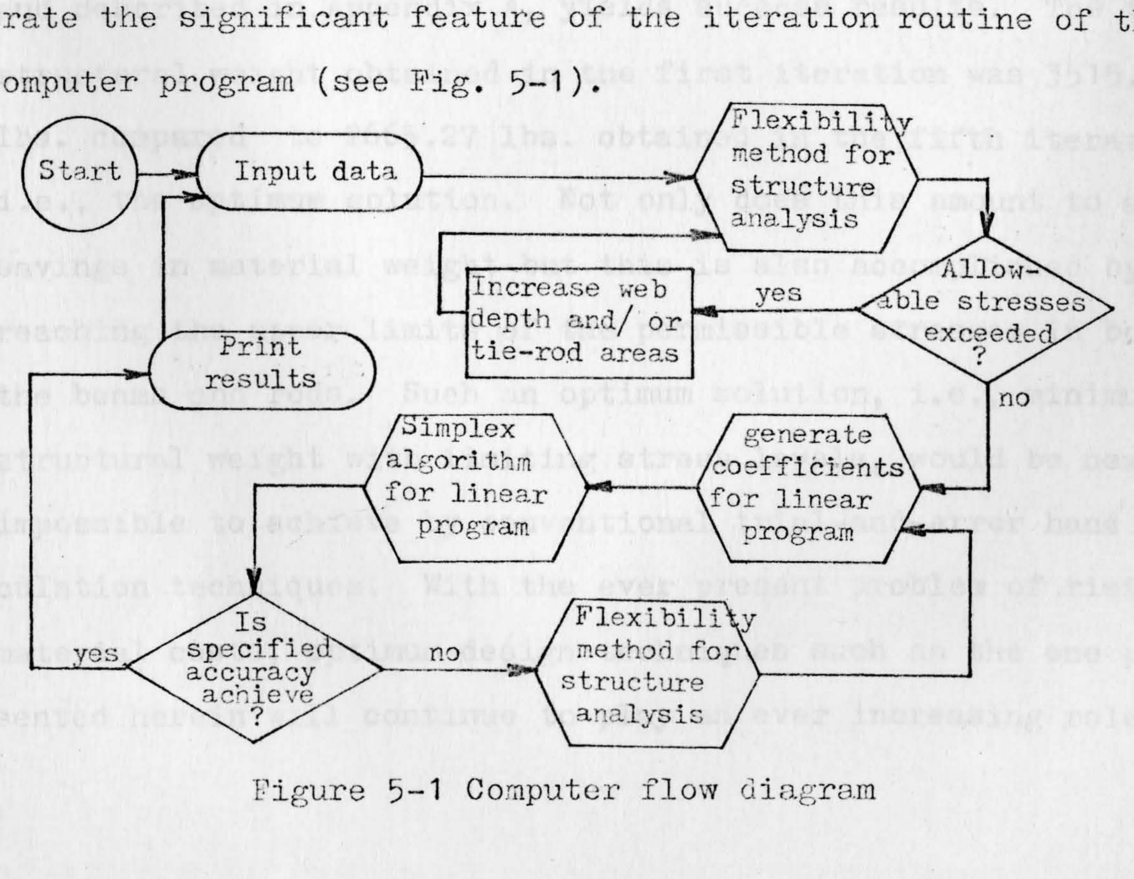


Figure 5-1 Computer flow diagram

5-2. Summary and unknown

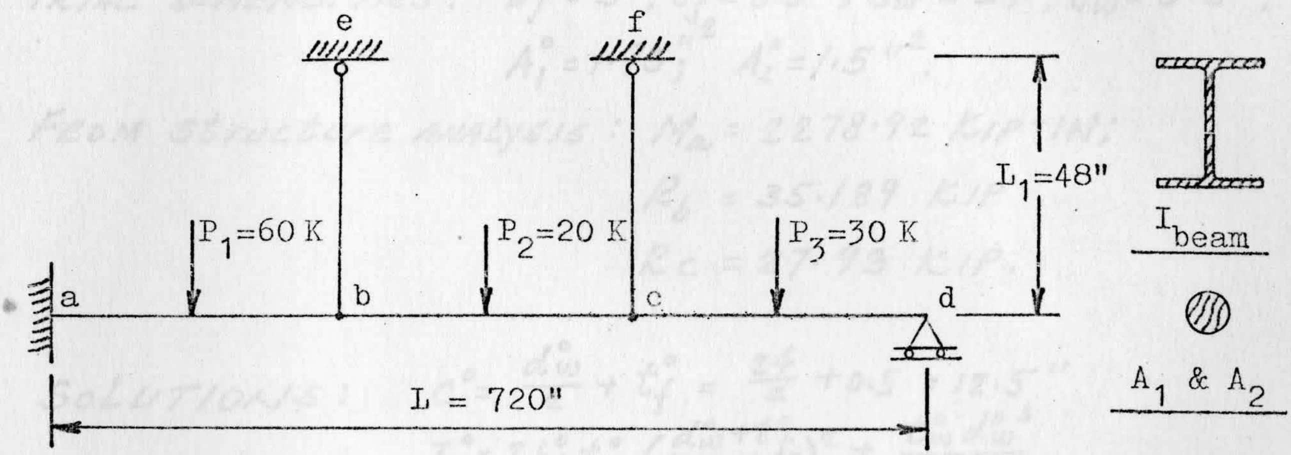
A systematic approach has been developed for obtaining an optimum design for an indeterminate composite structure. The design parameters in the objective function and the behavior variables of the constraint equations are treated as unknowm quantities. A trial design is chosen and analyzed. The nonlinear equations, both the objective function and the constraints, are then approximated by the first-order Taylor series linearization. Using the simplex method, a solution is obtained that minimizes the objective function subject to the constraints.

The iterative approach used in the computer program and described in Appendix A, yields success results. The total structural weight obtained in the first iteration was 3515.07 lbs. compared to 2665.27 lbs. obtained in the fifth iteration, i.e., the optimum solution. Not only does this amount to a 24% savings in material weight but this is also accomplished by reaching the upper limits of the permissible stresses in both the beams and rods. Such an optimum solution, i.e., minimum structural weight with limiting stress levels, would be next to impossible to achieve by conventional trial-and-error hand calculation techniques. With the ever present problem of rising material costs, optimum design techniques such as the one presented herein will continue to play an ever increasing role in structural design techniques.

APPENDIX A

Example: Optimum Design of Statically Indeterminate Composite Structures by Computer Solution

Example Problem



Given: As shown in diagram above; $E = 29x10^3$ ksi and $F_y = 36$ ksi

Find:
1. Solve for reactions using flexibility method
2. Maximum beam moment is determined along with tie rod force in terms of I_{beam}, A_1 and A_2
3. Utilize simplex method to find $I_{min.}$, $A_{1\ min.}$ and $A_{2\ min.}$ for beam geometry and loading shown

Example Problem (continued).

Example calculations:

GIVEN: $P_1 = 60$ KIPS, $P_2 = 20$ KIPS, $P_3 = 30$ KIPS; $L = 720''$; $L_1 = 48''$

$E = 29 \times 10^3$ KSI; $F_y = 36$ KSI.

$BF = 17''$, $TF = 1.7''$, $DW = 24''$, $TW = 0.5''$, $D_L = 10''$

$A1 = 3''^2$; $A2 = 3''^2$;

TRIAL DIMENSIONS: $b_f^o = 5''$, $t_f^o = 0.5''$, $d_w^o = 24''$, $t_w^o = 0.5''$;

$A_1^o = 1.75''^2$; $A_2^o = 1.5''^2$.

FROM structure ANALYSIS: $M_a = 2278.92$ KIP-IN;

$R_b = 35.189$ KIP

$R_c = 27.93$ KIP.

SOLUTIONS: $C^o = \frac{d_w^o}{2} + t_f^o = \frac{24}{2} + 0.5 = 12.5''$

$$I^o = 2 b_f^o t_f^o \left(\frac{d_w^o + t_f^o}{2}\right)^2 + \frac{t_w^o d_w^{o\,3}}{12}$$

$$= 2 \times 5 \times 0.5 (12.25)^2 + 576.0$$

$$= 1326.31''^4$$

(I) OBJECTIVE FUNCTION - EQ. 4-7

$C(1) = -2 L t_f^o = -720$

$C(2) = -2 L b_f^o = -7200$

$C(3) = -L t_w^o = -360$

$C(4) = -L d_w^o = -17280$

$C(5) = -L_1 = -48$

$C(6) = -L_1 = -48$

(Ia) By EQ. 4-4 [FOR MIN. $f(X_i)$ = MIN. $-f(X_i)$],

THEREFORE

$C(1) = 720$

$C(2) = 7200$

$C(3) = 360$

$C(4) = 17280$

$C(5) = 48$

$C(6) = 48$

(II) A-COEF. - EQ. 4-8

$a_{11} = 2.43$	$a_{21} = 0$	$a_{31} = 0$
$a_{12} = 25.29$	$a_{22} = 0$	$a_{32} = 0$
$a_{13} = 2.158$	$a_{23} = 0$	$a_{33} = 0$
$a_{14} = 18.655$	$a_{24} = 0$	$a_{34} = 0$
$a_{15} = 0$	$a_{25} = 11.49$	$a_{35} = 0$
$a_{16} = 0$	$a_{26} = 0$	$a_{36} = 12.41$

$a_{41} = -1$	$a_{51} = 0$	$a_{61} = 0$
$a_{42} = 17.4$	$a_{52} = 0$	$a_{62} = 1$
$a_{43} = 0$	$a_{53} = -1$	$a_{63} = 0$
$a_{44} = 0$	$a_{54} = 322.03$	$a_{64} = 0$
$a_{45} = 0$	$a_{55} = 0$	$a_{65} = 0$
$a_{46} = 0$	$a_{56} = 0$	$a_{66} = 0$

$a_{71} = 0$ $a_{81} = 1$ $a_{91} = 0$

$a_{72} = 0$ $a_{82} = -10$ $a_{92} = 0$

$a_{73} = 0$ $a_{83} = 0$ $a_{93} = 1$

$a_{74} = 1$ $a_{84} = 0$ $a_{94} = 0$

$a_{75} = 0$ $a_{85} = 0$ $a_{95} = 0$

$a_{76} = 0$ $a_{86} = 0$ $a_{96} = 0$

⟨III⟩ B-COEF. -EQ. 4-9

$b_1 = 59.59$

$b_2 = 15.85$

$b_3 = 21.6$

$b_4 = 12.58$

$b_5 = 137.02$

$b_6 = 1.45$

$b_7 = 0.25$

$b_8 = 0$

$b_9 = 14$

⟨IV⟩ FROM CALCULATIONS OF A, B AND C COEF. AND USING TABLEAU 4-1.

BF	TF	DW	TW	A1	A2	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}		RATIOS COMPUTE	
2.43	25.29	2.158	18.66	0	0	1	0	0	0	0	0	0	0	0	59.59	$\frac{59.59}{18.66}$	=3.19
0	0	0	0	11.49	0	0	1	0	0	0	0	0	0	0	15.85		
0	0	0	0	0	12.41	0	0	1	0	0	0	0	0	0	21.6		
-1	17.4	0	0	0	0	0	0	0	1	0	0	0	0	0	12.58		
0	0	-1	322.03	0	0	0	0	0	0	1	0	0	0	0	137.02	$\frac{137.02}{322.03}$	=0.43
0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1.45		
0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0.25	0.25	=0.25 PIVOT ROW
1	-10	0	0	0	0	0	0	0	0	0	0	0	1	0	0		
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	14		
720	7200	360	17280	48	48	-M	-M	-M	-M	-M	-M	-M	-M	-M			

PIVOT COL. (TW)

INITIAL STEP

X_7 TO X_{15} WILL BE ELIMINATED BY THE LARGE -M VALUES ∴ THEY WILL BE CANCELLED IN THE FOLLOWING TABLEAUS.

BF	TF	DW	TW	A1	A2		RATIO COMPUTATION
2.43	25.29	2.158	0	0	0	54.93	54.93/25.29 = 2.17
0	0	0	0	11.49	0	15.85	
0	0	0	0	0	12.41	21.6	
-1	17.4	0	0	0	0	12.58	12.58/17.4 = 0.72
0	0	-1	0	0	0	56.51	
0	1	0	0	0	0	1.45	= 1.45
0	0	0	1	0	0	0.25	
1	-10	0	0	0	0	0	
0	0	1	0	0	0	14	
720	7200	360	0	48	48		

← PIVOT ROW (row 4)

↑ PIVOT COL. (TF)

2 nd STEP

BF	TF	DW	TW	A1	A2		RATIO COMPUTATION
3.88	0	2.158	0	0	0	36.72	36.72/3.88 = 9.46 ← PIVOT ROW
0	0	0	0	11.49	0	15.85	
0	0	0	0	0	12.41	21.6	
-0.058	1	0	0	0	0	0.72	—
0	0	-1	0	0	0	56.51	
0.058	0	0	0	0	0	0.73	0.73/0.058 = 12.7
0	0	0	1	0	0	0.25	
1.58	0	0	0	0	0	0	
0	0	1	0	0	0	14	
4860	0	360	0	48	48		

↑ PIVOT COL. (BF)

3rd STEP

BF	TF	DW	TW	A1	A2		RATIO COMPUTATION
1	0	0.56	0	0	0	9.46	
0	0	0	0	11.49	0	15.85	15.85/11.49 = 1.38 ← PIVOT ROW
0	0	0	0	0	12.41	21.6	
0	1	0	0	0	0	1.26	
0	0	-1	0	0	0	56.51	
0	0	0	0	0	0	0.19	
0	0	0	1	0	0	0.25	
0	0	0	0	0	0	0	
0	0	1	0	0	0	14	
0	0	-2361.6	0	48	48		

↑ PIVOT COL. (A1)

4 th STEP

BF	TF	DW	TW	A1	A2		RATIO COMPUTATION
1	0	0.56	0	0	0	9.46	
0	0	0	0	1	0	1.38	
0	0	0	0	0	12.41	21.6	21.6/12.41=1.74
0	1	0	0	0	0	1.26	
0	0	-1	0	0	0	56.51	
0	0	0	0	0	0	0.19	
0	0	0	1	0	0	0.25	
0	0	0	0	0	0	0	
0	0	1	0	0	0	14	
0	0	-2361.6	0	0	48		

PIVOT ROW ←

↑ PIVOT COL.

5 th STEP

BF	TF	DW	TW	A1	A2		RATIO COMPUTATION
1	0	0.56	0	0	0	9.46	
0	0	0	0	1	0	1.38	
0	0	0	0	0	1	1.74	
0	1	0	0	0	0	1.26	
0	0	-1	0	0	0	56.51	
0	0	0	0	0	0	0.19	
0	0	0	1	0	0	0.25	
0	0	0	0	0	0	0	
0	0	1	0	0	0	14	
0	0	-2361.6	0	0	0		

6th STEP

SINCE THE COEFFICIENTS IN THE OBJECTIVE FUNCTION ARE REDUCED TO ZERO AND NEGATIVE VALUES, WHICH INDICATES THAT AN OPTIMAL SOLUTION HAS BEEN OBTAINED. HOWEVER, THESE VALUES HAVE NOT MET THE SPECIFIED ACCURACY AND MUST REPEAT STEPS 2 AND 3, WHICH DESCRIBED IN CHEAP 3, UNTIL THE SPECIFIED ACCURACY IS ACHIEVED. USING EQ. 4-6, A NEW SET OF DESIGN PARAMETERS ARE OBTAINED

$$b_f^{\circ} = 17 - 9.46 = 7.54'' \qquad t_f^{\circ} = 1.7 - 1.26 = 0.44''$$

$$d_W^{\circ} = 24 - 0 = 24'' \qquad t_W^{\circ} = 0.5 - 0.25 = 0.25''$$

$$A_1^{\circ} = 3 - 1.38 = 1.62''^2 \qquad A_2^{\circ} = 3 - 1.74 = 1.26''^2$$

Example Problem (continued).

Computer solution.

```
vm/370 online     ljh350 asvosu

login sr021020
ENTER PASSWORD:
optdsn
LOGON AT 11:24:16 EDT SATURDAY 05/08/76
ipl cms
CMS VERSION 2.0.20 - (12/08/75)
bs@asic contbm3
R; T=0.05/0.60 11:24:50

 PLEASE ENTER VALUES FOR P1,P2,P3,Z1,Z2,AND E1
? 60,20,30,720,48,29E3

 PLEASE ENTER VALUES FOR YIELD STRESS OF BAR AND BEAM
? 36,36

 PLEASE ENTER VALUES FOR U1 TO U6 AND D5
? 17,1.7,24,0.5,300,3.0,10
INVALID DATA...RETYPE
17,1.7,24,0.5,3.0,3.0,10
```

THIS IS THE ORIGINAL INPUT DATA SET

P1 (KIP)	P2 (KIP)	P3 (KIP)	Z1 (IN)	Z2 (IN)	E (KSI)
60	20	30	720	48	29000

UPPER LIMITS OF MEMBER SIZE

BF (IN)	TF (IN)	DW (IN)	TW (IN)	A1 (SQ. IN)	A2 (SQ. IN)
17	1.7	24	.5	3	3

THE LOWER LIMIT FOR DW = 10 (IN)

TRIAL DIMENSIONS

BF (IN)	TF (IN)	DW (IN)	TW (IN)	A1 (SQ. IN)	A2 (SQ. IN)
5	.5	5	.5	.5	.5

THIS IS THE ITERATIONS

ITERATION	BF (IN)	TF (IN)	DW (IN)	TW (IN)	BEAM STRESS (KSI)
1	5	.5	24	.5	21.6431
2	7.51513	.431906	24	.25	22.5044
3	7.712	.44322	24	.25	21.6573
4	7.72515	.443976	24	.25	21.604
5	7.72515	.443976	24	.25	21.604

ITERATION	A1 (IN)	A1 STRESS (KSI)	A2 (IN)	A2 STRESS (KSI)	TOTAL WEIGHT (LBS)
1	1.75	20.0943	1.5	18.6628	3515.07
2	1.61887	21.7625	1.26392	22.0854	2589.62
3	1.63096	21.5784	1.29171	21.621	2660.51
4	1.62933	21.5972	1.29296	21.6015	2665.27
5	1.62933	21.5972	1.29296	21.6015	2665.27

THE MAXIMUM MOMENT FOR BEAM IS 2278.92 (KIP-IN)
THE REACTION FORCE FOR A1 IS 35.189 (KIP)
THE REACTION FORCE FOR A2 IS 27.9299 (KIP)

THIS IS THE OPTIMIZATION SOLUTION

FLANGE IS 7.72515 X .443976
WEB IS .25 X 24

A1 IS 1.62933 SQUARE INCH
A2 IS 1.29296 SQUARE INCH

STRESS FOR BEAM IS 21.604 KSI
STRESS FOR A1 IS 21.5972 KSI
STRESS FOR A2 IS 21.6015 KSI

TOTAL WEIGHT IS 2665.27 LBS.

PLEASE ENTER (1) FOR CONTINUE AND (0) FOR STOP
? 1

APPENDIX B

Computer Program Listing

```
type contbm3 basic

 10 REM - OPTIMIZATION OF A THREE-SPAN CONTINUOUS BEAM
 20 REM
 30 DIM L(3,1), R(3,1), F(3,3), G(3,3), Q(3,3),Y(3,1), D(3,1)
 40 DIM A(20,20), B(20), C(20), V(3,3), K(3,1), E(3,1), O(3,1)
 50 DIM P(20), U(20), T(20,20)
 60 DIM H(20,10), X(20)
 70 LET M=9
 80 LET N=6
 90 LET M1=M+N
100 LET M1=M1+1
110 LET C1=M+1
120 PRINT
130 PRINT ' PLEASE ENTER VALUES FOR P1,P2,P3,Z1,Z2,AND E1'
140 INPUT P1,P2,P3,Z1,Z2,E1
150 PRINT
160 PRINT ' PLEASE ENTER VALUES FOR YIELD STRESS OF BAR AND BEAM'
170 INPUT Y6,Y7
180 PRINT
190 PRINT ' PLEASE ENTER VALUES FOR U1 TO U6 AND D5'
200 INPUT U1,U2,U3,U4,U5,U6,D5
210 LET C9=0
220 LET O2=0
230 LET O1 = 0.001
240 REM
250 REM - COMPUTE ALLOWABLE STRESS FOR BAR AND BEAM
260 REM
270 LET Y8=Y6*0.6
280 LET Y9=Y7*0.6
```

```
290 REM
300 REM - ASSUME TRIVAL VALUES
310 FOR I= 1 TO N
320 LET X(I)=1
330 NEXT I
340 LET W=5
350 LET W1=0.5
360 LET D8=5
370 LET D9=0.5
380 LET A1=0.5
390 LET A2=0.5
400 REM
410 REM - COMPUTE STRUCTURE ANALYSIS
420 REM
430 GOSUB 1400
440 IF (V1 - Y8) >0 THEN 480
450 IF (V2 - Y8 ) >0 THEN 500
460 IF (V3 - Y9) >0 THEN 520
470 GOTO 560
480 LET A1=A1+0.25
490 GOTO 430
500 LET A2=A2+0.25
510 GOTO 430
520 LET D8=D8+0.25
530 GOTO 430
540 GOSUB 1400
550 IF (C9 - 20)>=0 THEN 1100
560 LET C9=C9+1
570 LET H(C9,1)=W
580 LET H(C9,2)=D9
590 LET H(C9,3)=D8
600 LET H(C9,4)=W1
610 LET H(C9,5)=A1
620 LET H(C9,6)=A2
630 LET H(C9,7)=V1
640 LET H(C9,8)=V2
```

```
650 LET H(C9,9)=V3
660 LET W9=(( 2*W*D9 + D8*W1) * Z1 + (A1 + A2) * Z2)*490/1728
670 LET H(C9,10)=W9
680 IF O2=1 THEN 1150
690 REM
700 REM - COMPUTE A,B AND C COEFFICIENTS
710 GOSUB 2050
720 FOR I=1 TO M
725 LET L9=I
730 IF B(I)<0 THEN 1130
740 NEXT I
750 REM
760 REM - COMPUTE SIMPLEX ALGORITHM
770 REM
780 GOSUB 2650
785 REM
790 LET U(1)= U1 - U(1)
800 LET U(2)= U2 - U(2)
810 LET U(3)= U3 - U(3)
820 LET U(4)= U4 - U(4)
830 LET U(5)= U5 - U(5)
840 LET U(6)= U6 - U(6)
850 REM
860 REM - CHECK FOR CONVERGENCE
870 REM
880 LET G2=0
890  FOR I=1 TO N
900 IF X(I)=0 THEN 940
910 LET G1=ABS((U(I) -X(I))/X(I))
920  IF (G1-G2)<=0 THEN 940
930 LET G2=G1
940 NEXT I
950 IF(G2-O1) >0 THEN 980
960 LET O2=1
970 GOTO 540
980 FOR I=1 TO N
```

```
990 LET X(I) = U(I)
1000 NEXT I
1010 LET W=U(1)
1020 LET D9=U(2)
1030 LET D8=U(3)
1040 LET W1=U(4)
1050 LET A1=U(5)
1060 LET A2=U(6)
1070 GOTO 540
1080 REM
1090 REM- OUTPUT
1100 PRINT
1110 PRINT 'NOT CONVERGING IN 20 CYCLES'
1120 GOTO 1150
1130 PRINT
1140 PRINT L9, ' NOT CONVERGENT, NEGATIVE P COEFFICIENTS'
1150 PRINT
1151 PRINT ' THIS IS THE ORIGINAL INPUT DATA SET'
1152 PRINT
1160 PRINT ' P1', ' P2', ' P3', ' Z1', ' Z2', ' E'
1161 PRINT '(KIP)', '(KIP)', '(KIP)', '(IN)', '(IN)','(KSI)'
1162 PRINT
1170 PRINT P1,P2,P3,Z1,Z2,E1
1180 PRINT
1185 PRINT ' UPPER LIMITS OF MEMBER SIZE'
1190 PRINT
1191 PRINT ' BF',' TF', ' DW', ' TW', ' A1 ', ' A2 '
1192 PRINT '(IN)','(IN)','(IN)','(IN)','(SQ. IN)','(SQ. IN)'
1193 PRINT
1194 PRINT U1,U2,U3,U4,U5,U6
1195 PRINT
1196 PRINT 'THE LOWER LIMIT FOR DW = '; D5 ;'(IN)'
1197 PRINT
1200 PRINT 'TRIAL DIMENSIONS'
1201 PRINT
1202 PRINT ' BF',' TF',' DW',' TW',' A1 ',' A2 '
```

```
1203 PRINT '(IN)','(IN)','(IN)','(IN)','(SQ. IN)','(SQ. IN)'
1204 PRINT
1205 PRINT 5,0.5,5,0.5,0.5,0.5
1210 PRINT
1211 PRINT ' THIS IS THE ITERATIONS'
1212 PRINT
1220 PRINT 'ITERATION',' BF',' TF',' DW',' TW','BEAM STRESS'
1230 PRINT '    ','(IN)','(IN)','(IN)','(IN)','    (KSI)'
1240 PRINT
1250 FOR P7 = 1 TO C9
1260 PRINT P7,H(P7,1),H(P7,2),H(P7,3),H(P7,4),H(P7,9)
1263 REM
1265 NEXT P7
1266 PRINT
1267 PRINT 'ITERATION', 'A1', 'A1 STRESS', ' A2', 'A2 STRESS','TOTAL WEIGHT'
1268 PRINT '    ', '(IN)', '    (KSI)','(IN)', '    (KSI)','    (LBS)'
1269 PRINT
1270 FOR P8=1 TO C9
1271 PRINT P8,H(P8,5),H(P8,7),H(P8,6),H(P8,8),H(P8,10)
1280 LET I9 = P8
1290 NEXT P8
1300 IF (C9 -20) >=0 THEN 1380
1305 IF B(L9)<0 THEN 1380
1306 PRINT
1307 PRINT 'THE MAXIMUM MOMENT FOR BEAM IS ';M9;'(KIP-IN)'
1308 PRINT 'THE REACTION FORCE FOR A1 IS ';D1;'(KIP)'
1309 PRINT 'THE REACTION FORCE FOR A2 IS ';D2;'(KIP)'
1310 PRINT
1320 PRINT 'THIS IS THE OPTIMIZATION SOLUTION'
1330 PRINT
1340 PRINT 'FLANGE IS';H(I9,1);'Y';H(I9,2)
1350 PRINT 'WEB IS';H(I9,4);'X';H(I9,3)
1351 PRINT
1360 PRINT 'A1 IS';H(I9,5);'SQUARE INCH'
1361 PRINT 'A2 IS';H(I9,6);'SQUARE INCH'
1362 PRINT
```

```
1363 PRINT 'STRESS FOR BEAM IS';V3;'KSI'
1364 PRINT 'STRESS FOR A1 IS';V1;'KSI'
1365 PRINT 'STRESS FOR A2 IS';V2;'KSI'
1366 PRINT
1370 PRINT 'TOTAL WEIGHT IS '; H(19,10); ' LBS.'
1380 PRINT
1381 PRINT 'PLEASE ENTER (1) FOR CONTINUE AND (0) FOR STOP'
1382 INPUT P9
1383 IF P9=1 THEN 120
1384 PRINT
1390 GOTO 3320
1400 REM
1410 REM - INDETERMINATE ANALYSIS SUBROUTINE PROGRAM
1420 REM
1430 LET H1=D8/2 + D9
1440 LET O1=(D8+D9)/2
1450 LET O2=(W1*D8**3)/12
1460 LET O3=2*W*D9*O1**2
1470 LET O9=O3+O2
1480 LET K1 = Z2*O9/A1
1490 LET K2 = Z2*O9/A2
1500 LET R(1,1)=Z1**2*(55*P1+81*P2+35*P3)/1296
1510 LET R(2,1)=Z1**3*(114*P1+207*P2+93*P3)/11664
1520 LET R(3,1)=Z1**3*(93*P1+207*P2+114*P3)/11664
1530  MAT I=R
1540 LET O(1,1)=Z1/3
1550 LET O(1,2)=30*Z1**2/486
1560 LET O(1,3)=12*Z1**2/243
1570 LET O(2,1)=O(1,2)
1580 LET O(2,2)=(4*Z1**3/243) + K1
1590 LET O(2,3)=21*Z1**3/1458
1600 LET O(3,1)=O(1,3)
1610 LET O(3,2)=O(2,3)
1620 LET O(3,3)=(4*Z1**3/243) +K2
1630 MAT C=O
1640 MAT E=INV(C)
```

```
1650 MAT Y=F*L
1660 REM
1670 LET P1=Y(2,1)
1680 LET P2=Y(3,1)
1690 LET R2=P1*Z1/6 + P2*Z1/2 + P3*5*Z1/6
1700 LET R3=P1*Z1/3 + P2*Z1*2/3 + Y(1,1)
1710 LET R1=(R2-R3)/Z1
1720 LET V(1,1)=Z1/3
1730 LET V(1,2)=2*Z1/3
1740 LET V(1,3)=Z1
1750 LET V(2,1)=0
1760 LET V(2,2)=V(1,1)
1770 LET V(2,3)=V(1,2)
1780 LET V(3,1)=0
1790 LET V(3,2)=0
1800 LET V(3,3)=V(1,1)
1810 LET R4=P2*Z1/6 + P3*Z1/2
1820 LET R5=P3*Z1/6
1830 REM
1840 LET K(1,1)=P1
1850 LET K(2,1)=P2
1860 LET K(3,1)=P1
1870 LET P(1,1)=-R2
1880 LET P(2,1)=-R4
1890 LET P(3,1)=-R5
1900 MAT E=V*K
1910 MAT Q=E+P
1920 LET M9=0
1930 FOR I= 1 TO 3
1940 LET R9=Q(I,1)
1950 LET R8 =ABS(R9)
1960 IF (R8-M9)<=0 THEN 1980
1970 LET M9=R8
1980 NEXT I
1990 LET V1=P1/A1
2000 LET V2=P2/A2
```

```
2010 LET V3=M9*H1/O9
2020 RETURN
2030 REM
2040 REM - COMPUTE A,B,C COEFFICIENT SUBROUTINE PROGRAM
2050 REM
2060 FOR J=1 TO M1
2070 LET C(J)=1E10
2080 NEXT J
2090 LET C(1)=-2*D9*Z1
2100 LET C(2)=-2*W*Z1
2110 LET C(3)=-W1*Z1
2120 LET C(4)=-D8*Z1
2130 LET C(5)=-Z2
2140 LET C(6)=-Z2
2150 REM
2160 REM - A COEFFICIENTS
2170 REM
2180 FOR I=1 TO M
2190 LET B(I)=0
2200 FOR J=1 TO N
2210 LET A(I,J)=0
2220 NEXT J
2230 NEXT I
2240 REM
2250 LET A(1,1)=(-M9*H1*O3)/(W*O9*O9)
2260 LET A(1,2)=M9*(1/O9-(H1*(O3/D9+O3/O1)/(O9*O9)))
2270 LET A(1,3)=M9*(1/(2*O9)-(H1*(O3/O1+W1*D8**2/4)/(O9*O9)))
2280 LET A(1,4)=(-M9*H1*O2)/(W1*O9*O9)
2290 LET A(2,5)=-D1/(A1*A1)
2300 LET A(3,6)=-D2/(A2*A2)
2310 LET A(4,1)=1
2320 LET A(4,2)=-104.4/SQR(Y7)
2330 LET A(5,3)=1
2340 LET A(5,4)=-14E3/SQR(Y7*(16.5+Y7))
2350 LET A(6,2)=-1
2360 LET A(7,4)=-1
2370 LET A(8,1)=-1
```

```
2380 LET A(8,2)=10
2390 LET A(9,3)=-1
2400 LET F1=M9*H1/D9 - Y9
2410 LET F2=D1/A1 - Y8
2420 LET F3=D2/A2 - Y8
2430 REM
2440 REM - COMPUTE B COEFFICIENTS
2450 REM
2460 LET B(1)=-F1+A(1,1)*(W-U1)+A(1,2)*(D9-U2)+A(1,3)*(D8-U3)+A(1,4)*(W1-U4)
2470 LET B(2)=-F2+A(2,5)*(A1-U5)
2480 LET B(3)=-F3+A(3,6)*(A2-U6)
2490 LET B(4)=A(4,2)*(-U2)-U1
2500 LET B(5)=A(5,4)*(-U4)-U3
2510 LET B(6)=U2-0.25
2520 LET B(7)=U4-0.25
2530 IF (U1-10*U2)>=0 THEN 2550
2540 LET U8=U1/10
2550 LET B(8)=U1-10*U8
2560 LET B(9)=U3-D5
2570 FOR I=1 TO M
2580 FOR J=1 TO N
2590 LET A(I,J)=-A(I,J)
2600 NEXT J
2610 NEXT I
2620 RETURN
2630 REM
2640 REM - SIMPLEX ALGORITHM
2650 FOR I=1 TO C1
2660 LET P(I)=0
2670 LET U(I)=0
2680 FOR J=1 TO N1
2690 LET T(I,J)=0
2700 NEXT J
2710 NEXT I
2720 FOR I= 1 TO M
2730 FOR J=1 TO N
```

```
2740 LET T(I,J)=A(I,J)
2750 NEXT J
2760 LET I1=N+I
2770 LET T(I,I1)=1
2780 LET T(I,N1)=B(I)
2790 LET P(I)=N+I
2800 NEXT I
2810 FOR J=1 TO M1
2820 LET T(C1,J)=-C(J)
2830 NEXT J
2840 REM - PIVOT COLUMN
2850 LET Z=T(C1,1)
2860 LET K9=1
2870 FOR J=2 TO M1
2880 LET C2=T(C1,J)-Z
2890 IF C2<=0 THEN 2920
2900 LET Z=T(C1,J)
2910 LET K9=J
2920 NEXT J
2930 IF Z>0 THEN 2990
2940 FOR I=1 TO M
2950 LET B1=P(I)
2960 LET U(B1)=T(I,N1)
2970 NEXT I
2980 RETURN
2990 LET I=1
3000 IF T(I,K9)>0 THEN 3070
3005 LET L8=I
3010 IF (I-M)>=0 THEN 3035
3020 LET I=I+1
3030 GOTO 3000
3035 PRINT
3040 PRINT L8; K9, 'OBJECTIVE FUNCTION UNBOUNDED'
3050 RETURN
3060 REM PIVOT ROW
3070 LET S=T(I,N1)/T(I,K9)
```

```
3080 LET I2=I
3090 LET I=I+1
3100 IF (I-M) >0 THEN 3170
3110 IF T(I,K9)<=0 THEN 3160
3120 LET S1=T(I,N1)/T(I,K9)
3130 IF (S1-S)>=0 THEN 3160
3140 LET S=S1
3150 LET I2=I
3160 IF (I-M)<0 THEN 3090
3170 LET P(I2)=K9
3180 LET K3 = T(I2,K9)
3190 FOR J=1 TO N1
3200 LET T(I2,J)=T(I2,J)/K3
3210 NEXT J
3220 FOR I=1 TO C1
3230 LET K4=T(I,K9)
3240 IF K4=0 THEN 3290
3250 IF (I-I2)=0 THEN 3290
3260 FOR J=1 TO N1
3270 LET T(I,J)=T(I,J)-T(I2,J)*K4
3280 NEXT J
3290 NEXT I
3300 GOTO 2850
3310 RETURN
3320 END

R; T=1.85/11.85 11:58:54
```

APPENDIX C

Values of $\int_L M\, m\, dx$

m \ M	M, L	M_0, L	M_1, L	M_1, L	a, b, M_1, L	M_0, M_1, L	2°, M_1, L	L, 2°, +, M_1	M_1, L	M_0, M_1, L	M_0, C, M_1, L
m, L	mML	$\frac{1}{2}mM_0L$	$\frac{1}{2}mM_1L$	$\frac{1}{2}mM_1L$		$\frac{1}{2}mL(M_0+M_1)$	$\frac{2}{3}mM_1L$		$\frac{1}{3}mM_1L$	$\frac{1}{3}mL(2M_0-M_1)$	$\frac{1}{6}mL(M_0+4C+M_1)$
m_0, L	$\frac{1}{2}m_0ML$	$\frac{1}{3}m_0M_0L$	$\frac{1}{6}m_0M_1L$	$\frac{1}{4}m_0M_1L$		$\frac{1}{6}m_0L(2M_0+M_1)$	$\frac{1}{3}m_0M_1L$		$\frac{1}{12}M_1m_0L$	$\frac{1}{12}m_0L(5M_0-M_1)$	$\frac{1}{6}m_0L(M_0+2C)$
m_1, L	$\frac{1}{2}m_1ML$	$\frac{1}{6}m_1M_0L$	$\frac{1}{3}m_1M_1L$	$\frac{1}{4}m_1M_1L$		$\frac{1}{6}m_1L(2M_1+M_0)$	$\frac{1}{3}m_1M_1L$		$\frac{1}{4}m_1M_1L$	$\frac{1}{4}m_1L(M_0-M_1)$	
m_1, L	$\frac{1}{2}m_1ML$	$\frac{1}{4}m_1M_0L$	$\frac{1}{4}m_1M_1L$	$\frac{1}{3}m_1M_1L$		$\frac{1}{4}m_1L(M_0+M_1)$	$\frac{5}{12}m_1M_1L$				
a, b, m_1, L					$\frac{1}{3}m_1M_1L$	$\frac{1}{6}m_1L[M_0(1+b)+M_1(1+a)]$	$\frac{1}{3}m_1M_1L(1+ab)$	$\frac{1}{12}m_1M_1L(3a+b)$			
m_0, m_1, L	$\frac{1}{2}ML(m_0+m_1)$	$\frac{1}{6}M_0L(2m_0+m_1)$	$\frac{1}{6}M_1L(2m_1+m_0)$	$\frac{1}{4}M_1L(m_0+m_1)$	$\frac{1}{6}M_1L[m_0(1+b)+m_1(1+a)]$	$\frac{1}{6}L[m_0(2M_0+M_1)+m_1(2M_1+M_0)]$	$\frac{1}{3}M_1L(m_0+m_1)$	$\frac{1}{12}M_1L(m_0+3m_1)$	$\frac{1}{12}M_1L(m_0+3M_1)$	$\frac{1}{12}L[m_0(5M_0-M_1)+3m_1(M_0-M_1)]$	$\frac{1}{6}L[m_0(M_0+2C)+m_1(2C+M_1)]$
2°, m_1, L				$\frac{5}{12}m_1M_1L$	$\frac{1}{3}m_1M_1L(1+ab)$	$\frac{1}{3}m_1L(M_0+M_1)$	$\frac{8}{15}m_1M_1L$	$\frac{1}{5}m_1M_1L$			
L, 2°, +, m_1					$\frac{1}{12}m_1M_1L(3a+b^2)$	$\frac{1}{12}m_1L(M_0+3M_1)$	$\frac{1}{5}m_1M_1L$	$\frac{1}{5}m_1M_1L$			
L, 3°, +, m_1					$\frac{1}{20}m_1M_1L(1+a)(1+a^2)$	$\frac{1}{20}m_1L(M_0+4M_1)$					
3°, +, m_1, L					$\frac{1}{60}m_1M_1L(1+a)(7-3a^2)$	$\frac{1}{60}m_1L(7M_0+8M_1)$					

BIBLIOGRAPHY

BOOKS

1. Aguilar R. J. : Systems Analysis and Design, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973
2. Au, T., and T. E. Stelson : Introduction to System Engineering, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1969.
3. Gass, S. I. : Linear Programming Methods and Applications, 2nd ed., McGraw-Hill Book Company, New York, 1964.
4. Kleinlogel, A. : Beam Formulars, Frederick Ungar Publishing Company, New York, 1953.
5. Laursen, H. I. : Matrix Analysis of Structure, Mcgraw-Hill Book Company, New York, 1968
6. Manual of Steel Construction, 7th ed., American Institute of Steel Construction, New York, 1970.
7. Stark, R. M., and R. L. Nicholls : Mathmatical Foundations for Design, McGraw-Hill Book Company, New York, 1972.
8. Wagner, H. M. : Principles of Operations Research, 2nd ed., Prentice-Hall, Inc., Englewood Cliffs, 1975

PUBLICATIONS

9. Basic Language Reference Manual, GC20-1803-1, 2nd ed., IBM Corporation, Burlington, Mass., April, 1973.

10. Moses, F. : Optimum Structural Design Using Linear Programming, Proc. ASCE, Vol. 90, no. ST6, pp. 90-104, December, 1964.