MODAL SHAPES OF THE GENERAL

STIFFNESS MATRIX

by

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ABSTRACT

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The purpose of this thesis is to investigate the characteristics of the normal mode shapes associated with the general stiffness matrix of a long slender beam including the effects of axial force and transverse inertia loading.

Four separate problems are analyzed. These include the statical beam bending problem, the statical beam-column bending problem, the dynamical beam problem in free vibration, and the dynamical beam-column problem in free vibration. In each case, the orthogonality conditions of the modal shapes are established. Also, the existence of rigid body motions as possible modal shapes are investigated.

In general, it is found that each of the above four problems possesses two rigid body modal shapes, a translational and a rotational form. The remaining two deformed modal shapes are associated with the resonant frequency of free vibration of a beam, the critical buckling load of a column, and the resonant frequency of a beam-column.

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Great appreciation is given to my dear father, Ho-Siaw, and my brothers for supporting me during my studies.

TABLE OF CONTENTS

PAGE

LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Cross-sectional area of member
E	Young's modulus of elasticity
$\{F\}$	Vector of element forces
$\{f\}$	Vector of element forces in dimensionless form
$[G_o]$	Geometrical stiffness matrix
$[\hat{G}_o]$	Geometrical stiffness matrix in dimensionless form
I	Moment of inertia
$[I]$	The identity matrix
$[K]$	Elastic bending stiffness matrix
$[\hat{K}]$	Elastic bending stiffness matrix in dimensionless form
L	Length of member
M	Bending moment
$[M_o]$	Mass stiffness matrix
$[\hat{M}_o]$	Mass stiffness matrix in dimensionless form
P	Axial force
$[S]$	General stiffness matrix
$[\hat{S}]$	General stiffness matrix in dimensionless form
$[U]$	Modal matrix
V	Shear force
w	Transverse deflection
θ	Angular deflection
ω	Natural frequency of free vibration of the beam
Ω	Natural frequency of free vibration of the beam-column

SYMBOL	DEFINITION
ϕ	$= \frac{PL^2}{30EI}$
Ψ	$= \frac{\rho A L^4 \Omega^2}{420EI}$
$[\Lambda]$	Diagonal matrix of eigenvalues
ρ	Mass density per unit volume
$\{\Delta\}$	Vector of displacement
$\{\delta\}$	Vector of displacement in dimensionless form

SUBSCRIPTS

b	Beam
bc	Beam-column
cr	Critical buckling load
d	Dynamic
s	Static
$\wedge$	Dimensionless form indicator

LIST OF FIGURES

FIGURE PAGE

FIGURE PAGE

LIST OF TABLES

CHAPTER I

INTRODUCTION

1.1 Equations of Motion

The general stiffness matrix for a beam and/or a beam-column element is derived from the Bernoulli-Euler differential equation with the inclusion of the axial force for the beam-column. Rubenstein(1)* derived the required stiffness, mass, and axial force matrix utilizing static displacement functions for the beam-column element. Henshell(2) used the exact dynamic equations in obtaining the dynamic stiffness coefficients (i.e., mass matrix) for a beam element. Wang(3) used the 'exact' equation in deriving the geometric stiffness or axial force matrix for a beam-column element. The resulting matrix series allows for an efficient procedure for computer operations.

The general stiffness matrix takes the form

$$\left[S \right] = \left[K \right] - P\left[G_o \right] - \Omega^2 \left[M_o \right] \tag{1-1}$$

where $\left[K \right]$ is the elastic bending stiffness matrix, $\left[G_o \right]$ is the geometrical stiffness matrix associated with the compressive axial force (P), and $\left[M_o \right]$ is the mass matrix with Ω the natural frequency of free vibration. The stiffness matrix $\left[S \right]$

* Number in parenthesis refers to literature cited in the Bibliography.

is symmetric, but not necessarily positive definite. In general, it is positive indefinite, that is, its eigenvalues are positive, but also may include zero. These particular zero eigenvalues are associated with rigid body modal shapes.

By transforming this general stiffness matrix $[S]$ into diagonal form (i.e., spectral decomposition), that is, performing the eigenvalue-eigenvector problem, a complete set of modal shapes including both rigid body and deformable mode shapes are obtainable. This process requires the calculation of a matrix $[U]$ called the eigenvector matrix which satisfies the conditions

$$[U]^T[S][U] = [\Lambda] \tag{1-2a}$$

and

$$[U][U]^T = [U]^T[U] = [I] \tag{1-2b}$$

that is, $[U]$ is orthonormal. The matrix $[\Lambda]$ is a diagonal matrix of eigenvalues whose zeros are associated with rigid body mode shapes. Nonzero terms of the matrix $[\Lambda]$ when equated to zero yield values of critical buckling load and natural frequency. Since $[S]$ is a symmetric, it is diagonalized by an orthogonal matrix $[U]$. This condition is shown in equation (1-2b).

1.2 Form of the General Stiffness Matrix

The algebraic components of the stiffness matrix $\left[S\right]$ take the form

$$\left[K\right] = \frac{EI}{L^3}\begin{bmatrix} 12 & & \text{Symmetric} & \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L & -6L & 4L^2 \end{bmatrix} \tag{1-3a}$$

$$\left[G_0\right] = \begin{bmatrix} \frac{6}{5L} & & \text{Symmetric} & \\ \frac{1}{10} & \frac{2L}{15} & & \\ \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{bmatrix} \tag{1-3b}$$

$$\left[M_0\right] = \frac{\rho AL}{420}\begin{bmatrix} 156 & & \text{Symmetric} & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L & -22L & 4L^2 \end{bmatrix} \tag{1-3c}$$

The moments, shear forces, displacement, and rotations are related by the matrix equation (see Figure (I)).

$$\left\{F\right\} = \left[S\right]\left\{\Delta\right\} \tag{1-4a}$$

where

$$\left\{F\right\} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} \quad \text{and} \quad \left\{\Delta\right\} = \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} \tag{1-4b,c}$$

with V_1 and V_2, the joint shear forces

M_1 and M_2, the joint bending moments

W_1 and W_2, the joint displacements and

θ_1 and θ_2, the joint rotations

The positive sign convention for M_1, M_2, V_1, V_2, θ_1, θ_2, W_1 and W_2 used consistently throughout this work is shown in Figure (I).

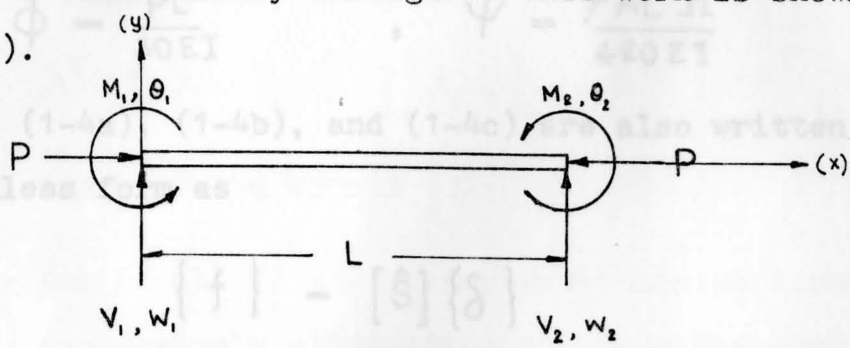


Figure (I) Problem Parameters and Sign Convention

For convenience, the equations (3a), (3b), and (3c) are recast in dimensionless form as

$$\left[\hat{K}\right] = \begin{bmatrix} 12 & & \text{Symmetric} & \\ 6 & 4 & & \\ -12 & -6 & 12 & \\ 6 & 2 & -6 & 4 \end{bmatrix} \qquad (1\text{-}5a)$$

$$\left[\hat{G}_o\right] = \begin{bmatrix} 36\phi & & \text{Symmetric} & \\ 3\phi & 4\phi & & \\ -36\phi & -3\phi & 36\phi & \\ 3\phi & -\phi & -3\phi & 4\phi \end{bmatrix} \qquad (1\text{-}5b)$$

$$[\hat{M}_g] = \begin{bmatrix} 156\psi & & \text{Symmetric} & \\ 22\psi & 4\psi & & \\ 54\psi & 13\psi & 156\psi & \\ -13\psi & -3\psi & -22\psi & 4\psi \end{bmatrix} \tag{1-5c}$$

where

$$\phi = \frac{\rho L^2}{30EI} \quad , \quad \psi = \frac{\rho A L^4 \Omega^2}{420EI} \tag{1-6a,6b}$$

Equations (1-4a), (1-4b), and (1-4c) are also written in dimensionless form as

$$\{f\} = [\hat{S}]\{\delta\} \tag{1-7a}$$

where

$$\{f\} = \frac{L}{EI}\begin{Bmatrix} V_1 L \\ M_1 \\ V_2 L \\ M_2 \end{Bmatrix} \quad , \quad \{\delta\} = \begin{Bmatrix} W_1/L \\ \theta_1 \\ W_2/L \\ \theta_2 \end{Bmatrix} \tag{1-7b,7c}$$

The modal shape problem is defined by the condition that the force vector is proportional to the displacement vector, that is,

$$\{f\} = [\hat{S}]\{\delta\} = \lambda\{\delta\} \tag{1-8a}$$

where λ's are defined as eigenvalues. Equation (1-8a) is rewritten in the form

$$\left[[\hat{S}] - \lambda[I]\right]\{\delta\} = \{0\} \tag{1-9}$$

For non-zero value of $\{\delta\}$, it follows that

$$\left|[\hat{S}] - \lambda[I]\right| = 0 \qquad (1\text{-}10a)$$

which yields the characteristic equation of this matrix $[\hat{S}]$ which is solved directly for the eigenvalues. The general form of equation (1-10a) becomes

$$\lambda^4 - I_1\lambda^3 + I_2\lambda^2 - I_3\lambda + I_4 = 0 \qquad (1\text{-}10b)$$

where I_1 = trace of the matrix $[\hat{S}]$ (1-10c)

I_2 = sum of all (2 x 2) determinant minors formed by successively eliminating all possible combinations of any two rows and the corresponding two columns (1-10d)

I_3 = sum of the (3 x 3) determinant minors of the principal diagonal elements (1-10e)

I_4 = the determinant of $[\hat{S}]$ (1-10f)

The roots of the equation (1-10b), λ_1, λ_2, λ_3, and λ_4, are the eigenvalues of $[\hat{S}]$

The eigenvalues of equation (1-10b) are individually substituted into equation (1-9) and the corresponding eigenvectors $\{\delta\}$ are obtained which directly define the modal shapes. These vectors are then combined to form the columns of the modal matrix $[U]$.

1.3 Four Special Cases of the Stiffness Matrix

The following four cases are investigated in this thesis:

Case I - Beam Bending Problem (Statical)

$$\left[\hat{S}_b^{(s)}\right] = \left[\hat{K}\right] \qquad (1\text{-}11a)$$

Case II - Beam-Column Bending Problem (Statical)

$$\left[\hat{S}_{bc}^{(s)}\right] = \left[\hat{K}\right] - \left[\hat{G}_o\right] \qquad (1\text{-}11b)$$

Case III - Vibrating Beam Problem (Dynamical)

$$\left[\hat{S}_b^{(d)}\right] = \left[\hat{K}\right] - \left[\hat{M}_o\right] \qquad (1\text{-}11c)$$

Case IV - Vibrating Beam-Column Problem (Dynamical)

$$\left[\hat{S}_{bc}^{(d)}\right] = \left[\hat{K}\right] - \left[\hat{G}_o\right] - \left[\hat{M}_o\right] \qquad (1\text{-}11d)$$

CHAPTER II

BEAM BENDING PROBLEM

2.1 Eigenvalue Matrix

For the statical beam bending problems, it follows that

$$\{f\} = [\hat{K}]\{\delta\} \qquad (2\text{-}1a)$$

or

$$\frac{L}{EI}\begin{Bmatrix} V_1L \\ M_1 \\ V_2L \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12 & & \text{Symmetric} & \\ 6 & 4 & & \\ -12 & -6 & 12 & \\ 6 & 2 & -6 & 4 \end{bmatrix} \qquad (2\text{-}1b)$$

The four matrix invariants of the matrix in equation (2-1b) are

$$I_1 = 32\ ,\ I_2 = 60\ ,\ I_3 = I_4 = 0 \qquad (2\text{-}2)$$

The characteristic equation becomes

$$\lambda^2(\lambda - 2)(\lambda - 30) = 0 \qquad (2\text{-}3b)$$

with the four roots determined as

$${}^{(1)}\lambda_1 = {}^{(1)}\lambda_2 = 0\quad ,\ {}^{(1)}\lambda_3 = 2\ ,\ {}^{(1)}\lambda_4 = 30 \qquad (2\text{-}3c)$$

The eigenvalue matrix takes the form

$$\left|\Lambda_b^{(s)}\right| = \begin{bmatrix} 0 & \text{Symmetric} & & \\ 0 & 0 & & \\ 0 & 0 & 2 & \\ 0 & 0 & 0 & 30 \end{bmatrix} \qquad (2\text{-}3d)$$

It should be noted that the four invariant properties of the latter matrix are identical to those given in equation (2-2) for the matrix $\left[\hat{K}\right]$

2.2 The Eigenvector Matrix

Utilizing equation (1-9), one obtains

$$\left[\left[\hat{K}\right] - \lambda\left[I\right]\right]\left\{\delta\right\} = \left\{0\right\} \qquad (2\text{-}4)$$

Substituting the four roots of individually into equation (2-4), the eigenvector matrix is constructed as

$$\left[U_b^{(1)}\right] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{10} & 0 & 2/\sqrt{10} \\ 0 & -2/\sqrt{10} & 1/\sqrt{2} & 1/\sqrt{10} \\ 1/\sqrt{2} & -1/\sqrt{10} & 0 & -2/\sqrt{10} \\ 0 & -2/\sqrt{10} & -1/\sqrt{2} & 1/\sqrt{10} \end{bmatrix} \qquad (2\text{-}5)$$

It should be noted that the eigenvector associated with the second zero value of λ is obtained by using the orthogonality equation (1-2b).

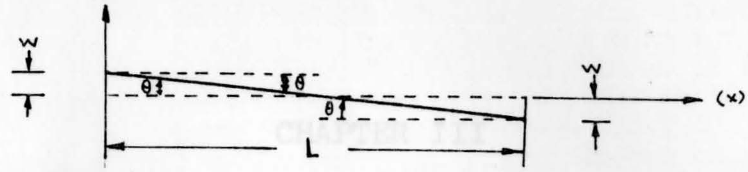
2.3 Solutions of the Moments, Shear Forces, and Normal Mode Shapes

The normal mode shapes, together with the joint moments, shears, displacements, and rotations values, are given for the four values of λ in Figure (IIA), (IIB), (IIC), and (IID) respectively

$W_1 = W_2 = W \approx L/\sqrt{2}$ $\qquad$ $V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta = 0$ $\qquad$ $M_1 = M_2 = 0$

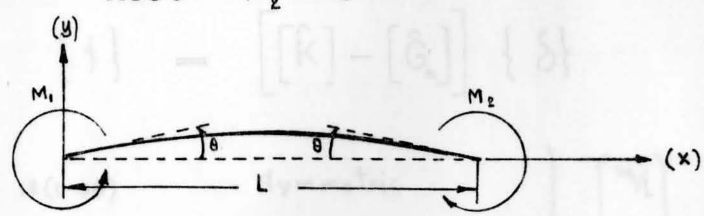
Figure (IIA) Modal Shape of the Beam for $^{(1)}\lambda_1 = 0$



$w_1 = -w_2 = w \approx L/\sqrt{10}$ $\qquad V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta \approx -2/\sqrt{10}$ $\qquad M_1 = M_2 = 0$

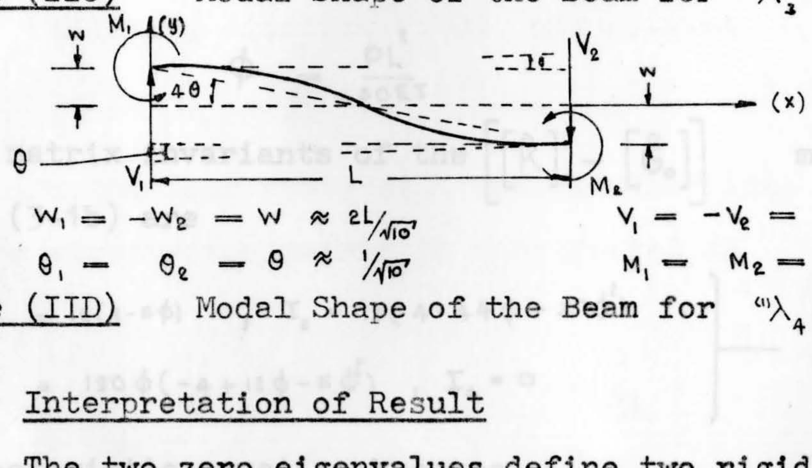
<u>Figure (IIB)</u> Modal Shape of the Beam for the Second Root ${}^{(1)}\lambda_2 = 0$



$w_1 = w_2 = w = 0$ $\qquad V_1 = V_2 = 0$

$\theta_1 = -\theta_2 = \theta \approx 1/\sqrt{2}$ $\qquad M_1 = -M_2 = 2\,\frac{EI}{L}\,\theta$

<u>Figure (IIC)</u> Modal Shape of the Beam for ${}^{(1)}\lambda_3 = 2$



$w_1 = -w_2 = w \approx 2L/\sqrt{10}$ $\qquad V_1 = -V_2 = 30\,\frac{EI}{L^3}\,w$

$\theta_1 = \theta_2 = \theta \approx 1/\sqrt{10}$ $\qquad M_1 = M_2 = 30\,\frac{EI}{L}\,\theta$

<u>Figure (IID)</u> Modal Shape of the Beam for ${}^{(1)}\lambda_4 = 30$

2.4 <u>Interpretation of Result</u>

The two zero eigenvalues define two rigid body mode shapes, one a rigid body translation, the other a rigid body rotation. In both cases, the associated joint moments and shear forces are all zero. The two nonzero eigenvalues define a pure bending mode shape (i.e. ${}^{(1)}\lambda_3 = 2$) and a combined bending and shear force mode shape (i.e. ${}^{(1)}\lambda_4 = 30$).

CHAPTER III

BEAM-COLUMN BENDING PROBLEM

3.1 Eigenvalue Matrix

For the statical beam-column bending problem, it follows that

$$\{f\} = \left[[\hat{K}] - [\hat{G}_o]\right] \{\delta\} \tag{3-1a}$$

or

$$\frac{L}{EI}\begin{Bmatrix} V_1L \\ M_1 \\ V_2L \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12(1-3\phi) & & \text{Symmetric} & \\ 3(2-3\phi) & 4(1-\phi) & & \\ -12(1-3\phi) & -3(2-\phi) & 12(1-3\phi) & \\ 3(2-\phi) & (2+\phi) & -3(2-\phi) & 4(1-\phi) \end{bmatrix} \begin{Bmatrix} w_1/L \\ \theta_1 \\ w_2/L \\ \theta_2 \end{Bmatrix} \tag{3-1b}$$

where

$$\phi = \frac{PL^2}{30EI}$$

The four matrix invariants of the $\left[[\hat{K}] - [\hat{G}_o]\right]$ matrix in equation (3-1b) are

$$\left.\begin{aligned} I_1 &= 16(2-5\phi) \quad , \quad I_2 = 15(4-44\phi+37\phi^2) \\ I_3 &= 180\phi(-4+12\phi-5\phi^2) \quad , \quad I_4 = 0 \end{aligned}\right\} \tag{3-2}$$

The characteristic equation becomes

$$\lambda^4 - 16(2-5\phi)\lambda^3 + 15(4-44\phi+37\phi^2)\lambda^2 + 180\,\phi(2-5\phi)(2-\phi) = 0 \tag{3-3a}$$

or in quadratic factored form as

$$\lambda\left[\lambda - (2-5\phi)\right]\left[\lambda^2 + 5(25\phi-6)\lambda + 180\,\phi(\phi-2)\right] = 0 \tag{3-3b}$$

with the four roots determined as

$$\left.\begin{aligned}
{}^{(2)}\lambda_1 &= 0 \\
{}^{(2)}\lambda_2 &= -\frac{15}{2}(5\phi-2)-\sqrt{\left[\frac{15}{2}(5\phi-2)\right]^2-180\,\phi(\phi-2)} \\
{}^{(2)}\lambda_3 &= (2-5\phi) \\
{}^{(2)}\lambda_4 &= -\frac{15}{2}(5\phi-2)+\sqrt{\left[\frac{15}{2}(5\phi-2)\right]^2-180\,\phi(\phi-2)}
\end{aligned}\right\} \tag{3-3c}$$

The eigenvalue matrix takes the form

$$\left[\Lambda^{(2)}_{bc}\right] = \begin{bmatrix} {}^{(2)}\lambda_1 & & \text{Symmetric} & \\ 0 & {}^{(2)}\lambda_2 & & \\ 0 & 0 & {}^{(2)}\lambda_3 & \\ 0 & 0 & 0 & {}^{(2)}\lambda_4 \end{bmatrix} \tag{3-3d}$$

3.2 The Eigenvector Matrix

Utilizing equation (1-9), one obtains

$$\left[\left[\hat{K}\right]-\left[\hat{G}_o\right]-\lambda\left[I\right]\right]\left\{\delta\right\} = \left\{0\right\} \tag{3-4}$$

Substituting the four roots of λ individually into equation (3-4), the eigenvector matrix is constructed as

$$\left[U^{(2)}_{bc}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -{}^{(2)}n_1/{}^{(2)}d_1 & 0 & {}^{(2)}n_3/{}^{(2)}d_2 \\ 0 & -{}^{(2)}n_2/{}^{(2)}d_1 & 1 & {}^{(2)}n_2/{}^{(2)}d_2 \\ 1 & {}^{(2)}n_1/{}^{(2)}d_1 & 0 & -{}^{(2)}n_3/{}^{(2)}d_2 \\ 0 & -{}^{(2)}n_2/{}^{(2)}d_1 & -1 & {}^{(2)}n_2/{}^{(2)}d_2 \end{bmatrix} \tag{3-5a}$$

where

$$^{(2)}n_1 = (6-23\phi) - \sqrt{5^2(5\phi-2)^2 - 80\phi(\phi-2)}$$

$$^{(2)}n_2 = 4(2-\phi)$$

$$^{(2)}n_3 = (6-23\phi) + \sqrt{5^2(5\phi-2)^2 - 80\phi(\phi-2)}$$

$$^{(2)}d_1 = \sqrt{\left\{(6-23\phi) - \sqrt{5^2(5\phi-2)^2 - 80\phi(\phi-2)}\right\}^2 + 16(2-\phi)^2} \qquad (3\text{-}5b)$$

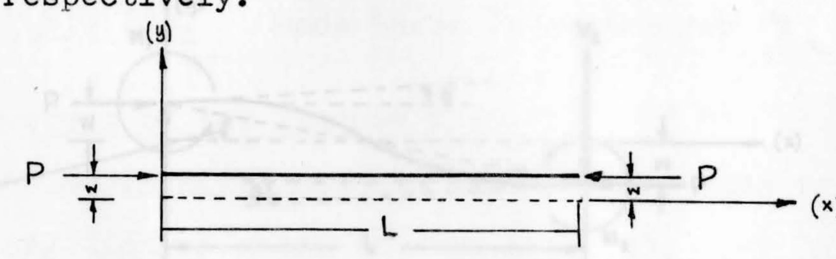
$$^{(2)}d_2 = \sqrt{\left\{(6-23\phi) + \sqrt{5^2(5\phi-2)^2 - 80\phi(\phi-2)}\right\}^2 + 16(2-\phi)^2}$$

Note

$$\frac{^{(2)}n_1}{^{(2)}d_1} = \frac{-^{(2)}n_2}{^{(2)}d_2}, \qquad \frac{^{(2)}n_2}{^{(2)}d_1} = \frac{^{(2)}n_3}{^{(2)}d_2}$$

3.3 Solutions for the Moments, Shear Forces and Variations of Normal Mode Shapes

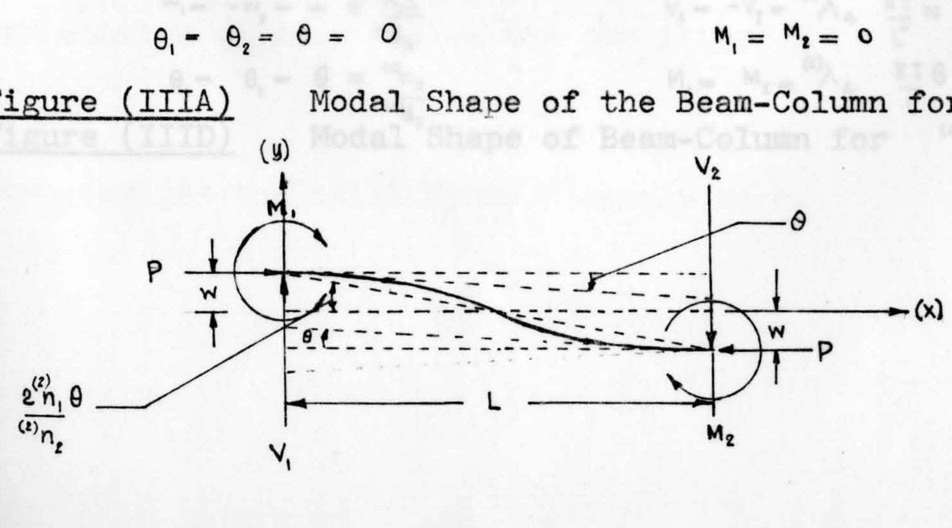
The normal mode shapes, together with the joint moments, shears, displacements, and rotations, are given for the four values of λ in Figure (IIIA), (IIIB), (IIIC), and (IIID), respectively.



$W_1 = W_2 = W \approx L/\sqrt{2}$ $\qquad V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta = 0$ $\qquad M_1 = M_2 = 0$

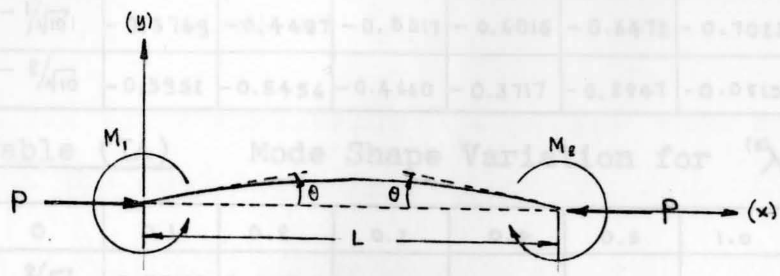
Figure (IIIA) Modal Shape of the Beam-Column for $^{(2)}\lambda_1$



$$w_1 = -w_2 = w \approx \frac{-{}^{(2)}n_1 L}{{}^{(2)}d_1} \qquad V_1 = -V_2 = -{}^{(2)}\lambda_2 \, \frac{EI}{L^3}\, w$$

$$\theta_1 = \theta_2 = \theta \approx \frac{-{}^{(2)}n_2}{{}^{(2)}d_1} \qquad M_1 = M_2 = -{}^{(2)}\lambda_2 \, \frac{EI}{L}\, \theta$$

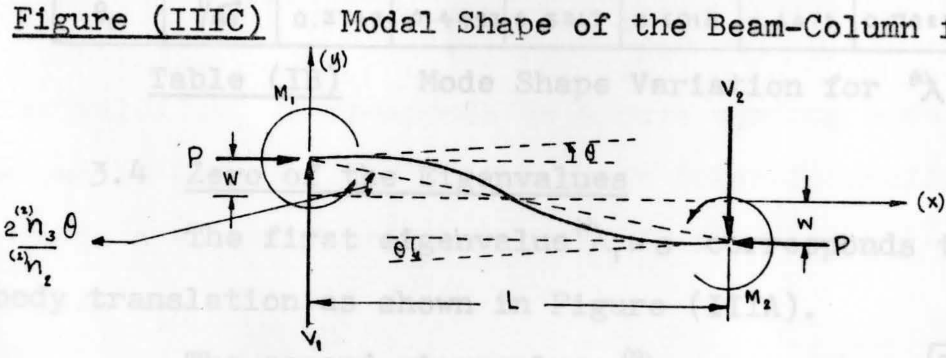
Figure (IIIB) Modal Shape of the Beam-Column for ${}^{(2)}\lambda_2$



$$w_1 = w_2 = w = 0 \qquad V_1 = V_2 = 0$$

$$\theta = -\theta_2 = \theta \approx \frac{1}{\sqrt{2}} \qquad M = -M_2 = {}^{(2)}\lambda_3 \, \frac{EI}{L}\, \theta$$

Figure (IIIC) Modal Shape of the Beam-Column for ${}^{(2)}\lambda_3$



$$w_1 = -w_2 = w \approx \frac{{}^{(2)}n_3 L}{{}^{(2)}d_2} \qquad V_1 = -V_2 = {}^{(2)}\lambda_4 \, \frac{EI}{L^3}\, w$$

$$\theta_1 = \theta_2 = \theta \approx \frac{{}^{(2)}n_2}{{}^{(2)}d_2} \qquad M_1 = M_2 = {}^{(2)}\lambda_4 \, \frac{EI}{L}\, \theta$$

Figure (IIID) Modal Shape of Beam-Column for ${}^{(2)}\lambda_4$

The first and third modal shapes do not change geometrically as the parameter ϕ increases. The variation in shape of the second and fourth modal shape as the parameter ϕ increases are given in Table (IA), and (IB), respectively.

${}^{(2)}\delta_2$ \ ϕ	0	0.1	0.2	0.3	0.4	0.5	1.0	1.9	2.0	2.1
w_1/L	$1/\sqrt{10}$	0.3749	0.4497	0.5317	0.6015	0.6472	0.7023	0.7070	$1/\sqrt{2}$	0.7070
θ_1	$-2/\sqrt{10}$	-0.5952	-0.5456	-0.4660	-0.3717	-0.2847	-0.0815	-0.0037	- 0	+0.0033
w_2/L	$-1/\sqrt{10}$	-0.3749	-0.4497	-0.5317	-0.6015	-0.6472	-0.7023	-0.7070	$-1/\sqrt{2}$	-0.7070
θ_2	$-2/\sqrt{10}$	-0.5952	-0.5456	-0.4660	-0.3717	-0.2847	-0.0815	-0.0037	0	+0.0033

<u>Table (IA)</u> Mode Shape Variation for ${}^{(2)}\lambda_2$

${}^{(2)}\delta_4$ \ ϕ	0	0.1	0.2	0.3	0.4	0.5	1.0	1.9	2.0	2.1
w_1/L	$2/\sqrt{10}$	0.5952	0.5456	0.4660	0.3717	0.2847	0.0815	0.0037	0	0.0033
θ_1	$1/\sqrt{10}$	0.3749	0.4497	0.5317	0.6015	0.6472	0.7023	0.7070	$1/\sqrt{2}$	-0.7070
w_2/L	$-2/\sqrt{10}$	-0.5952	-0.5456	-0.4660	-0.3717	-0.2847	-0.0815	-0.0037	0	-0.0033
θ_2	$1/\sqrt{10}$	0.3749	0.4497	0.5317	0.6015	0.6472	0.7023	0.7070	$1/\sqrt{2}$	-0.7070

<u>Table (IB)</u> Mode Shape Variation for ${}^{(2)}\lambda_4$

3.4 <u>Zero of the Eigenvalues</u>

The first eigenvalue ${}^{(2)}\lambda_1 = 0$ corresponds to a rigid body translation as shown in Figure (IIIA).

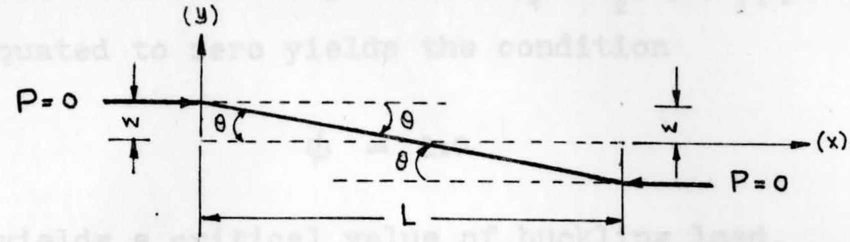
The second eigenvalue ${}^{(2)}\lambda_2 = -\frac{15}{2}(5\phi-2)-\sqrt{\left[\frac{15}{2}(5\phi-2)\right]^2-180\phi(\phi-2)}$,

when equated to zero yields the condition

$$\phi = 0 \tag{3-6}$$

which implies the axial force P equals zero.

For $\phi = 0$, the mode shape takes the form shown in Figure (IIIE).



$$w_1 = w_2 = w \approx L/\sqrt{10} \qquad V_1 = -V_2 = 0$$

$$\theta_1 = \theta_2 = \theta \approx -2/\sqrt{10} \qquad M_1 = M_2 = 0$$

Figure (IIIE) Modal Shape of Beam-Column for the Second Zero of ${}^{(2)}\lambda_2$

The third eigenvalue ${}^{(2)}\lambda_3 = (2 - 5\phi)$ when equated to zero yields the condition

$$\phi = \frac{2}{5} \tag{3-7a}$$

Noting $\phi = \dfrac{PL^2}{30EI}$, it follows that a critical value of axial force is obtained as

$$P_{cr}^{(1)} = 12 \frac{EI}{L^2} \tag{3-7b}$$

The value ${}^{(2)}\lambda_3$ corresponds to a pure bending mode shape as shown in Figure (IIIC). The exact Euler-Bernoulli theory yields a value of critical buckling for a simply-supported column as

$$P_{cr} = \pi^2 \frac{EI}{L^2} \tag{3-7c}$$

The value of P_{cr} by the matrix formulation given by equation (3-7b) is greater by 21.86%. For $\phi = \frac{2}{5}$, the same mode shape occurs as given in Figure (IIIC).

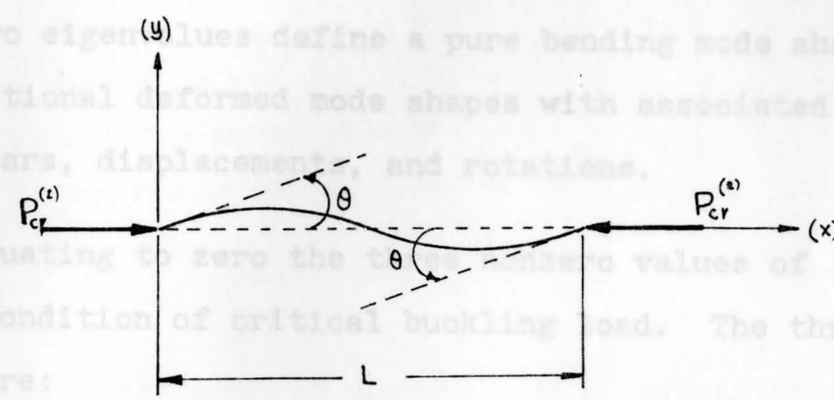
The fourth eigenvalue ${}^{(2)}\lambda_4 = -\frac{15}{2}(5\phi - 2) + \sqrt{\left[\frac{15}{2}(5\phi - 2)\right]^2 - 180\,\phi(\phi - 2)}$, when equated to zero yields the condition

$$\phi = 2.0 \tag{3-8a}$$

which yields a critical value of buckling load.

$$P_{cr}^{(2)} = 60\,\frac{EI}{L^2} \tag{3-8b}$$

The mode shape at critical load takes the form shown in Figure (IIIF).



$$w_1 = w_2 = w = 0 \qquad V_1 = V_2 = 0$$

$$\theta_1 = \theta_2 = \theta \approx 1/\sqrt{2} \qquad M_1 = M_2 = 0$$

Figure (IIIF) Modal Shape of the Beam-Column for the Fourth Zero of ${}^{(2)}\lambda_4$

The exact Euler-Bernoulli theory yields a value of critical buckling for the second mode shape of a simply supported column as

$$P_{cr} = 4\pi^2 \frac{EI}{L^2} \qquad (3\text{-}8c)$$

The value of P_{cr} by the matrix formulation given by equation (3-8b) is greater by 52%.

3.5 Interpretation of Results for the Beam-Column

The single zero eigenvalue is obtained for this problem which corresponds to a rigid body translation. The three nonzero eigenvalues define a pure bending mode shape and two additional deformed mode shapes with associated joint moments, shears, displacements, and rotations.

Equating to zero the three nonzero values of λ yields the condition of critical buckling load. The three conditions are:

a) ${}^{(2)}\lambda_2 = 0$ implies $\phi = 0$ or $P = 0$

b) ${}^{(2)}\lambda_3 = 0$ implies $\phi = 2/5$ or $P_{cr} = 12\frac{EI}{L^2}$

c) ${}^{(2)}\lambda_4 = 0$ implies $\phi = 2.0$ or $P_{cr} = 60\frac{EI}{L^2}$

For condition a), the mode shape corresponds to a rigid body rotation with zero joint axial force, moments, and shear forces. Condition b) produces a modal shape corresponding to the first buckling mode shape of a simply supported column. Condition c) produces a mode shape at critical load which corresponds to the second buckling mode (i.e. n=2) of a simply supported column.

CHAPTER IV

VIBRATING BEAM PROBLEM

4.1 Eigenvalue Matrix

For the dynamical vibrating beam problem, it follows that

$$\{f\} = \Big[[\hat{K}] - [\hat{M}_o]\Big]\{\delta\} \tag{4-1a}$$

or

$$\frac{L}{EI}\begin{Bmatrix} V_1L \\ M_1 \\ V_2L \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12(1-13\Psi) & & \text{Symmetric} & \\ 2(3-11\Psi) & 4(1-\Psi) & & \\ -6(2+9\Psi) & -(6+13\Psi) & 12(1-13\Psi) & \\ (6+13\Psi) & (2+3\Psi) & -2(3-11\Psi) & 4(1-\Psi) \end{bmatrix}\begin{Bmatrix} w_1/L \\ \theta_1 \\ w_2/L \\ \theta_2 \end{Bmatrix} \tag{4-1b}$$

where $\Psi = \dfrac{\rho A L^4 \omega^2}{420\,EI}$

The four matrix invariants of the $\Big[[\hat{K}] - [\hat{M}_o]\Big]$ matrix in equation (4-1b) are

$$\left.\begin{aligned} I_1 &= 32(1-10\Psi) \\ I_2 &= (60 - 7556\Psi + 22617\Psi^2) \\ I_3 &= -448\Psi(60 - 633\Psi + 133\Psi^2) \\ I_4 &= 735\Psi^2(\Psi-20)(7\Psi-12) \end{aligned}\right\} \tag{4-2}$$

The characteristic equation becomes

$$\lambda^4 - 32(1-10\Psi)\lambda^3 + (60 - 7556\Psi + 22617\Psi^2)\lambda^2 + 448\Psi(60 - 633\Psi + 133\Psi^2)\lambda + 735\Psi^2(\Psi-20)(7\Psi-12) = 0 \tag{4-3a}$$

or in quadratic factored form as

$$\left[\lambda^2 + (103\Psi - 30)\lambda + 21\Psi(\Psi-20)\right]\left[\lambda^2 + (217\Psi-2)\lambda + 35\Psi(7\Psi-12)\right] = 0 \tag{4-3b}$$

with the four roots determined as

$$\left.\begin{aligned}
{}^{(3)}\lambda_1 &= -\left(\frac{217}{2}\Psi - 1\right) - \sqrt{\left(\frac{217}{2}\Psi-1\right)^2 - 35\Psi(7\Psi-12)} \\
{}^{(3)}\lambda_2 &= -\left(\frac{103}{2}\Psi - 15\right) - \sqrt{\left(\frac{103}{2}\Psi-15\right)^2 - 21\Psi(\Psi-20)} \\
{}^{(3)}\lambda_3 &= -\left(\frac{217}{2}\Psi - 1\right) + \sqrt{\left(\frac{217}{2}\Psi-1\right)^2 - 35\Psi(7\Psi-12)} \\
{}^{(3)}\lambda_4 &= -\left(\frac{103}{2}\Psi - 15\right) + \sqrt{\left(\frac{103}{2}\Psi-15\right)^2 - 21\Psi(\Psi-20)}
\end{aligned}\right\} \tag{4-3c}$$

The eigenvalue matrix takes the form

$$\left[\Lambda_b^{(d)}\right] = \begin{bmatrix} {}^{(3)}\lambda_1 & & \text{Symmetric} & \\ 0 & {}^{(3)}\lambda_2 & & \\ 0 & 0 & {}^{(3)}\lambda_3 & \\ 0 & 0 & 0 & {}^{(3)}\lambda_4 \end{bmatrix} \tag{4-3d}$$

4.2 The Eigenvector Matrix

Utilizing equation (1-9), one obtains

$$\left[\left[\hat{K}\right] - \left[\hat{M}_o\right] - \lambda\left[I\right]\right]\left\{\delta\right\} = \left\{0\right\} \tag{4-4}$$

Substituting the four roots of λ individually into equation (4-4), the eigenvector matrix is constructed as

$$\left[U_b^{(d)}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} {}^{(3)}n_1/{}^{(3)}d_1 & {}^{(3)}n_3/{}^{(3)}d_2 & {}^{(3)}n_5/{}^{(3)}d_3 & {}^{(3)}n_6/{}^{(3)}d_4 \\ {}^{(3)}n_2/{}^{(3)}d_1 & {}^{(3)}n_4/{}^{(3)}d_2 & {}^{(3)}n_2/{}^{(3)}d_3 & {}^{(3)}n_4/{}^{(3)}d_4 \\ {}^{(3)}n_1/{}^{(3)}d_1 & {}^{(3)}n_3/{}^{(3)}d_2 & {}^{(3)}n_5/{}^{(3)}d_3 & {}^{(3)}n_6/{}^{(3)}d_4 \\ {}^{(3)}n_2/{}^{(3)}d_1 & {}^{(3)}n_4/{}^{(3)}d_2 & {}^{(3)}n_2/{}^{(3)}d_3 & {}^{(3)}n_4/{}^{(3)}d_4 \end{pmatrix} \qquad (4\text{-}5a)$$

where

$$\left.\begin{aligned}
{}^{(3)}n_1 &= \left\{ \left(1 + \frac{203}{2}\Psi\right) + \sqrt{\left(\frac{217}{2}\Psi - 1\right)^2 - 35\Psi(7\Psi - 12)} \right\} \\
{}^{(3)}n_2 &= 35\Psi \\
{}^{(3)}n_3 &= \left\{ \left(-6 + \frac{101}{3}\Psi\right) + \sqrt{\left(\frac{103}{3}\Psi - 10\right)^2 - \frac{28}{3}\Psi(\Psi - 20)} \right\} \\
{}^{(3)}n_4 &= 2(3\Psi - 4) \\
{}^{(3)}n_5 &= \left\{ \left(1 + \frac{203}{2}\Psi\right) - \sqrt{\left(\frac{217}{2}\Psi - 1\right)^2 - 35\Psi(7\Psi - 12)} \right\} \\
{}^{(3)}n_6 &= \left\{ \left(-6 + \frac{101}{3}\Psi\right) - \sqrt{\left(\frac{103}{3}\Psi - 10\right)^2 - \frac{28}{3}\Psi(\Psi - 20)} \right\} \\
{}^{(3)}d_1 &= \sqrt{\left\{ \left(1 + \frac{203}{2}\Psi\right) + \sqrt{\left(\frac{217}{2}\Psi - 1\right)^2 - 35\Psi(7\Psi - 12)} \right\}^2 + (35\Psi)^2}
\end{aligned}\right\} \qquad (4\text{-}5b)$$

$$^{(3)}d_2 = \sqrt{\left\{(-6+\tfrac{101}{3}\psi)+\sqrt{(\tfrac{103}{3}\psi-10)^2-\tfrac{28}{3}\psi(\psi-20)}\right\}^2+4(3\psi-4)^2}$$

$$^{(3)}d_3 = \sqrt{\left\{(1+\tfrac{203}{2}\psi)-\sqrt{(\tfrac{217}{2}\psi-1)^2-35\psi(7\psi-12)}\right\}^2+(35\psi)^2}$$

$$^{(3)}d_4 = \sqrt{\left\{(-6+\tfrac{101}{3}\psi)-\sqrt{(\tfrac{103}{3}\psi-10)^2-\tfrac{28}{3}\psi(\psi-20)}\right\}^2+4(3\psi-4)^2}$$

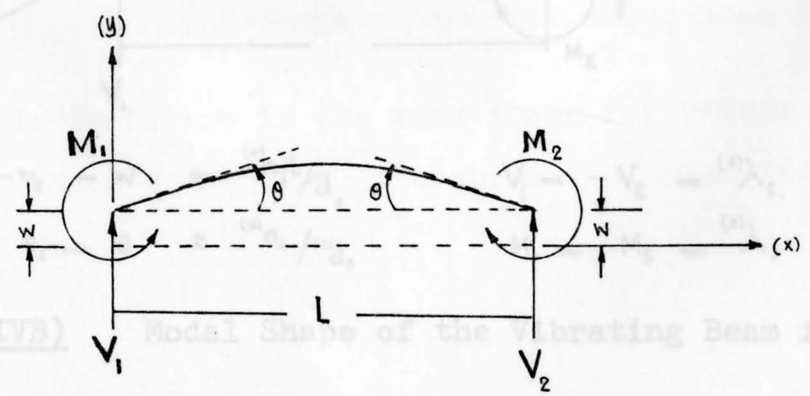
<u>Note</u>

$$\frac{^{(3)}n_1}{^{(3)}d_1} = \frac{^{(3)}n_2}{^{(3)}d_3}, \quad \frac{^{(3)}n_2}{^{(3)}d_1} = \frac{-^{(3)}n_5}{^{(3)}d_3}$$

$$\frac{^{(3)}n_3}{^{(3)}d_2} = \frac{-^{(3)}n_4}{^{(3)}d_4}, \quad \frac{^{(3)}n_4}{^{(3)}d_2} = \frac{^{(3)}n_6}{^{(3)}d_4}$$

4.3 <u>Solutions for the Moment, Shear Forces and Variations of Normal Mode Shapes</u>

For the eigenvalue $^{(3)}\lambda_1$, the normal mode shape together with moment and shear values are given in Figure (IVA).



$$w_1 = w_2 = w \approx \frac{{}^{(3)}n_1}{{}^{(3)}d_1} L \qquad V_1 = V_1 = {}^{(3)}\lambda_1 \frac{EI}{L^3} w$$

$$\theta_1 = -\theta_2 = \theta \approx \frac{{}^{(3)}n_2}{{}^{(3)}d_1} \qquad M_1 = -M_2 = {}^{(3)}\lambda_1 \frac{EI}{L} \theta$$

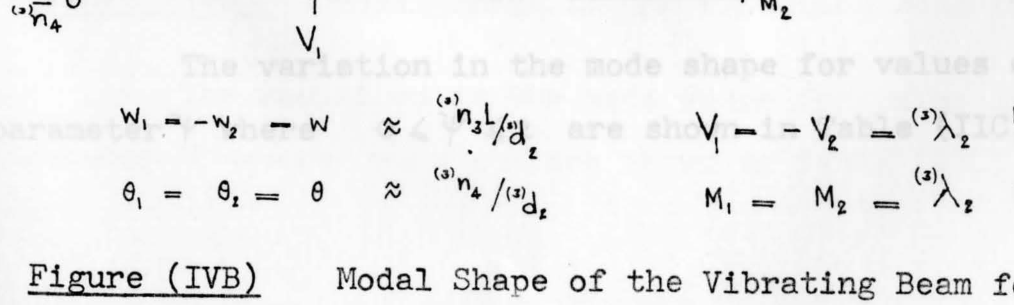
<u>Figure (IVA)</u> Modal Shape of the Vibrating Beam for ${}^{(3)}\lambda_1$

The variation in the mode shape for values of the parameter ψ where $0 < \psi < 2$, are shown in Table (IIA)

${}^{(3)}\delta_1$ \ ψ	0	0.001	0.005	0.0092	0.1	1.0	1.5	12/7	2
w_1/L	$1/\sqrt{2}$	0.7070	0.7059	0.7040	0.6989	0.69753	0.6975	0.6974	0.69736
θ_1	0	0.01122	0.0408	0.0582	0.1071	0.1158	0.11616	0.11624	0.11633
w_2/L	$1/\sqrt{2}$	0.7070	0.7059	0.7040	0.6989	0.69753	0.6975	0.6974	0.69736
θ_2	0	-0.01122	-0.0408	-0.0582	-0.1071	-0.1158	-0.11616	-0.11624	-0.11633

<u>Table (IIA)</u> Modal Shape Variation for ${}^{(3)}\lambda_1$

For the eigenvalue ${}^{(3)}\lambda_2$, the normal mode shape together with moment and shear values are given in Figure (IVB).



$$w_1 = -w_2 = w \approx {}^{(3)}n_3 L / {}^{(3)}d_2 \qquad V_1 = -V_2 = {}^{(3)}\lambda_2 \frac{EI}{L^3} w$$

$$\theta_1 = \theta_2 = \theta \approx {}^{(3)}n_4 / {}^{(3)}d_2 \qquad M_1 = M_2 = {}^{(3)}\lambda_2 \frac{EI}{L} \theta$$

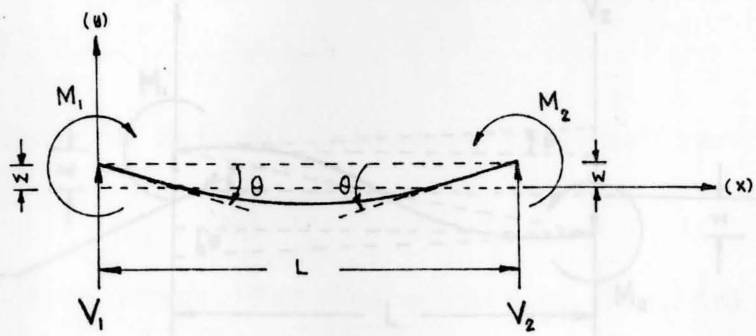
<u>Figure (IVB)</u> Modal Shape of the Vibrating Beam for ${}^{(3)}\lambda_2$

The variation in the mode shape for values of the parameter Ψ where $0 \leqslant \Psi \leqslant 21.0$ are shown in Table (IIB).

δ_2 \ Ψ	0	0.001	0.1	0.3	1.0	10.0	15.0	20.0	21.0
w_1/L	0.3162	0.3169	0.40765	0.6228	0.7066	0.7049	0.7047	0.70466	0.7046
θ_1	-0.6324	-0.63208	-0.57776	-0.3348	-0.0255	0.0550	0.0575	0.05872	0.05889
w_2/L	-0.3162	-0.3169	-0.4076	-0.6228	-0.7066	-0.7049	-0.7047	-0.70466	-0.7046
θ_2	-0.6324	-0.63208	-0.57776	-0.3348	-0.0255	0.0550	0.0575	0.05872	0.05889

Table (IIB) Modal Shape Variation for ${}^{(2)}\lambda_2$

For the eigenvalue ${}^{(3)}\lambda_3$, the normal mode shape together with moment and shear values are given in Figure (IVC).



$$w_1 = w_2 = w \approx -{}^{(3)}n_5 L / {}^{(3)}d_3 \qquad V_1 = V_2 = {}^{(3)}\lambda_3 \frac{EI}{L^3} w$$

$$\theta_1 = -\theta_2 = \theta \approx -{}^{(3)}n_2 / {}^{(3)}d_3 \qquad M_1 = -M_2 = {}^{(3)}\lambda_3 \frac{EI}{L} \theta$$

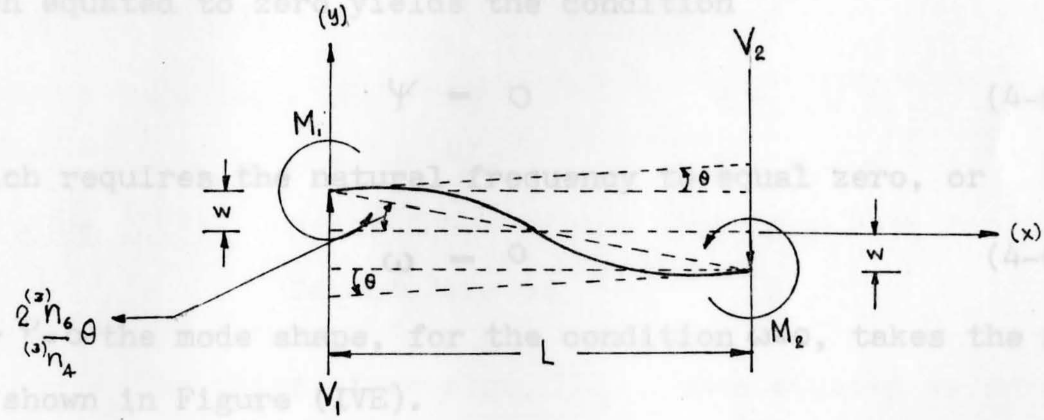
Figure (IVC) Modal Shape of the Vibrating Beam for ${}^{(2)}\lambda_3$

The variation in the mode shape for values of the parameter Ψ where $0 \leqslant \Psi \leqslant 2$ are shown in Table (IIC).

${}^{(3)}\delta_3$ \ ψ	0	0.001	0.005	0.0092	0.1	1.0	1.5	12/7	2.0
w_1/L	0	0.0112	0.0408	0.0582	0.1071	0.1158	0.1161	0.11624	0.11633
θ_1	$-1/\sqrt{2}$	-0.7070	-0.7059	-0.7040	-0.6989	-0.6975	-0.6975	-0.6974	-0.6974
w_2/L	0	0.0112	0.0408	0.0582	0.1071	0.1158	0.1161	0.11624	0.11633
θ_2	$1/\sqrt{2}$	0.7070	0.7059	0.7040	0.6989	0.6975	0.6975	0.6974	0.6974

Table (IIC) Modal Shape Variation for ${}^{(3)}\lambda_3$

For the eigenvalue ${}^{(3)}\lambda_4$, the normal mode shape together with moment and shear values are given in Figure (IVD).



$$w_1 = -w_2 = w \approx -{}^{(3)}n_6 L / {}^{(3)}d_4 \qquad V_1 = -V_2 = {}^{(3)}\lambda_4 \frac{EI}{L^3} w$$

$$\theta_1 = \theta_2 = \theta \approx -{}^{(3)}n_4 / {}^{(3)}d_4 \qquad M_1 = M_2 = {}^{(3)}\lambda_4 \frac{EI}{L}\theta$$

Figure (IVD) Modal Shape of the Vibrating Beam for ${}^{(3)}\lambda_4$

The variation in the mode shape for value of the parameter ψ where $0 \leq \psi \leq 2.0$ are shown in Table (IID).

$^{(3)}\delta_4$ \ Ψ	0	0.001	0.1	0.3	1.0	10.0	15.0	20.0	21.0
w_1/L	0.63245	0.63208	0.5777	0.334816	0.0255	0.0550	0.0575	0.05872	0.0588
θ_1	0.31622	0.316972	0.40767	0.622814	0.7066	−0.7049	−0.7047	−0.70466	−0.7046
w_2/L	−0.63245	−0.63208	−0.5777	−0.334816	−0.0255	−0.0550	−0.0575	−0.0587	−0.0588
θ_2	0.31622	0.316972	0.40767	0.622814	0.7066	−0.7049	−0.7047	−0.70466	−0.7046

<u>Table (IID)</u> Modal Shape Variation for $^{(2)}\lambda_4$

4.4 Zeros of the Eigenvalues

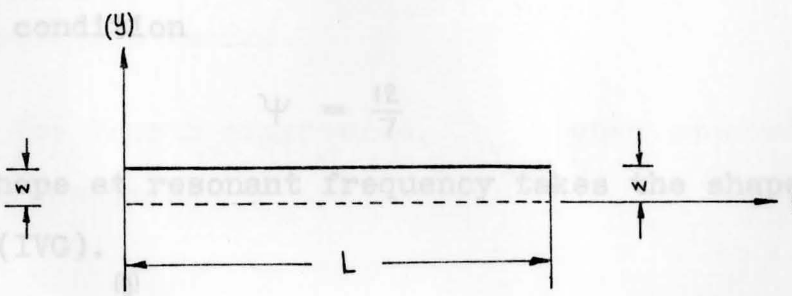
The first eigenvalue $^{(3)}\lambda_1 = -(\frac{217}{2}\Psi-1)-\sqrt{(\frac{217}{2}\Psi-1)^2-35\Psi(7\Psi-12)}$ when equated to zero yields the condition

$$\Psi = 0 \qquad (4\text{-}6a)$$

which requires the natural frequency to equal zero, or

$$\omega = 0 \qquad (4\text{-}6b)$$

For $\Psi=0$ the mode shape, for the condition $\omega=0$, takes the shape as shown in Figure (IVE).



$$w_1 = w_2 = w \approx L/\sqrt{2} \qquad V_1 = V_2 = 0$$

$$\theta_1 = \theta_2 = \theta = 0 \qquad M_1 = M_2 = 0$$

<u>Figure (IVE)</u> Modal Shape of the Vibrating Beam for the First Zero of $^{(2)}\lambda_1$

The second eigenvalue, ${}^{(2)}\lambda_2$, when equated to zero, yields the condition

$$\Psi = 0 \tag{4-7a}$$

which requires the natural frequency to equal zero or

$$\omega = 0 \tag{4-7b}$$

For $\Psi = 0$ the mode shape takes the shape as shown in Figure (IVF).

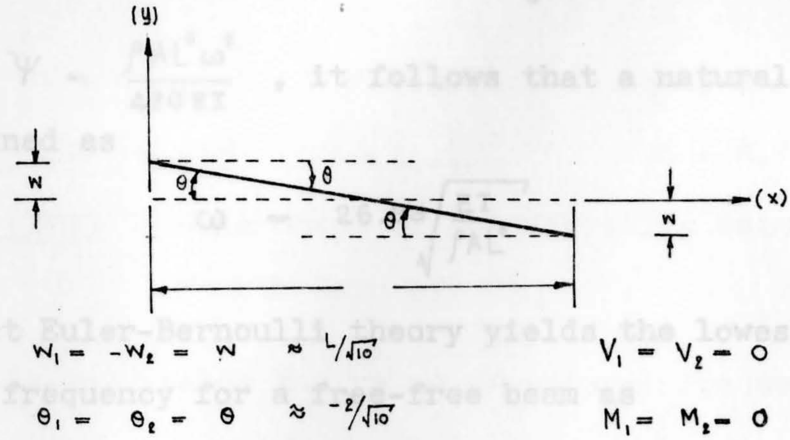
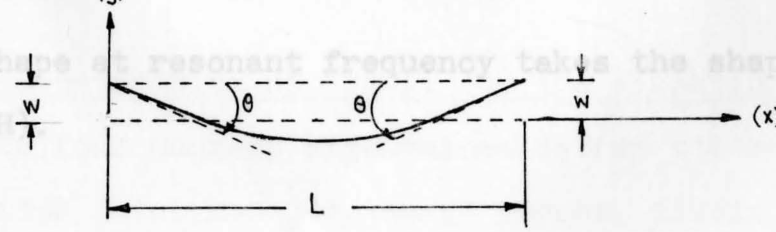


Figure (IVF) Modal Shape of the Vibrating Beam for the Second Zero of ${}^{(2)}\lambda_2$

The third eigenvalue, ${}^{(3)}\lambda_3$, when equated to zero, yields the condition

$$\Psi = \frac{12}{7} \tag{3-8a}$$

The mode shape at resonant frequency takes the shape shown in Figure (IVG).



$w_1 = w_2 = w \approx 0.1163L \qquad V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta \approx -0.6974 \qquad M_1 = M_2 = 0$

Figure (IVG) Modal Shape of the Vibrating Beam for the Third Zero of ${}^{(3)}\lambda_3$

Noting $\Psi = \dfrac{\rho A L^4 \omega^2}{420 EI}$, it follows that a natural frequency is obtained as

$$\omega = 26.83 \sqrt{\frac{EI}{\rho A L^4}} \qquad (3\text{-}8b)$$

the exact Euler-Bernoulli theory yields the lowest value of natural frequency for a free-free beam as

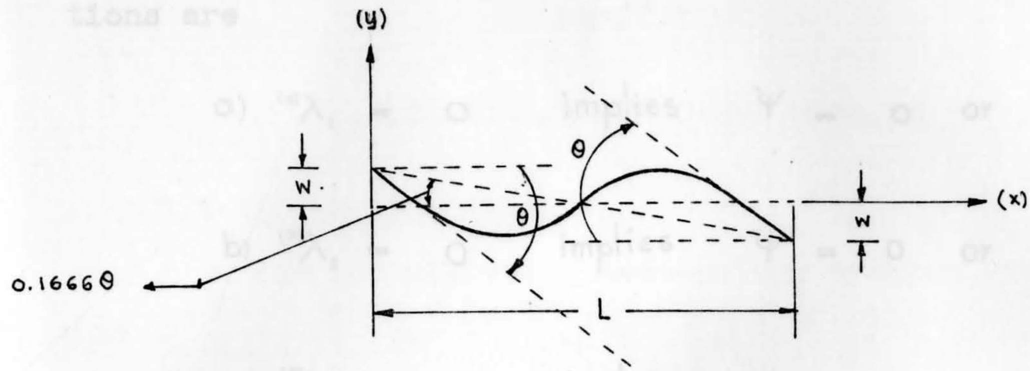
$$\omega = 22.3729 \sqrt{\frac{EI}{\rho A L^4}} \qquad (4\text{-}8c)$$

The value of ω obtained from the matrix formulation given by equation (4-8b) is greater by 19.93% than the exact value given in equation (4-8c).

The fourth eigenvalue, ${}^{(3)}\lambda_4$, when equated to zero, yields the condition

$$\Psi = 20.0 \qquad (4\text{-}9a)$$

The mode shape at resonant frequency takes the shape shown in Figure (IVH).



$w_1 = -w_2 = w \quad \approx 0.05872L \qquad V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta \quad \approx -0.70466 \qquad M_1 = M_2 = 0$

Figure (IVH) Mode Shape of the Vibrating Beam for the Fourth Zero of ${}^{(3)}\lambda_4$

Equation (4-ga) yields the value of natural frequency as

$$\omega = 91.65\sqrt{\frac{EI}{\rho AL^4}} \qquad (4\text{-}9b)$$

The exact Euler-Bernoulli theory yields a value of natural frequency for a free-free in its second mode as

$$\omega = 61.66\sqrt{\frac{EI}{\rho AL^4}} \qquad (4\text{-}9c)$$

The value of ω obtained from the matrix formulation given by equation (4-9b) is greater by 48.61% than the exact value given in equation (4-9c).

4.5 Interpretation of Result for the Vibrating Beam

The four nonzero eigenvalues define the mode shapes with associated joint moments, shear forces, displacements, and rotations. Equating to zero the four nonzero value λ

yields the conditions of natural frequency. The four conditions are

a) ${}^{(3)}\lambda_1 = 0$ implies $\psi = 0$ or $\omega = 0$

b) ${}^{(3)}\lambda_2 = 0$ implies $\psi = 0$ or $\omega = 0$

c) ${}^{(3)}\lambda_3 = 0$ implies $\psi = \frac{12}{7}$ or $\omega = 26.83\sqrt{\frac{EI}{\rho AL^4}}$

d) ${}^{(3)}\lambda_4 = 0$ implies $\psi = 20.0$ or $\omega = 91.65\sqrt{\frac{EI}{\rho AL^4}}$

For condition a), the mode shape corresponds to a rigid body translation with zero natural frequency, and zero joint moments and shears. In condition b), the mode shape corresponds to a rigid body rotation with zero natural frequency, and zero moments and shears. Condition c) produces a mode shape at natural frequency which corresponds to the first mode (i.e. n=1) of a free-free beam. Condition d) produces a mode shape at natural frequency which corresponds to the second mode (i.e. n=2) of a free-free beam.

CHAPTER V

VIBRATING BEAM-COLUMN PROBLEM

5.1 Eigenvalue Matrix

For the vibrating beam-column problem, it follows that

$$\{f\} = \Big[[\hat{K}] - [\hat{G}_o] - [\hat{M}_o]\Big]\{\delta\} \tag{5-1a}$$

$$\frac{L}{EI}\begin{Bmatrix} V_1 L \\ M_1 \\ V_2 L \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12(1-3\phi-13\psi) & & \text{symmetric} & \\ (6-3\phi-22\psi) & 4(1-\phi-\psi) & & \\ -6(2-6\phi+9\psi) & -(6-3\phi+13\psi) & 12(1-3\phi-13\psi) & \\ (6-3\phi+13\psi) & (2+\phi+3\psi) & -(6-3\phi-22\psi) & 4(1-\phi-\psi) \end{bmatrix} \begin{Bmatrix} w_1/L \\ \theta_1 \\ w_2/L \\ \theta_2 \end{Bmatrix}$$ (5-

with $\phi = \dfrac{PL^2}{30EI}$ and $\psi = \dfrac{\rho A L^4 \Omega^2}{420\,EI}$

The four matrix invariants of the $\Big[[\hat{K}] - [\hat{G}_o] - [\hat{M}_o]\Big]$ matrix in equation (5-1b) are

$$\left.\begin{aligned} I_1 &= (32 - 80\phi - 320\psi) \\ I_2 &= (60 - 660\phi - 7556\psi + 355\phi^2 + 18110\phi\psi + 22617\psi^2 \\ I_3 &= (-720\phi - 13440\psi + 2160\phi^2 + 143760\phi\psi + 141792\psi^2 \\ &\quad - 900\phi^3 - 119160\phi^2\psi - 185220\phi\psi^2 - 29792\psi^3) \\ I_4 &= (151200\phi\psi + 176400\psi^2 - 453600\phi^2\psi - 642600\phi\psi^2 \\ &\quad - 11720\psi^3 + 189000\phi^3\psi + 327600\phi^2\psi^2 + 88200\phi\psi^3 \\ &\quad + 5145\psi^4 \end{aligned}\right\} \tag{5-2}$$

The characteristic equation in quadratic factored form becomes

$$\left[\lambda^2 + (217\psi + 5\phi - 2)\lambda + 35\psi(30\phi - 12 + 7\psi)\right]\left[\lambda^2 + (103\psi + 75\phi - 30)\lambda + 3\left\{7\psi^2 + 10(9\phi - 14)\psi + 60\phi(\phi - 2)\right\}\right] = 0 \tag{5-3a}$$

with the four roots determined as

$$\left.\begin{aligned}
{}^{(4)}\lambda_1 &= -\left(\tfrac{217}{2}\psi + \tfrac{5}{2}\phi - 1\right) - \sqrt{\left(\tfrac{217}{2}\psi + \tfrac{5}{2}\phi - 1\right)^2 - 35\psi(30\phi - 12 + 7\psi)} \\
{}^{(4)}\lambda_2 &= -\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right) - \sqrt{\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right)^2 - 3\left\{7\psi^2 + 10(9\phi - 14)\psi + 60\phi(\phi - 2)\right\}} \\
{}^{(4)}\lambda_3 &= -\left(\tfrac{217}{2}\psi + \tfrac{5}{2}\phi - 1\right) + \sqrt{\left(\tfrac{217}{2}\psi + \tfrac{5}{2}\phi - 1\right)^2 - 35\psi(30\phi - 12 + 7\psi)} \\
{}^{(4)}\lambda_4 &= -\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right) + \sqrt{\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right)^2 - 3\left\{7\psi^2 + 10(9\phi - 14)\psi + 60\phi(\phi - 2)\right\}}
\end{aligned}\right\} \tag{5-3b}$$

The eigenvalue matrix takes the form

$$\left[\Lambda^{(4)}_{bc}\right] = \begin{bmatrix} {}^{(4)}\lambda_1 & & \text{Symmetric} & \\ 0 & {}^{(4)}\lambda_2 & & \\ 0 & 0 & {}^{(4)}\lambda_3 & \\ 0 & 0 & 0 & {}^{(4)}\lambda_4 \end{bmatrix} \tag{5-3c}$$

5.2 The Eigenvector Matrix

Utilizing equation (1-9), one obtains

$$\left[\left[\hat{K}\right] - \left[\hat{G}_0\right] - \left[\hat{M}_0\right]\right] - \lambda\left[I\right]\Big]\left\{\delta\right\} = \left\{0\right\} \tag{5-4}$$

Substituting the four of λ individually into equation (5-4), the eigenvector matrix is constructed as

$$\left[U_{bc}^{(d)}\right] = \frac{1}{\sqrt{2}} \begin{bmatrix} {}^{(4)}n_1/{}^{(4)}d_1 & {}^{(4)}n_3/{}^{(4)}d_2 & {}^{(4)}n_5/{}^{(4)}d_3 & {}^{(4)}n_6/{}^{(4)}d_4 \\ {}^{(4)}n_2/{}^{(4)}d_1 & {}^{(4)}n_4/{}^{(4)}d_2 & {}^{(4)}n_2/{}^{(4)}d_3 & {}^{(4)}n_4/{}^{(4)}d_4 \\ {}^{(4)}n_1/{}^{(4)}d_1 & {}^{(4)}n_3/{}^{(4)}d_2 & {}^{(4)}n_5/{}^{(4)}d_3 & {}^{(4)}n_6/{}^{(4)}d_4 \\ {}^{(4)}n_2/{}^{(4)}d_1 & {}^{(4)}n_4/{}^{(4)}d_2 & {}^{(4)}n_2/{}^{(4)}d_3 & {}^{(4)}n_4/{}^{(4)}d_4 \end{bmatrix} \tag{5-5a}$$

where

$$\left.\begin{aligned}
{}^{(4)}n_1 &= \left\{\left(1-\frac{5}{2}\phi+\frac{203}{2}\psi\right)+\sqrt{\left(\frac{5}{2}\phi+\frac{217}{2}\psi-1\right)^2-\left(1050\,\phi\psi-420\psi+245\psi^2\right)}\right\} \\
{}^{(4)}n_2 &= 35\psi \\
{}^{(4)}n_3 &= \left\{\left(6-23\phi-\frac{101}{3}\psi\right)-\sqrt{\left(10-25\phi-\frac{103}{2}\psi\right)^2-\left(80\phi^2-160\phi+120\,\phi\psi-\frac{560}{3}\psi+\frac{28}{3}\psi^2\right)}\right\} \\
{}^{(4)}n_4 &= \left(8-4\phi-6\psi\right) \\
{}^{(4)}n_5 &= \left\{\left(1-\frac{5}{2}\phi+\frac{203}{2}\psi\right)-\sqrt{\left(\frac{5}{2}\phi+\frac{217}{2}\psi-1\right)^2-\left(1050\,\phi\psi-420\psi+245\psi^2\right)}\right\} \\
{}^{(4)}n_6 &= \left\{\left(6-23\phi-\frac{101}{3}\psi\right)+\sqrt{\left(10-25\phi-\frac{103}{2}\psi\right)^2-\left(80\phi^2-160\phi+120\,\phi\psi-\frac{560}{3}\psi+\frac{28}{3}\psi^2\right)}\right\} \\
d_1 &= \sqrt{\left\{\left(1-\frac{5}{2}\phi+\frac{203}{2}\psi\right)+\sqrt{\left(1-\frac{5}{2}\phi-\frac{217}{2}\psi\right)^2-\left(1050\,\phi\psi-420\psi+245\psi^2\right)}\right\}^2+\left(35\psi\right)^2}
\end{aligned}\right\} \tag{5-5b}$$

$$^{(4)}d_2 = \sqrt{\left\{(6-23\phi-\tfrac{101}{3}\psi)-\sqrt{(10-25\phi-\tfrac{103}{3}\psi)^2-(80\phi^2-160\phi+120\phi\psi-\tfrac{560}{3}\psi+\tfrac{28}{3}\psi^2)}\right\}^2+(8-4\phi-6\psi)^2}$$

$$^{(4)}d_3 = \sqrt{\left\{(1-\tfrac{5}{2}\phi+\tfrac{203}{2}\psi)-\sqrt{(1-\tfrac{5}{2}\phi-\tfrac{217}{2}\psi)^2-(1050\phi\psi-420\psi+245\psi^2)}\right\}^2+(35\psi)^2}$$

$$^{(4)}d_4 = \sqrt{\left\{(6-23\phi-\tfrac{101}{3}\psi)+\sqrt{(10-25\phi-\tfrac{103}{3}\psi)^2-(80\phi^2-160\phi+120\phi\psi-\tfrac{560}{3}\psi+\tfrac{28}{3}\psi^2)}\right\}^2+(8-4\phi-6\psi)^2}$$

Note

$$\frac{^{(4)}n_1}{^{(4)}d_1} = \frac{^{(4)}n_2}{^{(4)}d_3}, \qquad \frac{^{(4)}n_2}{^{(4)}d_1} = \frac{-^{(4)}n_5}{^{(4)}d_3}$$

$$\frac{^{(4)}n_3}{^{(4)}d_2} = \frac{-^{(4)}n_4}{^{(4)}d_4}, \qquad \frac{^{(4)}n_4}{^{(4)}d_2} = \frac{^{(4)}n_6}{^{(4)}d_4}$$

5.3 Zeros of the Eigenvalues

The first eigenvalue

$$^{(4)}\lambda_1 = -(\tfrac{217}{2}\psi+\tfrac{5}{2}\phi-1)-\sqrt{(\tfrac{217}{2}\psi+\tfrac{5}{2}\phi-1)^2-35\psi(30\phi-12+7\psi)},$$

when equated to zero, yields the condition

either $\qquad \psi = 0 \qquad$ (5-6a)

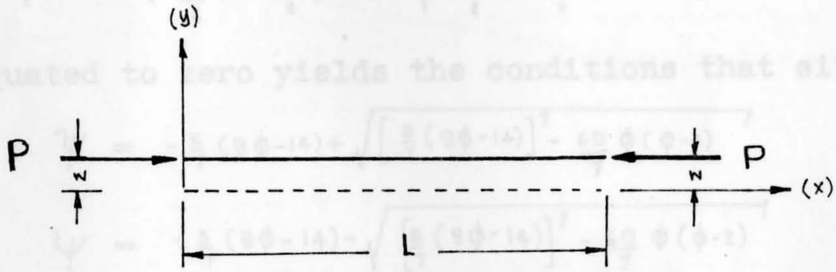
or $\qquad 3\phi-12+7\psi = 0 \qquad$ (5-6b)

together with the constraints that if $^{(4)}\lambda_1 = 0$, then

$$\left(\frac{217}{2}\psi + \frac{5}{2}\phi \right) \leqslant 1 \qquad (5\text{-}6c)$$

Thus, as ϕ increases over the range $0 \leqslant \phi \leqslant \frac{2}{5}$, it follows that ψ decreases. It is determined, by direct substitution, that equation (5-6a) satisfies equation (5-6c), and (5-6d) simultaneously and thus satisfies the condition ${}^{(4)}\lambda_1 = 0$. It is further determined that equation (5-6b) does not satisfy the inequality of equation (5-6c) and hence does not yield the condition ${}^{(4)}\lambda_1 = 0$. Thus, only the root $\psi = 0$ is applicable in this case.

The mode shape takes the shape shown in Figure (VA) which is identical to that given in Figure (IVE).



$$w_1 = w_2 = w \approx L/\sqrt{2} \qquad V_1 = V_2 = 0$$

$$\theta_1 = \theta_2 = \theta = 0 \qquad M_1 = M_2 = 0$$

Figure (VA) Modal Shape of the Vibrating Beam-Column for the First Zero of ${}^{(4)}\lambda_1$

The variation in the mode shape for values of the parameter ϕ where $0 \leq \phi \leq \frac{2}{5}$ in condition $\psi = 0$ shown in Table (IIIA) shows that the mode shape does not change as ϕ increases.

ϕ	0	1/10	2/10	3/10	4/10
${}^{(4)}\delta_1$ \ ψ	0	0	0	0	0
w_1/L	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
θ_1	0	0	0	0	0
w_2/L	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
θ_2	0	0	0	0	0

Table (IIIA) Modal Shape Variation for the First Zero of ${}^{(4)}\lambda_1$

The second eigenvalue

$${}^{(4)}\lambda_2 = -\left(\frac{103}{2}\psi + \frac{75}{2}\phi - 15\right) - \sqrt{\left(\frac{103}{2}\psi + \frac{75}{2}\phi - 15\right)^2 - 3\left\{7\psi^2 + 10(9\phi - 14)\psi + 60\phi(\phi - 2)\right.}\,,$$

when equated to zero yields the conditions that either

$$\psi = -\frac{5}{7}(9\phi - 14) + \sqrt{\left[\frac{5}{7}(9\phi - 14)\right]^2 - \frac{60}{7}\phi(\phi - 2)} \qquad \text{(5-7a)}$$

or

$$\psi = -\frac{5}{7}(9\phi - 14) - \sqrt{\left[\frac{5}{7}(9\phi - 14)\right]^2 - \frac{60}{7}\phi(\phi - 2)} \qquad \text{(5-7b)}$$

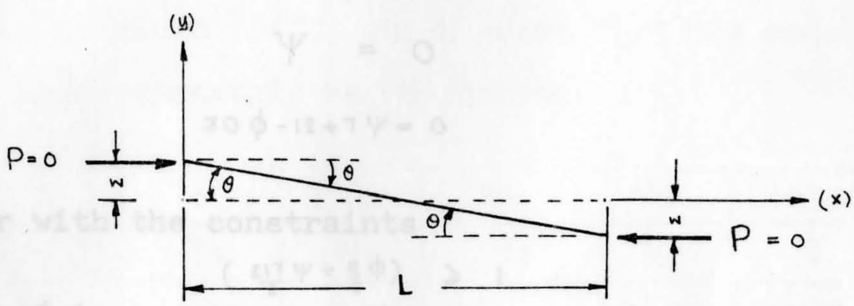
together with the constraints

$$\left(\frac{103}{2}\psi + \frac{75}{2}\phi\right) < 15 \qquad \text{(5-7c)}$$

Thus, as ϕ increases over the range $0 \leq \phi \leq \frac{2}{5}$, it follows that ψ decreases. (5-7d)

It is determined by direct substitution that equation (5-7b) satisfies equation (5-7c), and (5-7d) simultaneously only when $\phi = \psi = 0$ and thus satisfies the condition ${}^{(4)}\lambda_2 = 0$. It is further determined that equation (5-7a) does not satisfy the inequality of equation (5-7c) and hence does not yield the condition ${}^{(4)}\lambda_2 = 0$. Thus, only the root yielding $\psi = \phi = 0$ is applicable in this case.

The mode shape takes the shape shown in Figure (VB) which is identical to that given in Figure (IIB), Figure (IIIE), and Figure (IVF).



$w_1 = -w_2 = w \approx L/\sqrt{10}$ $\qquad V_1 = V_2 = 0$

$\theta_1 = \theta_2 = \theta \approx 2/\sqrt{10}$ $\qquad M_1 = M_2 = 0$

Figure (VB) Mode Shape of the Vibrating Beam-Column for the Second Zero of ${}^{(4)}\lambda_2$

The mode shape for the values of the parameters $\phi = 0$ and $\psi = 0$ is shown in Table (IIIB).

ψ	ϕ	w_1/L	θ_1	w_2/L	θ_2
0	0	$1/\sqrt{10}$	$-2/\sqrt{10}$	$-1/\sqrt{10}$	$-2/\sqrt{10}$

Table (IIIB) Mode Shape Variation for the Second Zero of ${}^{(4)}\lambda_2$

The third eigenvalue

$$^{(4)}\lambda_3 = -\left(\frac{217}{2}\psi + \frac{5}{2}\phi - 1\right) + \sqrt{\left(\frac{217}{2}\psi + \frac{5}{2}\phi - 1\right)^2 - 35\psi(30\phi - 12 + 7\psi)}\ ,$$

when equated to zero yields the condition

either $$\psi = 0 \qquad (5\text{-}8a)$$

or $$30\phi - 12 + 7\psi = 0 \qquad (5\text{-}8b)$$

together with the constraints

$$\left(\frac{217}{2}\psi + \frac{5}{2}\phi\right) \geq 1 \qquad (5\text{-}8c)$$

Thus, as ϕ increases over the range $0 \leq \phi < \frac{2}{5}$, it follows that ψ decreases. (5-8d)

It is determined by direct substitution that equation (5-8b) satisfies equation (5-8c) and (5-8d) simultaneously and thus satisfies the condition ${}^{(4)}\lambda_3 = 0$. It is further determined that equation (5-8a) does not satisfy the inequality of equation (5-8c) and hence does not yield the condition ${}^{(4)}\lambda_3 = 0$. Thus, only the root $\psi = \frac{12 - 30\phi}{7}$ is applicable in this case.

The mode shape takes the shape shown in Figure (VC) which is identical to that given in Figure (IVF).

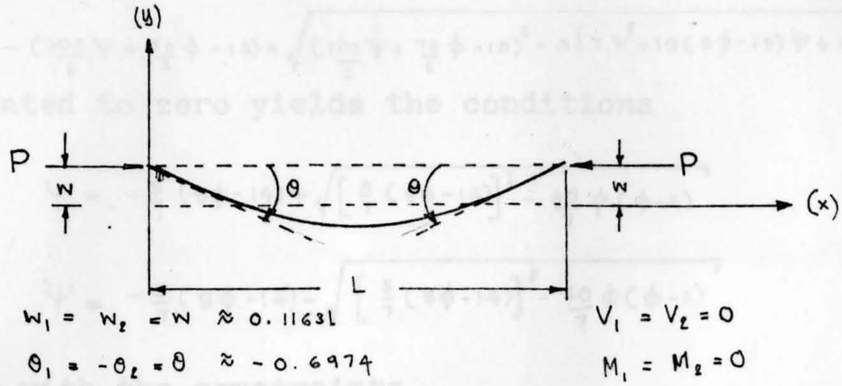


Figure (VC) Mode Shape of the Vibrating Beam-Column for the Third Zero of ${}^{(4)}\lambda_3$

The variation in the mode shape for values of the parameter ϕ where $0 \leqslant \phi \leqslant \frac{2}{5}$ in condition $\psi = \frac{12 - 30\phi}{7}$ are shown in Table (IIIC) which shows that the mode shape changes proportionately as ϕ increases

ϕ	0	1/10	2/10	3/10	4/10
${}^{(4)}\delta_3$ \ ψ	12/7	9/7	6/7	3/7	0
w_1/L	0.11624	0.08718	0.05812	0.02905	0
θ_1	-0.6974	-0.70171	-0.70471	-0.7065	-0.7071
w_2/L	0.11624	0.08718	0.05812	0.02905	0
θ_2	0.6974	0.70171	0.70471	0.7065	0.7071

Table (IIIC) Mode Shape Variations for the Third Zero of ${}^{(4)}\lambda_3$

The fourth eigenvalue

$$ {}^{(4)}\lambda_4 = -\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right) + \sqrt{\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi - 15\right)^2 - 3\left\{7\psi^2 + 10(9\phi - 14)\psi + 60\phi(\phi - 2)\right\}} \;, $$

when equated to zero yields the conditions

either $$\psi = -\tfrac{5}{7}(9\phi - 14) + \sqrt{\left[\tfrac{5}{7}(9\phi - 14)\right]^2 - \tfrac{60}{7}\phi(\phi - 2)} \qquad (5\text{-}9a)$$

or $$\psi = -\tfrac{5}{7}(9\phi - 14) - \sqrt{\left[\tfrac{5}{7}(9\phi - 14)\right]^2 - \tfrac{60}{7}\phi(\phi - 2)} \qquad (5\text{-}9b)$$

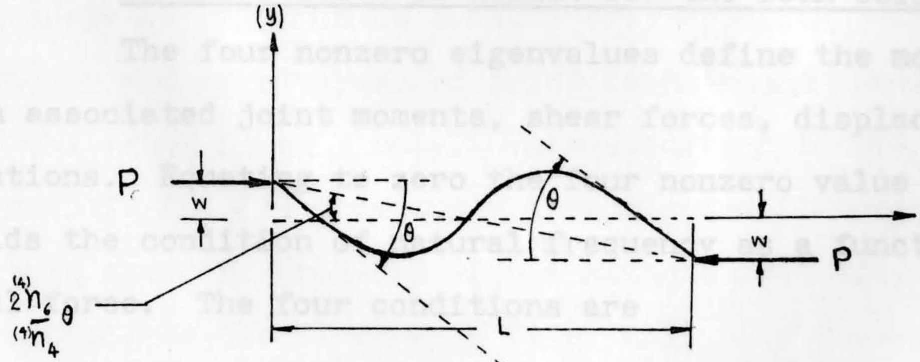
together with the constraints

$$\left(\tfrac{103}{2}\psi + \tfrac{75}{2}\phi\right) > 15 \qquad (5\text{-}9c)$$

Thus, as ϕ increases over the range $0 < \phi < 2.0$, it follows that ψ decreases. (5-9d)

It is determined by direct substitution that equation (5-9a) satisfies equation (5-9c), and (5-8d) simultaneously and thus satisfies the condition ${}^{(4)}\lambda_4 = 0$. It is further determined that equation (5-9b) does not satisfy the inequality of equation (5-9c), and hence does not yield the condition ${}^{(4)}\lambda_4 = 0$. Thus, only the root $\psi = -\frac{5}{7}(9\phi - 14) + \sqrt{\left[\frac{5}{7}(9\phi - 14)\right]^2 - \frac{60}{7}\phi(\phi - 2)}$ is applicable in this case.

The mode shape takes the shape shown in Figure (VD).



$$w_1 = -w_2 = w \approx {}^{(4)}n_6 L/{}^{(4)}d_4 \qquad V_1 = V_2 = 0$$

$$\theta_1 = \theta_2 = \theta \approx {}^{(4)}n_4/{}^{(4)}d_4 \qquad M_1 = M_2 = 0$$

Figure (VD) Mode Shape of the Vibrating Beam-Column for the Fourth Zero of ${}^{(4)}\lambda_4$

The variation in the mode shape for values of the parameter ϕ where $0 \leqslant \phi \leqslant 2.0$ are shown in Table (IIID).

ϕ	0	0.1	0.5	1.0	1.5	1.9	2.0
${}^{(4)}\delta_4$ \ ψ	20.0	18.80091	14.029642	8.1894942	2.917635	0.341420	0
w_1/L	0.05872	0.058501	0.05727396	0.0540398	0.0430188	0.01184268	0
θ_1	-0.70466	-0.7046826	-0.7047834	-0.705038	-0.705796	-0.7070076	$-1/\sqrt{2}$
w_2/L	-0.0587	-0.058501	-0.0572739	-0.054039	-0.0430188	-0.01184268	-0
θ_2	-0.7046	-0.7046826	-0.7047834	-0.705038	-0.705796	-0.7070076	$-1/\sqrt{2}$

Table (IIID) Mode Shape Variations for the Fourth Zero of ${}^{(4)}\lambda_4$

5.4 Interpretation of Result for the Beam-Column

The four nonzero eigenvalues define the mode shapes with associated joint moments, shear forces, displacement and rotations. Equating to zero the four nonzero value of λ, yields the condition of natural frequency as a function of axial force. The four conditions are

a) ${}^{(4)}\lambda_1 = 0$ implies $\psi = 0$ when $0 \leq \phi \leq \frac{2}{5}$

b) ${}^{(4)}\lambda_2 = 0$ implies $\psi = 0$ when $\phi = 0$

c) ${}^{(4)}\lambda_3 = 0$ implies $\psi = \frac{12-30\phi}{7}$ when $0 < \phi < \frac{2}{5}$

d) ${}^{(4)}\lambda_4 = 0$ implies $\psi = -\frac{5}{7}(9\phi - 14) + \sqrt{\left[\frac{5}{7}(9\phi-14)\right]^2 - \frac{60}{7}\phi(\phi-2)}$

when $0 < \phi < 2$

For condition a), the mode shape corresponds to a rigid body translation with zero natural frequency and zero values of joint moments and shear forces with the parameter variation $0 \leqslant \phi \leqslant \frac{2}{5}$. This condition is the same as in Cases I, II and III previously investigated. For condition b), the mode shape corresponds to a rigid body rotation with zero natural frequency and zero values of joint moments and shear forces with the value of $\phi = 0$ only. This condition is the same for Cases I, II and III previously investigated. Condition c), the mode shape is produced with a natural frequency which corresponds to the first mode (i.e. n=1) of a free-free beam-column. For this condition, the mode shape changes proportionately as ϕ increases over the range $0 \leqslant \phi \leqslant \frac{2}{5}$, with a simultaneous decrease in natural frequency. For $\phi = 0$, this case simplifies to that of Case III, Condition c), Chapter IV. For $\Psi = 0$, this case simplifies to that of Case II, Condition b), Chapter III. For condition d), the mode shape is produced with a natural frequency which corresponds to the second mode (i.e. n=2) of a free-free beam-column. For this condition, the mode shape changes as ϕ increases with a simultaneous decrease in natural frequency. For $\phi = 0$, this case simplifies to that of Case III, Condition d), Chapter IV. For $\Psi = 0$, the case simplifies to that of Case II, Condition c), Chapter III.

CHAPTER VI

DISCUSSION AND CONCLUSION

6.1 Discussion

A summary of the normal mode shapes for Cases I, II, III, and IV for λ_i's $i = 1, 2, 3, 4$ are shown in Figure (VIA). In general, four deformed mode shapes are defined for each case except for the beam-bending problem with one rigid body translational mode shape, and one rigid body rotational mode shape, and the beam-column bending problem with one rigid body translational mode shape.

The zeros of the λ_i's $i = 1, 2, 3, 4$ produce mode shapes for the four cases as summarized in Figure (VIB). For all four cases, two rigid body mode shapes exist for each case, one a rigid body translational mode shape and the other a rigid body rotational mode shape.

The zeros of the eigenvalues in Cases II, III and IV produce approximate values of critical buckling load of column, natural frequency of beam and resonant frequency of a beam-column, respectively. These values are compared with the theoretical values as given by the exact Euler-Bernoulli theory in Table (IVA).

	Boundary Condition		Critical Buckling Load		Difference %
			Exact Theory	Approximate Matrix Solution	
COLUMN	Simply supported both ends, $n=1$	$P_{cr}^{(1)}$	$\pi^2 \frac{EI}{L^2}$	$12 \frac{EI}{L^2}$	+21.86
	Simply supported both end, $n=2$	$P_{cr}^{(2)}$	$4\pi^2 \frac{EI}{L^2}$	$60 \frac{EI}{L^2}$	+52.0
	Boundary Condition		Natural Frequencies		Difference %
			Exact Theory	Approximate Matrix Solution	
VIBRATING BEAM	Free-free both ends, $n=1$	ω_1	$22.37\sqrt{\frac{EI}{\rho AL^4}}$	$26.83\sqrt{\frac{EI}{\rho AL^4}}$	+ 19.93
	Free-free both ends, $n=2$	ω_2	$61.66\sqrt{\frac{EI}{\rho AL^4}}$	$91.65\sqrt{\frac{EI}{\rho AL^4}}$	+ 48.61

Table (IVA) Summary of Numerical Results for Critical Buckling Loads and Natural Frequencies

For the vibrating beam-column, the relationships between the natural frequency of free vibration and the axial force are shown in Figure (VIC). The end points of the two curves shown in Figure (VIC) correspond to the summary conditions of Table (IVA). It should be noted that all values of critical buckling load and natural frequency obtained using the stiffness matrix approach are greater than those given by the exact theory.

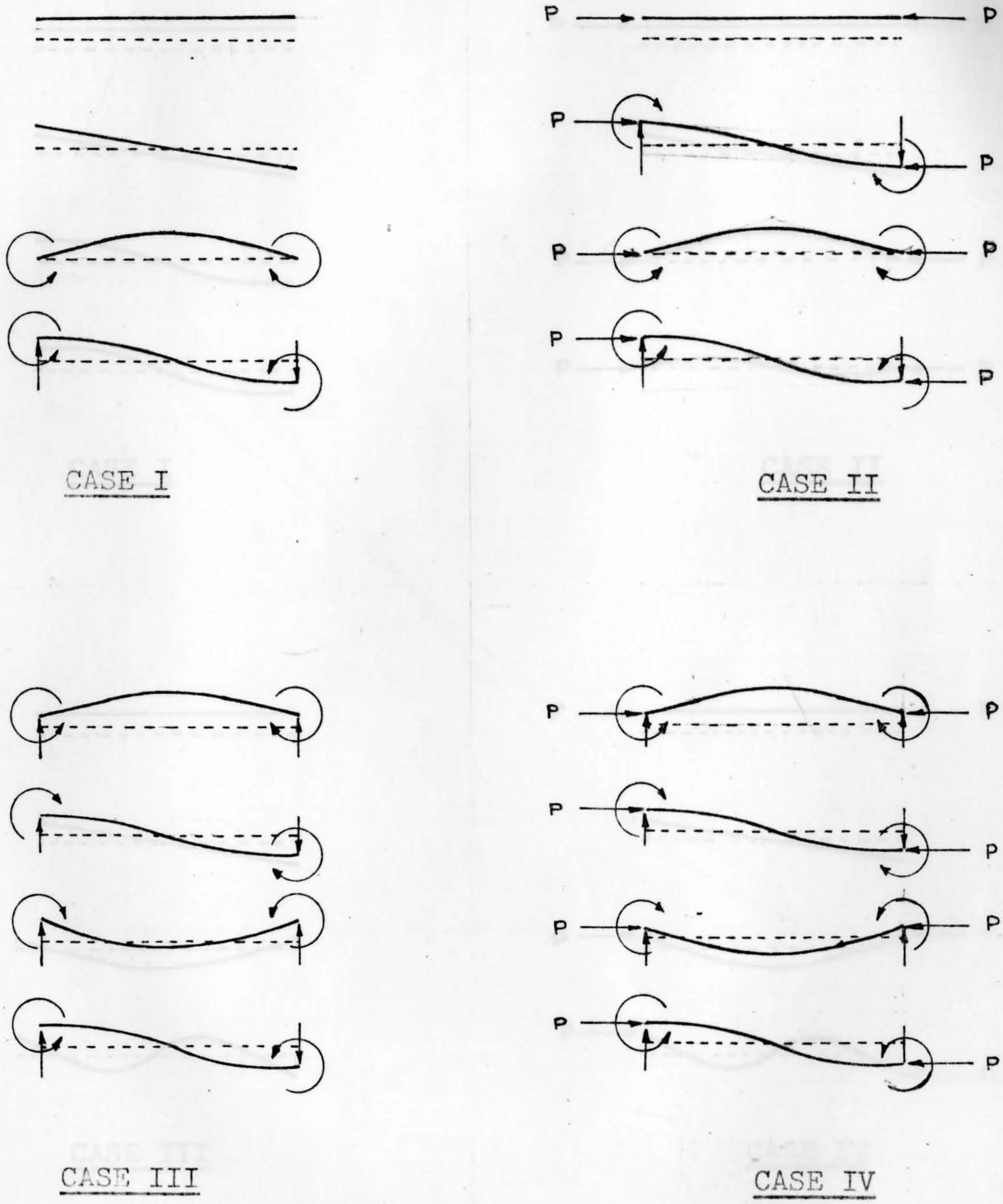


Figure (VIA) Summary of the Normal Mode Shapes for $\lambda_i \neq 0 \quad i = 1, 2, 3, 4$

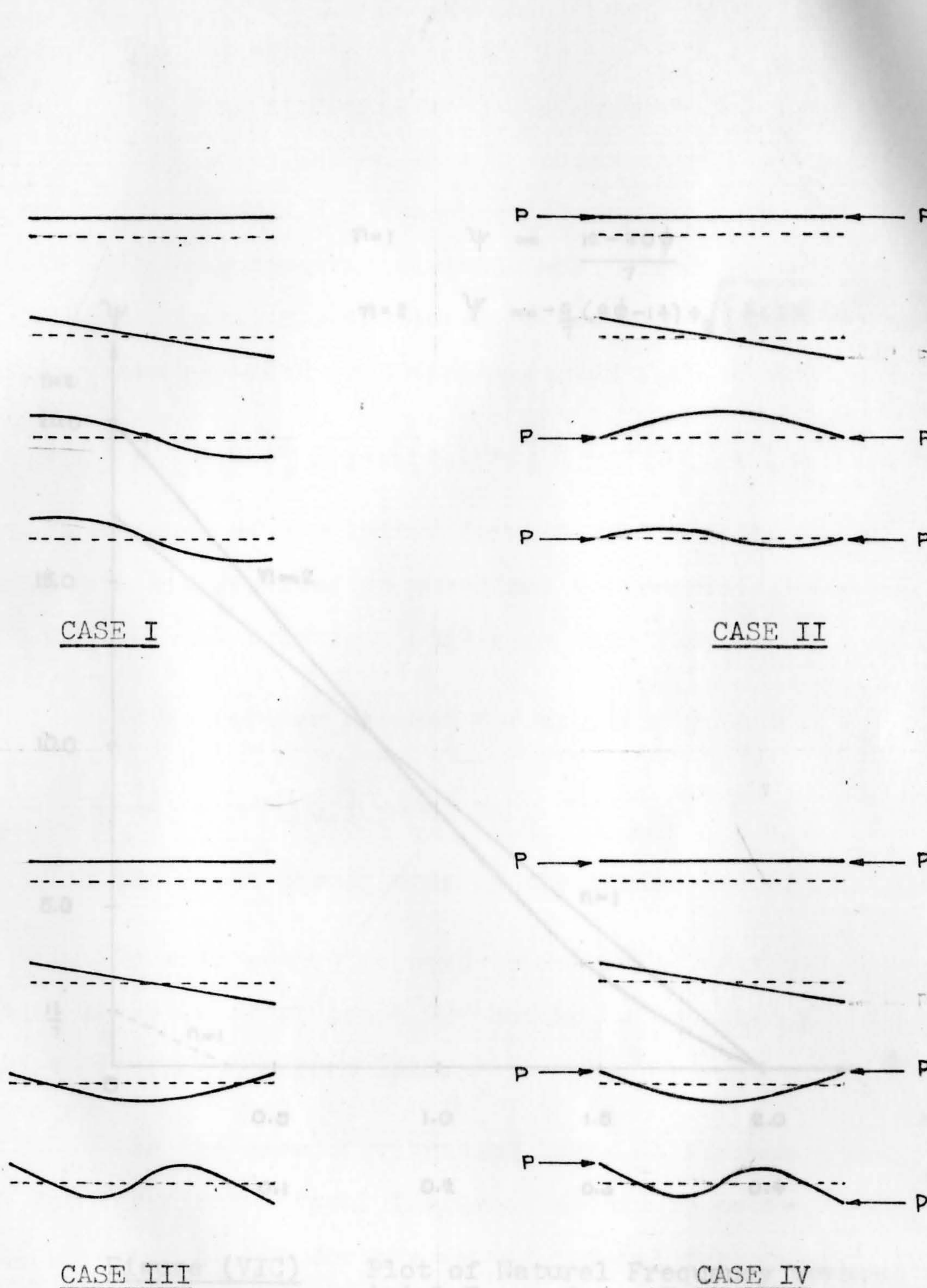


Figure (VIB) Summary of the Normal Mode Shapes for $\lambda_i = 0 \quad i = 1, 2, 3, 4$

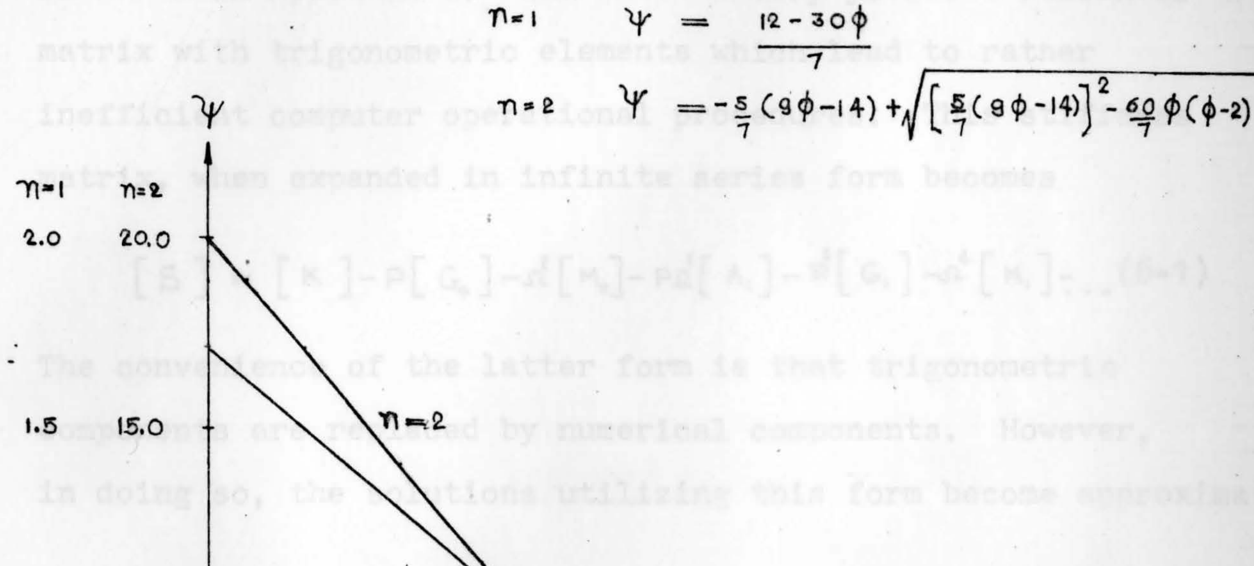


Figure (VIC) Plot of Natural Frequency versus Axial Force for a Free-Free Beam-Column

6.2 Conclusion

The form of the general stiffness matrix relating end harmonic forces and moments to displacements and rotations is shown in Appendix I. The exact theory yields a stiffness matrix with trigonometric elements which lead to rather inefficient computer operational procedures. This stiffness matrix, when expanded in infinite series form becomes

$$[S] \approx [K] - P[G_0] - \Omega^2[M_0] - P\Omega^2[A_1] - P^2[G_1] - \Omega^4[M_1] - \ldots \quad (6\text{-}1)$$

The convenience of the latter form is that trigonometric components are replaced by numerical components. However, in doing so, the solutions utilizing this form become approximate.

It is further assumed for simplicity that

$$[S] \approx [K] - P[G_0] - \Omega^2 [M_0] \quad (6\text{-}2)$$

that, is the first three terms of the series are utilized.

If this method is used to determine critical buckling loads, an error of at least 22% should be expected for the lowest critical buckling load.

For the case of vibrating beams, a minimum error of 20% for the first natural frequency should be expected and a minimum error of 49% for the second natural frequency.

If the percentage error obtained by the approximation of equation (6-2) is too large, the higher order form of the series given by equation (6-1) should be utilized.

APPENDIX I

The form of the general stiffness matrix relating end harmonic forces and moments to displacements and rotations is

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & & \text{Symmetric} & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \qquad \text{(A-1)}$$

where

$$S_{11} = S_{33} = B\left[(p_1^2 p_2^3 + p_1^4 p_2) S c + (p_1 p_2^4 + p_1^3 p_2^2) C s \right] \qquad \text{(A-2)}$$

$$S_{21} = -S_{43} = B\left[(p_1 p_2^3 - p_1^3 p_2) + (p_1^3 p_2 - p_1 p_2^3) C c + 2 p_1^2 p_2^2 S s \right] \qquad \text{(A-3)}$$

$$S_{22} = S_{44} = B\left[(p_1 p_2^2 + p_1^3) C s - (p_1^2 p_2 + p_2^2) S c \right] \qquad \text{(A-4)}$$

$$S_{32} = -S_{41} = B\left[(p_1 p_2^3 + p_1^3 p_2)(c - C) \right] \qquad \text{(A-5)}$$

$$S_{31} = B\left[(-p_1^2 p_2^3 - p_1^4 p_2) S - (p_1^3 p_2^2 + p_1 p_2^4) s \right] \qquad \text{(A-6)}$$

$$S_{42} = B\left[(p_1^2 p_2 + p_2^3) S - (p_1 p_2^2 + p_1^3) s \right] \qquad \text{(A-7)}$$

$$B = \frac{EI}{2 p_1 p_2 - 2 p_1 p_2 C c + (p_1^2 - p_2^2) S s} \qquad \text{(A-8)}$$

subject to the condition that

$$2p_1p_2 - 2p_1p_2\,Cc + (p_1^2 - p_2^2)\,Ss \neq 0 \tag{A-9}$$

$$p_1 = \left[-\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \lambda^4} \right]^{1/2} \tag{A-10}$$

$$p_2 = \left[\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \lambda^4} \right]^{1/2} \tag{A-11}$$

$$k^2 = \frac{P}{EI} \tag{A-12}$$

$$\lambda^4 = \frac{\rho A \Omega^2}{EI} \tag{A-13}$$

$$s = \operatorname{Sin} p_2 L \tag{A-14}$$

$$c = \operatorname{Cos} p_2 L \tag{A-15}$$

$$S = \operatorname{Sinh} p_1 L \tag{A-16}$$

$$C = \operatorname{Cosh} p_1 L \tag{A-17}$$

$$\left[A_1\right] = \frac{\rho A L^3}{EI} \begin{bmatrix} \frac{1}{3{,}150} & \text{symmetric} & & \\ \frac{L}{1{,}260} & \frac{L^2}{3{,}150} & & \\ -\frac{1}{3{,}150} & \frac{L}{1{,}680} & \frac{1}{3{,}150} & \\ -\frac{L}{1{,}680} & -\frac{L^2}{3{,}600} & -\frac{L}{1{,}260} & \frac{L^2}{3{,}150} \end{bmatrix} \qquad \text{(A-18)}$$

$$\left[G_1\right] = \frac{1}{EI} \begin{bmatrix} \frac{L}{700} & \text{symmetric} & & \\ \frac{L^2}{1{,}400} & \frac{11L}{6{,}300} & & \\ -\frac{L}{700} & -\frac{L^2}{1{,}400} & \frac{L}{700} & \\ \frac{L^2}{1{,}400} & -\frac{13L^3}{12{,}600} & -\frac{L^2}{1{,}400} & \frac{11L^3}{6{,}300} \end{bmatrix} \qquad \text{(A-19)}$$

$$\left[M_1\right] = \frac{(\rho A)^2 L^3}{100EI} \begin{bmatrix} \frac{59}{1{,}617} & \text{symmetric} & & \\ \frac{223L}{29{,}106} & \frac{71L^2}{43{,}659} & & \\ \frac{11{,}279}{38{,}808} & \frac{1{,}681L}{232{,}848} & \frac{59}{1{,}617} & \\ -\frac{1{,}681L}{232{,}848} & -\frac{1097L^2}{698{,}544} & -\frac{223L}{29{,}106} & \frac{71L^2}{43{,}659} \end{bmatrix} \qquad \text{(A-20)}$$

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