NONSYMMETRIC MATRICES WITH APPLICATIONS

IN LINEAR ELASTICITY

by

Chiravut Santaputra

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science in Engineering

in the

Civil Engineering

Program

Paul X Bellini 6 JAN 78

Advisor Date

Leon Rand January 11 1978

Dean Date

YOUNGSTOWN STATE UNIVERSITY

January, 1978

ABSTRACT

NONSYMMETRIC MATRICES WITH APPLICATIONS IN LINEAR ELASTICITY

Chiravut Santaputra
Master of Science in Engineering
Youngstown State University, 1978

The purpose of this thesis is to investigate and summarize some of the properties and characteristics of nonsymmetric matrices containing real components. Nonsymmetric matrices are associated with practical engineering problems which arise in the field of linear elasticity and the theory of deformable solids.

This thesis is divided into two sections. The first section presents series of solutions of elastic solid problems illustrating typical conditions in which nonsymmetric matrices are generated. The second section investigates the characteristics of nonsymmetric matrices, including the concepts of the skew coordinate axes, biorthogonal coordinates, replacement by a symmetric matrix and a skew symmetric matrix, eigenvalue-eigenvector problem, and the concept of the super matrix.

ACKNOWLEDGEMENTS

The author wishes to acknowledge his deep appreciation and gratitude to Dr. Paul X. Bellini, his thesis advisor, whose time, efforts, guidance, and encouragement directly contributed in the completion of this thesis.

The author also wishes to thank his review committee, Dr. Michael K. Householder and Professor John F. Ritter for giving their valuable time toward the completion of the requirements of this work.

Great appreciation is given to his dear father and mother, Mr. Siri and Mrs. Chandhanee Santaputra for supporting and encouraging him during his studies.

I wish to especially thank Miss Malinee Emaruchi for her excellent work and her unlimited patience when typing this thesis.

TABLE OF CONTENTS

LIST OF NOTATIONS

SYMBOL	DEFINITION
$[J]$	Jacobain matrix
$[D]$	Deformation matrix
$[e]$	Linear strain matrix
$[\omega]$	Rotation matrix
$[I]$	Unit matrix
$[\Lambda]$	Diagonal matrix of eigenvalues
$[U],[V]$	Biorthogonal matrices
$[A_s]$	Super matrix
$\dot{i}_1, \dot{i}_2, \dot{i}_3$	Unit vectors in X_1, X_2, X_3 direction
$\dot{\tilde{i}}_1^*, \dot{\tilde{i}}_2^*, \dot{\tilde{i}}_3^*$	Unit vectors tangent to curvilinear coordinate axes
$\dot{\tilde{J}}_1^*, \tilde{J}_2^*, \tilde{J}_3^*$	Biorthogonal unit vectors set
I_1, I_2, I_3	First, second, and third tensor invariants
τ_{ij}	Stress component
u_1, u_2, u_3	Displacement in X_1, X_2, X_3 direction
E	Young's modulus of elasticity
γ	Weight density per unit volume
μ	Poisson's ratio
G	Shear modulus of elasticity
I	Second moment of area
J	Polar moment of inertia
$[\ ]^{-1}$	Inverse of the matrix
$[\ ]^T$	Transpose of the matrix
$[\tilde{\ }]$	Complex conjugate of the matrix
$Tr[\ ]$	Trace of the matrix

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Nonsymmetric Matrices

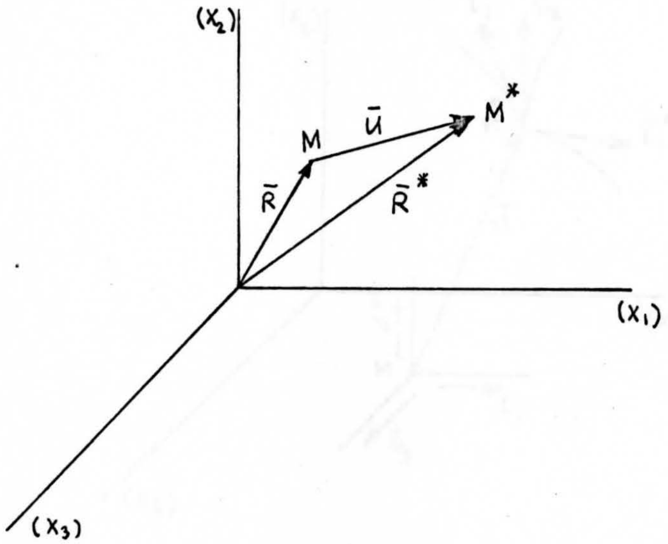
Matrices that occur in the field of linear elasticity are both symmetric and nonsymmetric. The three symmetric matrices most Elasticians[1,2,3]* traditionally associate with linear elasticity are stress matrix, the linear strain matrix, and the Hooke's Law matrix relating stress and strain. The matrix[A] defining the skew curvilinear axes of the deformed body, and the Jacobian matrix [J] which functionally defines the shape of the deformed body are two nonsymmetric matrices. These matrices play a primary role in defining the geometric shape of the elastic body in its deformed equilibrium state. Thus, it is very useful to study the properties and characteristics of the nonsymmetric matrices.

The coordinates of the points in an undeformed solid are usually defined with respect to the rectangular coordinate frame. After the body deforms, due to applied surface forces, straight lines in the body parallel to the coordinate axes deform into curvilinear lines. The most efficient method to define the shape of the deformed solid

* Numbers in parenthesis referred to literature cited in the Bibliography.

is to formulate unit vectors tangent to these deformed curvilinear lines.

Let $M(x_1,x_2,x_3)$ be a point in an undeformed solid. After the solid deforms the point $M(x_1,x_2,x_3)$ moves to $\overset{*}{M}(\overset{*}{x}_1,\overset{*}{x}_2,\overset{*}{x}_3)$ with the displacements u_1,u_2,u_3 in the direction of x_1,x_2,x_3 respectively, (See Figure(1-1))



Figure(1-1) Rectangular Coordinates-Deformed Geometry

thus,

$$\overset{*}{x}_i = x_i + u_i \qquad i = 1,2,3 \tag{1-1a}$$

$$\bar{R}^* = \overset{*}{x}_1\dot{\bar{i}}_1 + \overset{*}{x}_2\dot{\bar{i}}_2 + \overset{*}{x}_3\dot{\bar{i}}_3 \tag{1-1b}$$

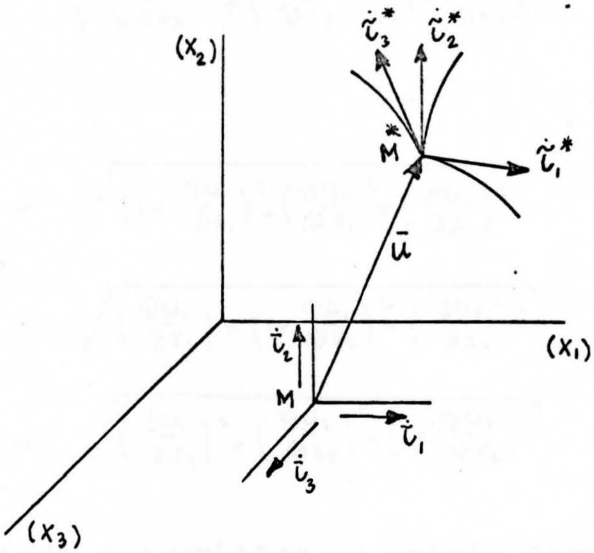
where $\dot{\bar{i}}_i$ is the unit vector in the x_i direction. It follows that

$$\frac{\partial \bar{R}^*}{\partial x_i} = \tilde{x}_i^* \qquad i = 1,2,3$$

where $\tilde{x}_i^*$ is a vector tangent to the deformed curved line which in the undeformed body is parallel to the coordinate axes. Further $\dot{\tilde{\iota}}_i^*$ is defined as a unit vector in the $\tilde{X}_i^*$ direction

$$\dot{\tilde{\iota}}_i^* = \frac{\tilde{X}_i^*}{|\tilde{X}_i^*|}$$

where $|\tilde{X}_i^*|$ defines the magnitude of the vector $\tilde{X}_i^*$ (See Figure (1-2))



Figure(1-2) Curvilinear Coordinates

Noting Equations (1-1a) and (1-1b) one obtains

$$\bar{R}^* = (X_1+u_1)\dot{\iota}_1 + (X_2+u_2)\dot{\iota}_2 + (X_3+u_3)\dot{\iota}_3 \tag{1-1c}$$

and

$$\tilde{X}_1^* = \frac{\partial \bar{R}^*}{\partial X_1} = \left(1+\frac{\partial u_1}{\partial X_1}\right)\dot{\iota}_1 + \frac{\partial u_2}{\partial X_1}\dot{\iota}_2 + \frac{\partial u_3}{\partial X_1}\dot{\iota}_3$$

hence,

$$\dot{\tilde{\iota}}_1^* = \frac{\left(1+\frac{\partial u_1}{\partial X_1}\right)\dot{\iota}_1 + \frac{\partial u_2}{\partial X_1}\dot{\iota}_2 + \frac{\partial u_3}{\partial X_1}\dot{\iota}_3}{\sqrt{\left(1+\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2}} \tag{1-2a}$$

$$\dot{\iota}_2^* = \frac{\frac{\partial u_1}{\partial x_2}\dot{\iota}_1 + \left(1+\frac{\partial u_2}{\partial x_2}\right)\dot{\iota}_2 + \frac{\partial u_3}{\partial x_2}\dot{\iota}_3}{\sqrt{\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(1+\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2}} \tag{1-2b}$$

$$\dot{\iota}_3^* = \frac{\frac{\partial u_1}{\partial x_3}\dot{\iota}_1 + \frac{\partial u_2}{\partial x_3}\dot{\iota}_2 + \left(1+\frac{\partial u_3}{\partial x_3}\right)\dot{\iota}_3}{\sqrt{\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2 + \left(1+\frac{\partial u_3}{\partial x_3}\right)^2}} \tag{1-2c}$$

Defining

$$\left.\begin{aligned}
1+E_1 &= \sqrt{\left(1+\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} \\
1+E_2 &= \sqrt{\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(1+\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2} \\
1+E_3 &= \sqrt{\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2 + \left(1+\frac{\partial u_3}{\partial x_3}\right)^2}
\end{aligned}\right\} \tag{1-2d}$$

the unit vectors are written in matrix form as

$$\begin{Bmatrix} \dot{\iota}_1^* \\ \dot{\iota}_2^* \\ \dot{\iota}_3^* \end{Bmatrix} = \begin{bmatrix} \frac{1}{1+E_1} & 0 & 0 \\ 0 & \frac{1}{1+E_2} & 0 \\ 0 & 0 & \frac{1}{1+E_3} \end{bmatrix} \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \dot{\iota}_1 \\ \dot{\iota}_2 \\ \dot{\iota}_3 \end{Bmatrix} \tag{1-3a}$$

or in symbolic form as

$$\{\dot{\iota}^*\} = \left[\frac{1}{1+E}\right][J]^T\{\dot{\iota}\} = [A]^T\{\dot{\iota}\} \tag{1-3b}$$

The matrix $\left[\frac{1}{1+E}\right]$ for small deformation, small rotation linear elasticity[4] becomes

$$\left[\frac{1}{1+E}\right] = \begin{bmatrix} \dfrac{1}{1+\dfrac{\partial u_1}{\partial x_1}} & 0 & 0 \\ 0 & \dfrac{1}{1+\dfrac{\partial u_2}{\partial x_2}} & 0 \\ 0 & 0 & \dfrac{1}{1+\dfrac{\partial u_3}{\partial x_3}} \end{bmatrix}$$

Expansion of each term in the above matrix in a power series and noting Equation (1-3b) gives

$$\begin{Bmatrix} \dot{\tilde{i}}_1^* \\ \dot{\tilde{i}}_2^* \\ \dot{\tilde{i}}_3^* \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & 1 & \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_3} & \dfrac{\partial u_2}{\partial x_3} & 1 \end{bmatrix} \begin{Bmatrix} \dot{\tilde{i}}_1 \\ \dot{\tilde{i}}_2 \\ \dot{\tilde{i}}_3 \end{Bmatrix} \qquad (1\text{-}3c)$$

or in symbolic form as

$$\{\dot{\tilde{i}}^*\} = [A]^T\{\dot{\tilde{i}}\} \qquad (1\text{-}3d)$$

The matrix $[A]$ is in general nonsymmetric and defines the direction of a skew reference frame in the deformed body.

The Jacobian matrix $[J]$ as defined in Equation (1-3b) is a second nonsymmetric matrix found in linear elasticity. Matrix $[J]$ is usually rewritten as

$$[J] = [D] + [I] \qquad (1\text{-}4a)$$

where $[D]$ is defined as the deformation matrix which is in general nonsymmetric. The matrix $[D]$ is further reduced to the sum of a symmetric matrix $[e]$ and a skew symmetric matrix $[\omega]$ in the form

$$[D] = [e] + [\omega] \tag{1-4b}$$

where $[e]$ is the symmetric linear strain matrix and $[\omega]$ is the skew symmetric rotation matrix.

1.2 <u>Biorthogonal Coordinate</u>

The three unit vectors tangent to the deformed curvilinear lines of material body in the deformed equilibrium state are not in general perpendicular to each other. They usually form a skew angular coordinate frame. In a skew angular reference system it is necessary to construct a second set of vectors defined as $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ (5). This set of vectors together with the initial set of base vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ along the original skew axes system satisfy the following biorthogonal conditions

$$\left.\begin{array}{lll} (\bar{u}_i \cdot \bar{v}_k) & = 0 & i \neq k \\ (\bar{u}_i \cdot \bar{v}_i) & = 1 & i = 1, 2, \ldots, n \end{array}\right\} \tag{1-5a}$$

The family of $\bar{u}$ vectors and $\bar{v}$ vectors are associated with the matrices $[U]$ and $[V]$, respectively. The matrix form of Equation (1-5a) becomes

$$[V]^T[U] = [U]^T[V] = [I] \tag{1-5b}$$

A nonsymmetric matrix $[A]$ defined with respect to the dual coordinate basis satisfies the condition that

$$[V]^T[A]^T[U] = [U]^T[A][V] = [\Lambda] \tag{1-6}$$

where $[\Lambda]$ is a diagonal matrix of eigenvalues. If the original coordinate vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ are orthogonal the second set of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ are also orthogonal and coincide with the first set. Thus,

$$[V] = [U] \qquad (1\text{-}7a)$$

and Equation (1-5b) reduces to

$$[U]^T[U] = [U][U]^T = [I] \qquad (1\text{-}7b)$$

and Equation (1-6) becomes

$$[U]^T[A][U] = [\Lambda] \qquad (1\text{-}7c)$$

The latter case occurs if the matrix $[A]$ is symmetric.

1.3 Replacement of a Nonsymmetric Matrix by a Symmetric Matrix and a Skew Symmetric Matrix

Any real nonsymmetric matrix may be replaced by the sum of a symmetric matrix and a skew symmetric matrix in the form

$$[C] = [A] + [B] \qquad (1\text{-}8a)$$

where

$$[A] = \frac{1}{2}\left[[C] + [C]^T\right] \qquad (1\text{-}8b)$$

and

$$[B] = \frac{1}{2}\left[[C] - [C]^T\right] \qquad (1\text{-}8c)$$

where $[A]$ is a symmetric matrix and $[B]$ is a skew symmetric matrix. Three special cases arise from Equation (1-8a):

Case I - $[C]$ is symmetric;

$$[B] = [0] \tag{1-9a}$$

Case II - $[C]$ is skew symmetric;

$$[A] = [0] \tag{1-9b}$$

Case III - $[C]$ is orthogonal;

$$[A][B] = [B][A] \tag{1-9c}$$

From Equations (1-4a) and (1-4b), the Jacobian matrix which indicates the shape of the body in the deformed equilibrium state is written as

$$[J] = [e] + [\omega] + [I]$$

or

$$[J] = \Big[[e] + [I]\Big] + [\omega] \tag{1-10a}$$

and

$$[J]^T = \Big[[e] + [I]\Big] - [\omega] \tag{1-10b}$$

where $[A] = [e]+[I]$ and $[B] = [\omega]$

Of particular importance in a deformable solids problem is the condition when $[J]$ is an orthogonal matrix. It may be shown in this case that

$$[A][B][C] = [C][B][A] = [B][A][C] \; \ldots \qquad (1\text{-}11)$$

These latter equations require that $[A]$, $[B]$ and $[C]$ have the same eigenvectors. One real eigenvector associated with the components of the skew symmetric matrix $[B]$ is a vector normal to a plane about which rotation takes place.

1.4 Eigenvalue-Eigenvector Problem

The nonlinear strain tensor(4) in the mechanics of deformable solids is given as

$$\frac{1}{2}\Big[[J]^T[J] - [I]\Big] = [e] + \frac{1}{2}\Big[[e]^2 + [e][\omega] - [\omega][e] + [\omega]^2\Big] \qquad (1\text{-}12)$$

For the special case of linear elasticity one obtains

$$\frac{1}{2}\Big[[J]^T[J] - [I]\Big] \approx [e]$$

where the last four terms of Equation (1-12) are higher ordered terms. The eigenvector directions of $\frac{1}{2}\Big[[J]^T[J] - [I]\Big]$ are the principal directions of the strain matrix $[e]$ (i.e. in the direction of principal strain). Thus, given the nonsymmetric matrix $[J]$ it is expedient to investigate the eigenvalue-eigenvector problem of $[J]$ premultiplied by its transpose. Hence, given a nonsymmetric matrix $[A]$ with real components, it follows that the multiplication of matrix $[A]$ and its transpose produces a symmetric matrix, that is,

$$[A]^T[A] = [B] = [B]^T \qquad (1\text{-}13a)$$

$$[A][A]^T = [C] = [C]^T \qquad (1\text{-}13b)$$

where $[B]$ and $[C]$ are both symmetric matrices. In general $[B] \neq [C]$ and as the result $[B][C] \neq [C][B]$. The eigenvalues of $[B]$ and $[C]$ are all real numbers and the eigenvectors are orthogonal, while for matrix $[A]$ the eigenvalues may be real or complex and the eigenvectors may be real or complex. The diagonalized form of matrices $[A]$, $[B]$ and $[C]$ defined as $[\Lambda_A]$, $[\Lambda_B]$ and $[\Lambda_C]$, respectively, may be shown to be

$$[\Lambda_B] = [\Lambda_C] = [\hat{\Lambda}] \tag{1-14}$$

since both $[B]$ and $[C]$ possess the same characteristic equation. A special case arises when $[B][C] = [C][B]$. It follows that $[B] = [C]$. Three special cases are involved.

Case I - $[A]$ is orthogonal, $[A]^{-1} = [A]^T$, eigenvalues are real or complex and the absolute value of each eigenvalue must be one. All complex eigenvalues appear in complex conjugate pairs. If $[A]$ is both orthogonal and symmetric all eigenvalues are real and must be only ± 1, the eigenvectors are real and orthogonal.

Case II - $[A]$ is skew symmetric, $[A] = -[A]^T$, eigenvalues are real or complex. If $[A]$ is odd ordered matrix at least one eigenvalue is zero. All complex eigenvalues have zero real parts and appear in complex conjugate pairs.

Case III - $[A]$ is symmetric, $[A] = [A]^T$, eigenvalues are real and eigenvectors are orthogonal.

This thesis investigates the mathematical patterns in the above cases.

1.5 The Super Matrix

The concept of multiplying a nonsymmetric matrix by its transpose and producing a symmetric matrix as considered in Section 1.4 leads to the formulation of a new matrix defined as the super matrix[6] which is symmetric. Given a nonsymmetric matrix $[A]$ of order $(n \times n)$, one constructs $[A_s]$, the super matrix, in the form of a partitioned matrix as

$$[A_s] = \left[\begin{array}{c|c} [0] & [A] \\ \hline [A]^T & [0] \end{array}\right] \tag{1-15a}$$

$[A_s]$ is a symmetric matrix which has an order $(2n \times 2n)$, twice that of matrix $[A]$. In addition the matrix $[A_s]^2$ becomes

$$[A_s]^2 = \left[\begin{array}{c|c} [A][A]^T & [0] \\ \hline [0] & [A]^T[A] \end{array}\right] \tag{1-15b}$$

which has the same eigenvectors as the matrix $[A_s]$. Also, $[A_s]^2$ has eigenvalues that are the square of the eigenvalues of $[A_s]$. The form of Equation (1-15b) relates directly to the concept of Section 1.4 in Equation (1-13a) and (1-13b).

CHAPTER II

ENGINEERING PROBLEMS INVOLVING NONSYMMETRIC MATRICES

2.1 Summary of Illustrative Problems

In this chapter we summarize the Jacobian matrix and its properties as applied to series of six realistic engineering problems that occur in the theory of elastic solid. As shown in Chapter I, the Jacobian matrix defines a set of vectors which are tangent to the curvilinear lines in the deformable static equilibrium state, as will be shown the Jacobian matrix may take on properties of unit matrix, symmetry, orthogonality, or general nonsymmetry. The following problems will be used to illustrate these conditions[7]:

1) Uniaxial extension of a three dimensional slender rod.

2) Three dimensional extension of a long slender rod under its weight distribution.

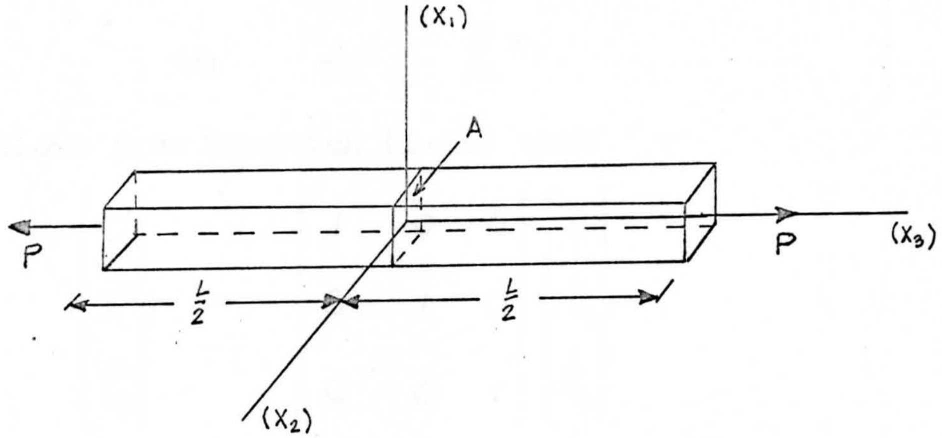
3) Pure bending of a slender rod in three dimensions.

4) Plane stress analysis of pure bending.

5) Plane stress analysis of bending and constant shear.

6) Plane stress analysis of bending and linearly varying shear.

2.2 Uniaxial Extension of a Three Dimensional Slender Rod



Figure(2-1) Long Slender Rod in Tension

Consider a long slender rod of length L and cross sectional area A subject to axial force P as shown in Figure(2-1); the stress state within the rod is constant with

$$\tau_{33} = \frac{P}{A}$$

and

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = 0$$

Satisfying the three equations of stress equilibrium, the six equations of Hooke's Law for a linear elastic material, and the six linear strain displacement equations, it follows that the displacement field for the elastic body becomes

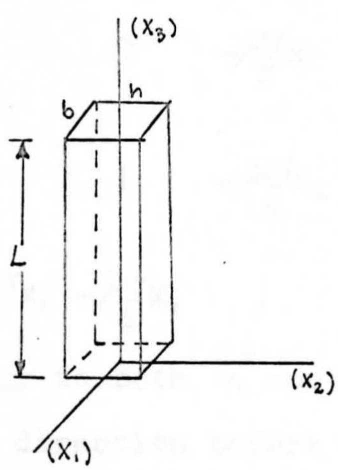
$$\left.\begin{aligned} u_1 &= -\frac{\mu P}{AE} X_1 \\ u_2 &= -\frac{\mu P}{AE} X_2 \\ u_3 &= \frac{P}{AE} X_3 \end{aligned}\right\} \qquad (2\text{-}1)$$

It follows from Equation (1-3c) that

$$\begin{Bmatrix} \dot{i}_1^* \\ \dot{i}_2^* \\ \dot{i}_3^* \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \end{Bmatrix} \qquad (2\text{-}2)$$

The [A] matrix in this case possesses the properties of symmetry, orthogonality as well as being unit matrix. Thus, lines parallel to coordinate axes before deformation remain parallel to these coordinate axes after deformation. Finally all planes parallel to the coordinate planes before deformation remain parallel to these coordinate planes after deformation.

2.3 Three Dimensional Extension of a Long Slender Rod Under Its Weight Distribution



Figure(2-2) Long Slender Rod under Its Own Weight

Consider a long slender rod of length L subject to its weight distribution with weight per unit volume γ as shown in Figure(2-2).The stress state within the rod is

$$\tau_{33} = \gamma X_3$$

and

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = 0$$

Satisfying the three equations of stress equilibrium, the six equations of Hooke's Law for a linear elastic material, and the linear strain displacement equations, it follows that the displacement field becomes

$$\left.\begin{aligned} u_1 &= -\frac{\mu\gamma}{E} X_1 X_3 \\ u_2 &= -\frac{\mu\gamma}{E} X_2 X_3 \\ u_3 &= \frac{\mu\gamma}{2E}(X_1^2 + X_2^2) + \frac{\gamma}{2E}(X_3^2 - L^2) \end{aligned}\right\} \quad (2\text{-}3)$$

It follows from Equation (1-3c) that

$$\begin{Bmatrix} \dot{\tilde{\upsilon}}_1^* \\ \dot{\tilde{\upsilon}}_2^* \\ \dot{\tilde{\upsilon}}_3^* \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{\mu\gamma}{E}X_1 \\ 0 & 1 & -\frac{\mu\gamma}{E}X_2 \\ -\frac{\mu\gamma}{E}X_1 & -\frac{\mu\gamma}{E}X_2 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\bar{\upsilon}}_1 \\ \dot{\bar{\upsilon}}_2 \\ \dot{\bar{\upsilon}}_3 \end{Bmatrix} \quad (2\text{-}4)$$

vector $\dot{\tilde{\upsilon}}_3^*$ is perpendicular to both $\dot{\tilde{\upsilon}}_1^*$ and $\dot{\tilde{\upsilon}}_2^*$. Straight lines parallel to the X_3 direction before deformation remain straight lines after deformation. The $[A]$ matrix is nonsymmetric. However, if $\frac{\gamma}{E}$ is small in comparison to unity the matrix [A] is most nearly orthogonal, and the three unit vectors $\tilde{\dot{\iota}}_i^*$ $i=1,2,3$ form an orthogonal set.

The coordinates of a point in the deformed body are

$$X_1^* = X_1 - \frac{\mu\gamma}{E} X_1 X_3$$

$$X_2^* = X_2 - \frac{\mu\gamma}{E} X_2 X_3$$

$$X_3^* = X_3 + \frac{\mu\gamma}{2E}(X_1^2 + X_2^2) + \frac{\gamma}{2E}(X_3^2 - L^2)$$

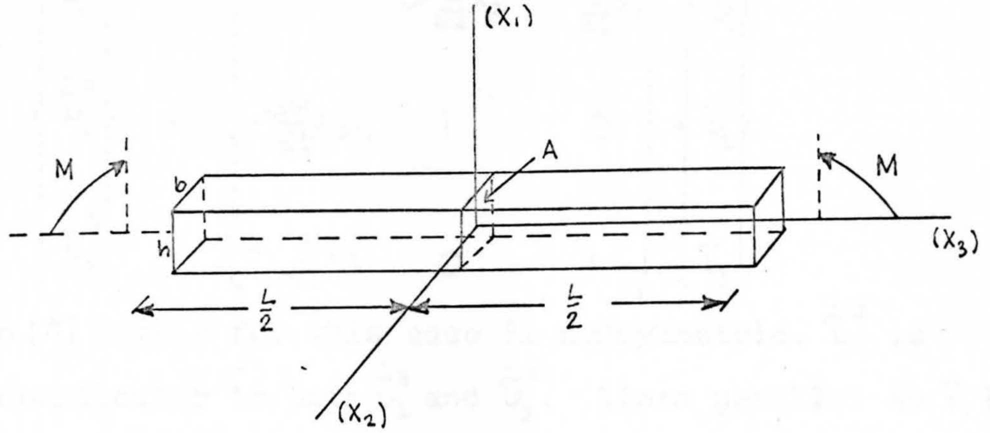
For the line X_2 = constant = $\hat{X}_2$ and X_1 = constant = $\hat{X}_1$ (i.e. a line parallel to the X_3 axis) it follows that

$$\frac{X_1^* - \hat{X}_1}{-\frac{\mu\gamma}{E}\hat{X}_1} = \frac{X_2^* - \hat{X}_2}{-\frac{\mu\gamma}{E} X_2} = \frac{X_3^* - \left[\frac{\mu\gamma}{2E}(\hat{X}_1^2 - \hat{X}_2^2) - \frac{\gamma}{2E} L^2\right]}{1} = X_3$$

$$0 \le X_3 \le L$$

where $\frac{\gamma}{E}$ is small in comparison to unity. The latter equation is the equation of a straight line passing through the point $\left(\hat{X}_1, \hat{X}_2, \frac{\gamma}{2E}\left(\mu(\hat{X}_1^2 + \hat{X}_2^2) - L^2\right)\right)$ and having direction numbers $\left\{-\frac{\mu\gamma}{E}\hat{X}_1 : -\frac{\mu\gamma}{E}\hat{X}_2 : 1\right\}$. These latter direction numbers are just the constant numerical components of the vector $\tilde{\dot{\iota}}_3^*$.

2.4 Pure Bending of a Slender Rod in Three Dimension



Figure(2-3) Long Slender Rod in Pure Bending

Consider a long slender rod of length L subject to bending moment M as shown in Figure(2-3). The stress state within the rod is

$$\tau_{33} = \frac{M}{I} X_1$$

and

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = \tau_{23} = 0$$

Satisfying the three equations of stress equilibrium, the six equations of Hooke's Law for a linear elastic material, and the linear strain displacement equations, the displacement field becomes

$$\left.\begin{aligned} u_1 &= \frac{\mu M}{2EI}(X_2^2 - X_1^2) - \frac{M}{2EI} X_3^2 \\ u_2 &= -\frac{\mu M}{EI} X_1 X_2 \\ u_3 &= \frac{M}{EI} X_1 X_3 \end{aligned}\right\} \qquad (2\text{-}5)$$

it follows from Equation (1-3c) that

$$\begin{Bmatrix} \dot{\tilde{\iota}}_1^* \\ \\ \dot{\tilde{\iota}}_2^* \\ \\ \dot{\tilde{\iota}}_3^* \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{\mu M}{EI}X_2 & \frac{M}{EI}X_3 \\ \\ \frac{\mu M}{EI}X_2 & 1 & 0 \\ \\ -\frac{M}{EI}X_3 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\tilde{\iota}}_1 \\ \\ \dot{\tilde{\iota}}_2 \\ \\ \dot{\tilde{\iota}}_3 \end{Bmatrix} \tag{2-6}$$

The [A] matrix for this case is nonsymmetric, $\dot{\tilde{\iota}}_1^*$ is perpendicular to both $\dot{\tilde{\iota}}_2^*$ and $\dot{\tilde{\iota}}_3^*$. Lines parallel to $\tilde{\iota}_1$ before deformation remain lines after deformation and all planes normal to $\dot{\iota}_3$ before deformatiom remain planes after deformation. The matrix [A] is most nearly orthogonal provided the quantity $\frac{M}{EI}$ is small in comparison to unity. The coordinates of a point in the deformed body are

$$X_1^* = X_1 - \frac{\mu M}{EI}(X_2^2 - X_1^2) - \frac{M}{2EI}X_3^2$$

$$X_2^* = X_2 - \frac{\mu M}{EI}X_1X_2$$

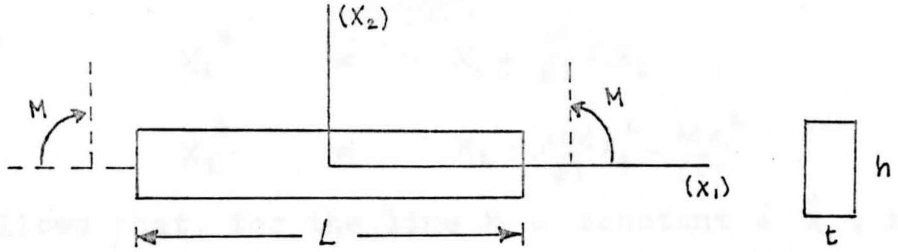
$$X_3^* = X_3 + \frac{M}{EI}X_1X_3$$

For the plane X_3 = constant = $\hat{X}_3$ (i.e. plane parallel to X_1X_2 plane), it follows that

$$-\frac{M}{EI}\hat{X}_3 . X_1^* + X_3^* = \hat{X}_3$$

The latter equation is the equation of a plane whose direction cosines are $(-\frac{M}{EI}\hat{X}_3, 0, 1)$.

2.5 Plane Stress Analysis of a Beam in Pure Bending



Figure(2-4) Thin Beam in Pure Bending

Consider a long thin beam of length L subject to bending moment M as shown in Figure(2-4), the stress state in the X_1X_2 plane is

$$\tau_{11} = \frac{M}{I}X_2$$

$$\tau_{22} = \tau_{12} = 0$$

Satisfying the equation of stress equilibrium, the equation of Hooke's Law for a linear elastic material, and the strain displacement equations, the displacement field becomes

$$\left.\begin{aligned} U_1 &= \frac{M}{EI}X_1X_2 \\ U_2 &= -\mu\frac{M}{EI}X_2^2 - \frac{M}{2EI}X_1^2 \end{aligned}\right\} \qquad (2\text{-}7)$$

It follows from Equation (1-3c) that

$$\begin{Bmatrix} \dot{\tilde{\iota}}_1^* \\ \dot{\tilde{\iota}}_2^* \end{Bmatrix} = \begin{bmatrix} 1 & \frac{M}{EI}X_1 \\ -\frac{M}{EI}X_1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\bar{\iota}}_1 \\ \dot{\bar{\iota}}_2 \end{Bmatrix} \qquad (2\text{-}8)$$

The [A] matrix possesses the property of orthogonality, assuming the quantity $\frac{M}{EI}$ is small in comparison to unity. Thus, any two perpendicular lines parallel to the coordinate axes before deformation remain perpendicular lines after deformation. Noting

$$X_1^* = X_1 + \frac{M}{EI} X_1 X_2$$

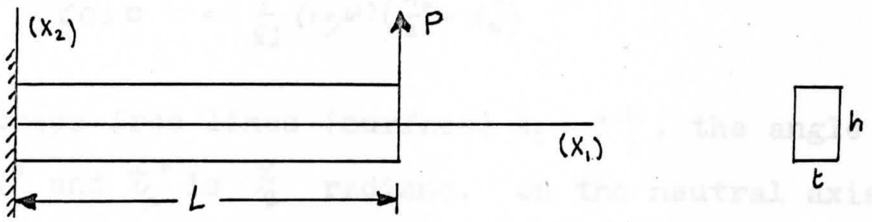
$$X_2^* = X_2 - \frac{\mu M}{EI} X_2^2 - \frac{M X_1^2}{2EI}$$

it follows that, for the line X_1 = constant = $\hat{X}_1$, and neglecting higher order terms in $\frac{M}{EI}$,

$$\frac{X_1^* - \hat{X}_1}{\frac{M}{EI}} = \frac{X_2^* - 0}{1}$$

Hence, straight lines parallel to X_2 before deformation remain straight lines after deformation.

2.6 Plane Stress Analysis of Beam Bending with Constant Shear



Figure(2-5) Cantilever Thin Beam with Concentrated Load

Consider a thin long slender beam subject to force P as shown in Figure(2-5), the stress state in the plane is

$$\tau_{11} = \frac{PL}{I} X_2 + \frac{P}{I} X_1 X_2$$

$$\tau_{22} = 0$$

$$\tau_{12} = -\frac{P}{2I} X_2^2 - \frac{3P}{2th}$$

Satisfying the equations of stress equilibrium, the equations of Hooke's Law for a linear elastic material, and the linear strain displacement equations, the displacement field becomes

$$\left.\begin{aligned} u_1 &= \frac{P}{EI}\left[x_1 x_2\left(\frac{x_1}{2} - L\right) + (1+\mu)\left(\frac{h^2}{4}x_2 - \frac{x_2^3}{3}\right) + \mu\frac{x_2^3}{6}\right] \\ u_2 &= -\frac{P}{EI}\left[\mu\frac{x_2^2}{2}(x_1 - L) + \left(\frac{x_1^3}{6} - \frac{x_1^2}{2}L\right)\right] \end{aligned}\right\} \qquad (2\text{-}9)$$

it follows from Equation (1-3c) that

$$\begin{Bmatrix} \dot{\tilde{\iota}}_1^* \\ \\ \dot{\tilde{\iota}}_2^* \end{Bmatrix} = \begin{bmatrix} 1 & \frac{P}{EI}\left[-\frac{\mu x_2^2}{2} - \left(\frac{x_1^2}{2} - x_1 L\right)\right] \\ \frac{P}{EI}\left[\left(\frac{x_1^2}{2} - x_1 L\right) + (1+\mu)\left(\frac{h^2}{4} - x_2^2\right) + \frac{\mu x_2^2}{2}\right] & 1 \end{bmatrix} \begin{Bmatrix} \dot{\bar{\iota}}_1 \\ \\ \dot{\bar{\iota}}_2 \end{Bmatrix} \qquad (2\text{-}10)$$

The [A] matrix is nonsymmetric. The cosine of the angle between $\dot{\tilde{\iota}}_1^*$ and $\dot{\tilde{\iota}}_2^*$ is given as

$$\cos\theta = \frac{P}{EI}(1+\mu)\left(\frac{h_2^2}{4} - x_2^2\right)$$

On the stress-free lines (surface) $x_2 = \pm\frac{h}{2}$, the angle between $\dot{\tilde{\iota}}_1^*$ and $\dot{\tilde{\iota}}_2^*$ is $\pi/2$ radians. On the neutral axis one obtains

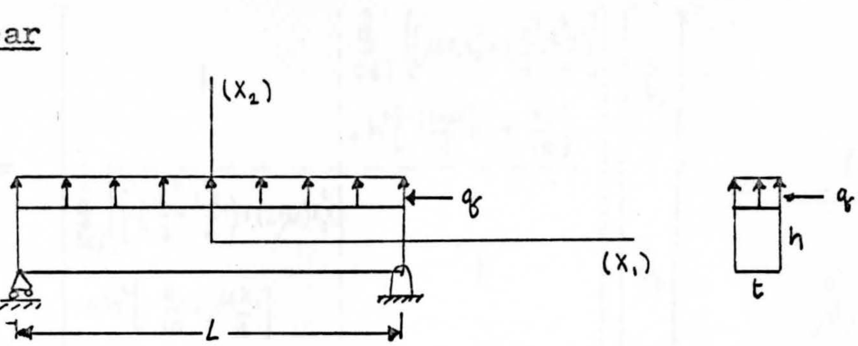
$$(\dot{\tilde{\iota}}_1^* \cdot \dot{\tilde{\iota}}_2^*) = \cos\theta = (1+\mu)\frac{P}{EI}\frac{h^2}{4}$$

or

$$\cos\theta = \frac{3}{2}\cdot\frac{P}{A}\cdot\frac{1}{G} = \frac{3}{2}\cdot\frac{\tau_{12}|_{x_2=0}}{G} = \frac{3\gamma_{12}|_{x_2=0}}{2}$$

Thus, lines parallel to x_2 before deformation do not remain perpendicular to the curvilinear neutral axis after deformation.

2.7 Plane Stress Analysis of Beam Bending with Linearly Varying Shear



Figure(2-6) Simply Supported Beam under Uniform Load

Consider a simply supported thin beam of length L with uniformly applied stress q as shown in Figure(2-6). For convenience the force per unit length $\hat{q}$ is defined with $\hat{q} = qt$. The stress state in the plane $X_1 X_2$ is

$$\tau_{11} = -\frac{\hat{q}}{2I}\left(x_1^2 - \frac{L^2}{4}\right)x_2 + \frac{\hat{q}}{2I}\left(\frac{2}{3}x_2^3 - \frac{h^2}{10}x_2\right)$$

$$\tau_{12} = -\frac{\hat{q}}{2I}\left(\frac{h^2}{4} - x_2^2\right)x_1$$

$$\tau_{22} = -\frac{\hat{q}}{2I}\left(\frac{x_2^3}{3} - \frac{h^2}{4}x_2 - \frac{h^3}{12}\right).$$

Satisfying the equations of stress equilibrium, the equations of Hooke's Law for a linear elastic material, and the linear strain displacement equations, the displacement field becomes

$$\left.\begin{aligned} u_1 &= \frac{\hat{q}}{2EI}\left[\left[-\frac{x_1^3 x_2}{3} - \frac{L^2 x_1 x_2}{4} + \frac{2x_1 x_2^3}{3} + \mu\frac{x_1 x_2^3}{3}\right] - h^2\left[\frac{x_1 x_2}{10} + \mu\frac{x_1 x_2}{4}\right.\right. \\ &\quad \left.\left. + \frac{h x_1}{12} + \mu\frac{hL}{24}\right]\right] \\ u_2 &= \frac{\hat{q}}{2EI}\left[\left[-\frac{x_2^4}{12} + \mu\frac{x_1^2 x_2^2}{2} - \mu\frac{L^2 x_2^2}{8} - \mu\frac{x_2^4}{6} + \frac{x_1^4}{12} - \frac{L^2 x_1^2}{8} + \frac{5L^4}{192}\right] + \right. \\ &\quad \left. h^2\left[\frac{x_2^2}{8} + \frac{h x_2}{12} + \mu\frac{x_2^2}{20} - (2+\mu)\frac{x_1^2}{8} + \frac{x_1^2}{20} + (2+\mu)\frac{L^2}{36} - \frac{L^2}{80}\right]\right] \end{aligned}\right\} \quad (2\text{-}11)$$

it follows from Equation (1-3c) that

$$
\left\{\begin{array}{c} \dot{\tilde{\iota}}_1^{*} \\ \\ \dot{\tilde{\iota}}_2^{*} \end{array}\right\}
=
\left[\begin{array}{c|c}
1 & \dfrac{\hat{q}}{2EI}\left[\left[\mu x_1 x_2^2 + \left(\dfrac{x_1^3}{3} - \dfrac{L^2 x_1}{4}\right)\right] + h^2\left[-\dfrac{(2+\mu)}{4}x_1 + \dfrac{x_1}{10}\right]\right] \\
\hline
\dfrac{\hat{q}}{2EI}\left[\left[-\left(\dfrac{x_1^3}{3} - \dfrac{L^2 x_1}{4}\right) + (2+\mu) x_2^2 x_1\right] + h^2\left[-\dfrac{x_1}{10} - \mu\dfrac{x_1}{4}\right]\right] & 1
\end{array}\right]
\left\{\begin{array}{c} \dot{\iota}_1 \\ \\ \dot{\iota}_2 \end{array}\right\}
\qquad (2\text{-}12)
$$

The [A] matrix is nonsymmetric. The cosine of the angle between $\dot{\tilde{\iota}}_1^{*}$ and $\dot{\tilde{\iota}}_2^{*}$ is

$$\cos\hat{\theta} = \frac{2(1+\mu)\hat{q}h^2 x_1}{8EI}\left[1 - \left(\frac{x_2}{h/2}\right)^2\right].$$

On the stress-free lines $x_2 = \pm\frac{h}{2}$, the angle between $\dot{\tilde{\iota}}_1^{*}$ and $\dot{\tilde{\iota}}_2^{*}$ is $\frac{\pi}{2}$ radians. In general, one obtain

$$\cos\theta = \frac{1}{G}\,\frac{\hat{q}h^2 x_1}{8I}\left[1 - \left(\frac{x_2}{h/2}\right)^2\right]$$

$$= \frac{\tau_{12}\big|_{x_2=0}}{G}\left[1 - \left(\frac{x_2}{h/2}\right)^2\right]$$

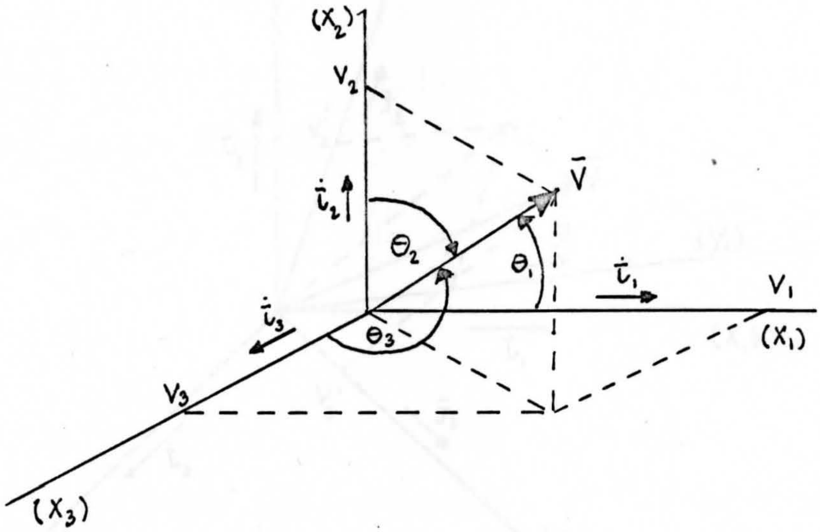
$$= \gamma_{12}\big|_{x_2=0}\left[1 - \left(\frac{x_2}{h/2}\right)^2\right]$$

Thus, lines parallel to x_2 before deformation do not remain perpendicular to the curvilinear neutral axis after deformation except the line $x_1 = 0$.

CHAPTER III

BIORTHOGONAL COORDINATES

3.1 General Transformation Matrix



Figure(3-1) Rectangular Vector Components

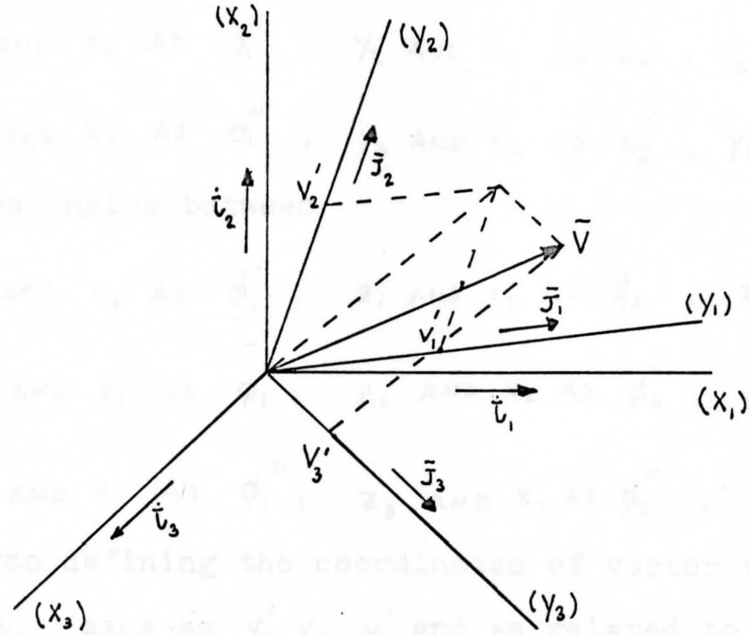
Given the X_1, X_2, X_3 axes defining a three dimensional orthogonal coordinate frame which has the unit vectors $\dot{\iota}_1, \dot{\iota}_2, \dot{\iota}_3$, respectively. Also, vector $\bar{V}$ has the orthogonal components V_1, V_2, V_3 related to the X_1, X_2, X_3 axes. (See Figure(3-1)). It follows that

$$\bar{V} = V_1\dot{\iota}_1 + V_2\dot{\iota}_2 + V_3\dot{\iota}_3$$

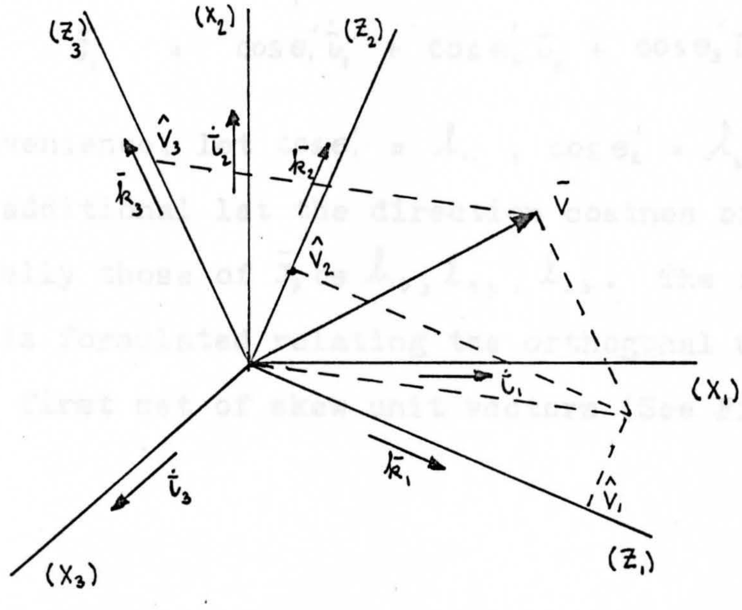
or in matrix form

$$\{V\} = \{V_1 \quad V_2 \quad V_3\}\begin{Bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \end{Bmatrix} \tag{3-1a}$$

$$\{V\} = \{V\}^T\{\dot{i}\} \tag{3-1b}$$



Figure(3-2a) First Skew Coordinate Axes



Figure(3-2b) Second Skew Coordinate Axes

Given y_1, y_2, y_3 and z_1, z_2, z_3 as additional sets of skew coordinate axes which have the unit vectors $\bar{J}_1, \bar{J}_2, \bar{J}_3$ and $\bar{k}_1, \bar{k}_2, \bar{k}_3$, respectively. Defining the angles between

y_1 AND x_1 AS θ_1' , y_1 AND x_2 AS θ_2' , y_1 AND x_3 AS θ_3'

y_2 AND x_1 AS θ_1'' , y_2 AND x_2 AS θ_2'' , y_2 AND x_3 AS θ_3''

y_3 AND x_1 AS θ_1''' , y_3 AND x_2 AS θ_2''' , y_3 AND x_3 AS θ_3'''

and the angles between

z_1 AND x_1 AS ϕ_1' , z_1 AND x_2 AS ϕ_2' , z_1 AND x_3 AS ϕ_3'

z_2 AND x_1 AS ϕ_1'' , z_2 AND x_2 AS ϕ_2'' , z_2 AND x_3 AS ϕ_3''

z_3 AND x_1 AS ϕ_1''' , z_3 AND x_2 AS ϕ_2''' , z_3 AND x_3 AS ϕ_3'''

and also defining the coordinates of vector $\bar{V}$ as related to y_1, y_2, y_3 axes as v_1', v_2', v_3' and as related to z_1, z_2, z_3 axes as $\hat{v}_1, \hat{v}_2, \hat{v}_3$, one obtains

$$\bar{J}_1 = \cos\theta_1' \dot{\bar{\iota}}_1 + \cos\theta_2' \dot{\bar{\iota}}_2 + \cos\theta_3' \dot{\bar{\iota}}_3$$

For convenience, let $\cos\theta_1' = \ell_{11}$, $\cos\theta_2' = \ell_{21}$ and $\cos\theta_3' = \ell_{31}$, and in additional let the direction cosines of $\bar{J}_2$ be $\ell_{12}, \ell_{22}, \ell_{32}$ and finally those of $\bar{J}_3$ be $\ell_{13}, \ell_{23}, \ell_{33}$. The following matrix is formulated relating the orthogonal unit vectors and the first set of skew unit vectors (See Figure(3-2a)) as

$$\begin{Bmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{Bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{Bmatrix} \dot{\bar{\imath}}_1 \\ \dot{\bar{\imath}}_2 \\ \dot{\bar{\imath}}_3 \end{Bmatrix} \qquad (3\text{-}2a)$$

or symbolically

$$\{\bar{J}\} = [L]^T\{\dot{\imath}_1\} \qquad (3\text{-}2b)$$

The vector $\bar{V}$ related to the first skew axes set becomes

$$\bar{V} = V_1'\bar{J}_1 + V_2'\bar{J}_2 + \hat{V}_3\bar{J}_3$$

or in matrix form

$$\{V\} = \{V_1' \;\; V_2' \;\; V_3'\} \begin{Bmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{Bmatrix} \qquad (3\text{-}3a)$$

$$\{V\} = \{V'\}^T\{\bar{J}\} \qquad (3\text{-}3b)$$

Combining Equations (3-1b) and (3-3b) gives

$$\{V_1 \;\; V_2 \;\; V_3\} \begin{Bmatrix} \dot{\imath}_1 \\ \dot{\imath}_2 \\ \dot{\imath}_3 \end{Bmatrix} = \{V_1' \;\; V_2' \;\; V_3'\} \begin{Bmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{Bmatrix} \qquad (3\text{-}4a)$$

$$\{V\}^T\{\dot{\imath}\} = \{V'\}^T\{\bar{J}\} \qquad (3\text{-}4b)$$

Substituting Equation (3-2b) into Equation (3-4b) yields

$$\{V\}^T[L]^{-T}\{\bar{J}\} = \{V'\}^T\{\bar{J}\}$$

$$\{V\}^T[L]^{-T} = \{V'\}^T$$

$$[L]^{-1}\{V\} = \{V'\} \qquad (3\text{-}5a)$$

$$\{V\} = [L]\{V'\}$$

In a similar manner one obtains

$$\{v\} = [\hat{L}]\{\hat{v}\} \qquad (3\text{-}5b)$$

Equating Equations (3-5a) and (3-5b) gives

$$[L]\{v'\} = [\hat{L}]\{\hat{v}\}$$

$$\{v'\} = [L]^{-1}[\hat{L}]\{\hat{v}\} \qquad (3\text{-}6)$$

Assuming $|[L]| \neq 0$, and defining

$$[L]^{-1}[\hat{L}] = [T] \qquad (3\text{-}7)$$

one obtains

$$\{v'\} = [T]\{\hat{v}\} \qquad (3\text{-}8)$$

The matrix $[T]$ in the latter equation is the matrix that transforms $\{\hat{v}\}$ into $\{v'\}$. In general it is nonsymmetric. A special case arises when y_1, y_2, y_3 axes and the z_1, z_2, z_3 axes are both orthogonal. It follows that

$$[L]^T = [L]^{-1} \qquad (3\text{-}9a)$$

$$[\hat{L}]^T = [\hat{L}]^{-1} \qquad (3\text{-}9b)$$

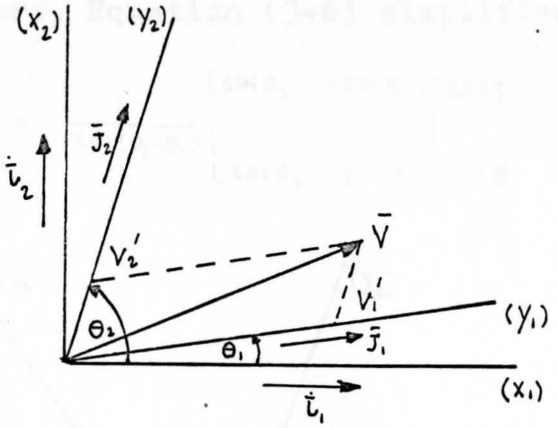
Equation (3-7) becomes

$$[T]^{-1} = [\hat{L}]^T[L] \qquad (3\text{-}10a)$$

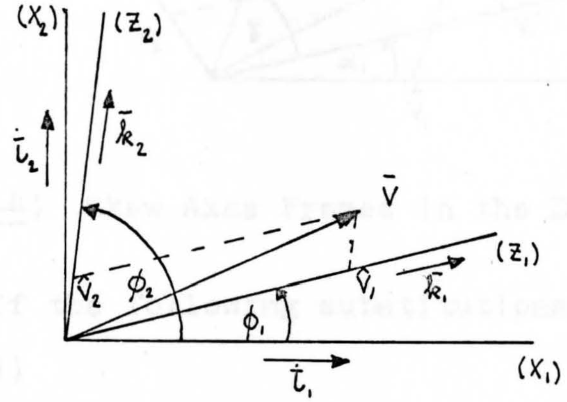
or

$$[T]^T = [T]^{-1} \qquad (3\text{-}10b)$$

Thus, $[T]$ is an orthogonal matrix.



Figure(3-3a) First Skew Coordinate Axes in Two Dimensions



Figure(3-3b) Second Skew Coordinate Axes in Two Dimentions

For the case of two dimensions only

$$[L] = \begin{bmatrix} \cos\theta_1 & \cos(\frac{\pi}{2}-\theta_1) \\ \cos\theta_2 & \cos(\frac{\pi}{2}-\theta_2) \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{bmatrix} \qquad (3\text{-}11a)$$

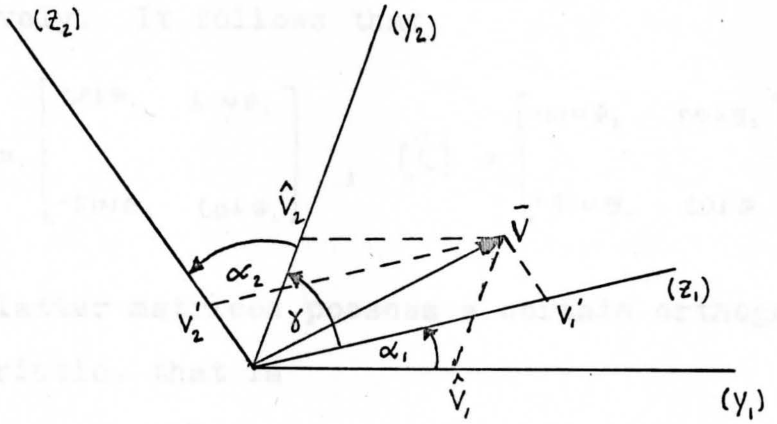
$$[\hat{L}] = \begin{bmatrix} \cos\phi_1 & \sin\phi_1 \\ \cos\phi_2 & \sin\phi_2 \end{bmatrix} \qquad (3\text{-}11b)$$

It follows that,

$$[L]^{-1} = \frac{1}{\sin(\theta_2-\theta_1)} \begin{bmatrix} \sin\theta_2 & -\sin\theta_1 \\ -\cos\theta_2 & \cos\theta_1 \end{bmatrix} \qquad (3\text{-}11c)$$

and therefore, Equation (3-6) simplifies to the form

$$\left\{V'\right\} = \frac{1}{\sin(\theta_2-\theta_1)}\begin{bmatrix}\sin\theta_2 & -\sin\theta_1\\ -\cos\theta_2 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\cos\phi_1 & \sin\phi_1\\ \cos\phi_2 & \sin\phi_2\end{bmatrix}\left\{\hat{V}\right\} \tag{3-12}$$



Figure(3-4) Skew Axes Frames in the Dimensions

If the following substitutions are made (See Figure(3-4))

$$\theta_1 = 0$$

$$\theta_2 = \gamma$$

$$\phi_1 = \alpha_1$$

$$\phi_2 = \alpha_2 + \gamma$$

One obtains from Equation (3-12) the following result

$$\begin{Bmatrix}V_1'\\ V_2'\end{Bmatrix} = \frac{1}{D}\begin{bmatrix}\sin\alpha_2\cot\gamma + \cos\alpha_2 & \sin\alpha_2\csc\gamma\\ -\sin\alpha_1\csc\gamma & \cos\alpha_1 - \sin\alpha_1\cot\gamma\end{bmatrix}\begin{Bmatrix}\hat{V}_1\\ \hat{V}_2\end{Bmatrix} \tag{3-13a}$$

where

$$D = \cos(\alpha_2-\alpha_1) + \cot\gamma\sin(\alpha_2-\alpha_1) \tag{3-13b}$$

This result has been obtained for this special two dimensional case by Kardestuncer[8]. Assuming $\phi_1 = \theta_2 + \frac{\pi}{2}$ and $\phi_2 = \theta_1 + \frac{\pi}{2}$ the notion of a biorthogonal basis set is defined. Axes y_1 and y_2 are perpendicular to axes z_2 and z_1, respectively. It follows that

$$[L] = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad [\hat{L}] = \begin{bmatrix} -\sin\theta_2 & \cos\theta_2 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad (3\text{-}14a)$$

The two latter matrices possess a certain orthogonality characteristic, that is

$$[L][\hat{L}^*]^T = [I] \qquad (3\text{-}14b)$$

where $[\hat{L}^*]$ is the normalized form of $[\hat{L}]$ with respect to matrix $[L]$. This concept is called a biorthogonal condition and is taken up in detail later in this Chapter.

For the case where $\theta_2 = \theta_1 + \frac{\pi}{2}$ and $\phi_2 = \phi_1 + \frac{\pi}{2}$, y_1 and z_1 are perpendicular to y_2 and z_2, respectively, one obtains

$$[L] = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos(\theta_1 + \frac{\pi}{2}) & \sin(\theta_1 + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad (3\text{-}15a)$$

and

$$[\hat{L}] = \begin{bmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{bmatrix} \qquad (3\text{-}15b)$$

where

$$|[L]| = |[\hat{L}]| = 1$$

Since $[L]$ and $[\hat{L}]$ are both orthogonal, it follows that

$$[T]^{-1} = [T]^{T} \tag{3-16}$$

and $[T]$ is an orthogonal matrix.

3.2 Biorthogonal Transformation

When the equation of quadratic surface is defined related to a skew coordinate reference system, the matrix $[A]$ associated with this geometry is by mathematical consequence a nonsymmetric matrix. The diagonal form of this matrix $[A]$ is defined as the eigenvalue matrix $[\Lambda]$. Let $[U]$ be the matrix which transforms $[A]$ into $[\Lambda]$, where $[U]$ is associated with the eigenvectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ which form a set of skew angular axes. As shown by Lanczos[5], one obtains

$$[A][U] = [U][\Lambda] \tag{3-17a}$$

or

$$[\Lambda] = [U]^{-1}[A][U] \tag{3-17b}$$

The transpose of matrix $[A]$ has identical eigenvalues as $[A]$. Let $[V]$ be the matrix which transforms $[A]^T$ into $[\Lambda]$, where $[V]$ is associated with the eigenvectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ which also form another set of skew angular axes; one obtains

$$[A]^T[V] = [V][\Lambda] \tag{3-17c}$$

or

$$[\Lambda] = [V]^{-1}[A]^T[V] \tag{3-17d}$$

Taking the transpose of both sides of Equation (3-17c) gives

$$[V]^T[A] = [\Lambda][V]^T$$

Postmultiplying the latter equation with $[U]$ yields

$$[V]^T[A][U] = [\Lambda][V]^T[U]$$

Substituting Equation (3-17a) into the previous equation gives

$$\left[[V]^T[U]\right][\Lambda] = [\Lambda]\left[[V]^T[U]\right] \qquad (3\text{-}18)$$

Thus, the matrices $\left[[V]^T[U]\right]$ and $[\Lambda]$ commute and have the same principal axes. Since $[\Lambda]$ is a diagonal matrix, then $\left[[V]^T[U]\right]$ must be a diagonal matrix also. It follows that,

$$(\bar{V}_i \cdot \bar{u}_k) = 0 \qquad i \neq k \qquad (3\text{-}19a)$$

$$(\bar{V}_i \cdot \bar{u}_k) = 1 \qquad i = k \qquad (3\text{-}19b)$$

if one normalizes the vector $\bar{V}_i$ with respect to $\bar{u}_i$. Equations (3-19a) and (3-19b) define the notion of a biorthogonal set of vectors. In symbolic matrix form one writes

$$[V]^T[U] = [I] \qquad (3\text{-}20a)$$

or

$$[U]^{-1} = [V]^T \qquad (3\text{-}20b)$$

or

$$[V]^{-1} = [U]^T \tag{3-20c}$$

Equating Equations (3-17b) and (3-17d) gives

$$[\Lambda] = [U]^{-1}[A][U] = [V]^{-1}[A]^T[V] \tag{3-21a}$$

and noting Equations (3-20b) and (3-20c) yields

$$[\Lambda] = [V]^T[A][U] = [U]^T[A]^T[V] \tag{3-21b}$$

This transformation is called the biorthogonal transformation. For the special case when $[A]$ is a symmetric matrix, $[A] = [A]^T$, it follows that $[U] = [V]$. Equations (3-21b) and (3-20a) simplify to the form

$$[\Lambda] = [U]^T[A][U] \tag{3-22a}$$

and

$$[U]^T[U] = [I] \tag{3-22b}$$

respectively. This transformation is characterized by an orthogonal rotation of axes which is associated with an orthogonal transformation.

Numerical Example

Given $$[A] = \begin{bmatrix} 33 & 16 & 72 \\ -24 & -10 & -57 \\ -8 & -4 & -17 \end{bmatrix}$$

The three invariants of the matrix [A] are

$$I_1 = 33-10-17 = 6$$

$$I_2 = \begin{vmatrix} -10 & -57 \\ -4 & -17 \end{vmatrix} + \begin{vmatrix} 33 & 72 \\ -8 & -17 \end{vmatrix} + \begin{vmatrix} 33 & 16 \\ -24 & -10 \end{vmatrix} = 11$$

$$I_3 = 33\begin{vmatrix} -10 & -57 \\ -4 & -17 \end{vmatrix} - 16\begin{vmatrix} -24 & -67 \\ -8 & -17 \end{vmatrix} + 72\begin{vmatrix} -24 & -10 \\ -8 & -4 \end{vmatrix} = 6$$

The characteristic equation becomes

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

with eigenvalues as

$$\lambda = 1, 2, 3$$

For $\lambda = 1$ the eigenvector is $\{u_1\} = \begin{Bmatrix} -15 \\ 12 \\ 4 \end{Bmatrix}$

For $\lambda = 2$ the eigenvector is $\{u_2\} = \begin{Bmatrix} -16 \\ 13 \\ 4 \end{Bmatrix}$

For $\lambda = 3$ the eigenvector is $\{u_3\} = \begin{Bmatrix} -4 \\ 3 \\ 1 \end{Bmatrix}$

The [U] matrix is constructed as

$$[U] = \begin{bmatrix} -15 & -16 & -4 \\ 12 & 13 & 3 \\ 4 & 4 & 1 \end{bmatrix}.$$

Noting

$$[A]^T = \begin{bmatrix} 33 & -24 & -8 \\ 16 & -10 & -4 \\ 72 & -57 & -17 \end{bmatrix},$$

the eigenvalues of $[A]$ and $[A]^T$ are the same, hence

$$\lambda = 1, 2, 3$$

For $\lambda = 1$ the eigenvector is $\qquad \{V_1^*\} = \begin{Bmatrix} 1/4 \\ 0 \\ 1 \end{Bmatrix}.$

For $\lambda = 2$ the eigenvector is $\qquad \{V_2^*\} = \begin{Bmatrix} 0 \\ -1/3 \\ 1 \end{Bmatrix}.$

For $\lambda = 3$ the eigenvector is $\qquad \{V_3^*\} = \begin{Bmatrix} 4/3 \\ 4/3 \\ 1 \end{Bmatrix}.$

The matrix $[V^*]$ becomes

$$[V^*] = \begin{bmatrix} 1/4 & 0 & 4/3 \\ 0 & -1/3 & 4/3 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is noted that Equation (3-19a) is satisfied although Equation (3-19b) is not satisfied. Normalizing matrix $[\overset{*}{V}]$ with respect to [U] to satisfy this latter condition yields

$$[V] = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 4 & -3 & -3 \end{bmatrix}$$

The numerical forms of Equation (3-21b) becomes

$$[V]^T[A][U] = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 33 & 16 & 72 \\ -24 & -10 & -57 \\ -8 & -4 & -17 \end{bmatrix}\begin{bmatrix} -15 & -16 & -4 \\ 12 & 13 & 3 \\ 4 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = [\Lambda]$$

$$[U]^T[A]^T[V] = \begin{bmatrix} -15 & 12 & 4 \\ -16 & 13 & 4 \\ -4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 33 & -24 & -8 \\ 16 & -10 & -4 \\ 72 & -57 & -17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = [\Lambda]$$

Also Equation (3-20a) is satisfied by the numerical form

$$[V]^T[U] = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} -15 & -16 & -4 \\ 12 & 13 & 3 \\ 4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I]$$

3.3 Replacement of a Nonsymmetric Matrix by a Symmetric Matrix and a Skew Symmetric Matrix

Any nonsymmetric matrix may be replaced by the sum of a symmetric and a skew symmetric matrix in the form

$$[C] = [A] + [B] \tag{3-23a}$$

where

$$[A] = [A]^T = \frac{1}{2}\left[[C] + [C]^T\right] \tag{3-23b}$$

and

$$[B] = -[B]^T = \frac{1}{2}\left[[C] - [C]^T\right] \tag{3-23c}$$

and

$$[C]^T = [A] - [B] \tag{3-23d}$$

Three general cases of matrix $[C]$ are observed:

Case I - $[C]$ is symmetric, $[B] = [0]$, and $[C] = [A]$

Case II - $[C]$ is skew symmetric, $[A] = [0]$, and $[C] = [B]$

Case III - $[C]$ is orthogonal; this case is the most important of the three and is considered in detail. Noting the orthogonality conditions on $[C]$, it follows that

$$[C][C]^T = \left[[A]+[B]\right]\left[[A]-[B]\right] = [A]^2 + [B][A] - [A][B] - [B]^2 = [I] \tag{3-24a}$$

and

$$[C]^T[C] = \left[[A]-[B]\right]\left[[A]+[B]\right] = [A]^2 - [B][A] + [A][B] - [B]^2 = [I] \tag{3-24b}$$

Subtracting Equations (3-24a) and (3-24b) yields

$$[A][B] = [B][A] \tag{3-25}$$

Noting Equation (3-23a), postmultiplying and premultiplying by $[A]$ yields, respectively

$$[C][A] = [A]^2 + [B][A] \tag{3-26a}$$

and

$$[A][C] = [A]^2 + [A][B] \tag{3-26b}$$

Combining the latter two equations with Equation (3-25) gives

$$[C][A] = [A][C] \tag{3-27}$$

Similarly, one obtains

$$[C][B] = [B][C] \tag{3-28}$$

Combining Equations (3-25), (3-27), and (3-28) yields

$$[A][B][C] = [C][B][A] = [A][C][B] = \ldots$$

Thus, matrices $[A]$, $[B]$ and $[C]$ are diagonalized to $[\Lambda_A]$, $[\Lambda_B]$, $[\Lambda_C]$ by the same transformation matrix which is formed from their eigenvectors. Let the matrix that transforms $[C]$ into $[\Lambda_C]$ be $[W]$ which is composed of complex components. This matrix $[W]$ also transforms $[A]$ and $[B]$ into $[\Lambda_A]$ and $[\Lambda_B]$. Premultiplying and postmultiplying Equation (3-23a) by $[\tilde{W}]^T$

and $[W]$, respectively gives

$$[\tilde{W}]^T[C][W] = [\tilde{W}]^T[A][W] + [\tilde{W}]^T[B][W] \qquad (3\text{-}29a)$$

and

$$[\Lambda_C] = [\Lambda_A] + [\Lambda_B] \qquad (3\text{-}29b)$$

The symbol $[\tilde{\ }]$ denotes complex conjugate form. All elements of $[\Lambda_A]$ are real since $[A]$ is symmetric and all elements of $[\Lambda_B]$ are imaginary or zero since $[B]$ is a skew symmetric matrix.

An alternate approach to the problem is to determine the real matrix $[M]$ that transforms $[A]$ into $[\Lambda_A]$, and observe the operation of $[M]$ on the matrices $[C]$ and $[B]$ in the form

$$[M]^T[C][M] = [M]^T[A][M] + [M]^T[B][M] \qquad (3\text{-}30)$$

Matrix $[C^*] = [M]^T[C][M]$ remains orthogonal since

$$[C^*][C^*]^T = [M]^T[C][M][M]^T[C]^T[M] = [I]$$

Also the matrices $[M]^T[A][M]$ and $[M]^T[B][M]$ remain symmetric and skew symmetric, respectively since

$$[M]^T[A][M] = [\Lambda_A] = \tfrac{1}{2}[M]^T\left[[C]+[C]^T\right][M] = \tfrac{1}{2}\left[[C^*]+[C^*]^T\right]$$

and

$$[M]^T[B][M] = [B^*] = \tfrac{1}{2}[M]^T\left[[C]-[C]^T\right][M] = \tfrac{1}{2}\left[[C^*]-[C^*]^T\right]$$

Using the complex analysis approach, one characteristic value of [C] must be either ± 1, with other two values as complex conjugate pairs. The diagonal matrix $[\Lambda_c]$ is written for convenience as

$$[\Lambda_c] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a+ib & 0 \\ 0 & 0 & a-ib \end{bmatrix} \tag{3-31a}$$

where $a^2+b^2=1$. It follows that,

$$[\Lambda_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \qquad [\Lambda_b] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & -ib \end{bmatrix} \tag{3-31b}$$

and hence by Equation (3-29b), one obtains

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a+ib & 0 \\ 0 & 0 & a-ib \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & -ib \end{bmatrix} \tag{3-31c}$$

Using the alternate real analysis approach, one obtains

$$[\overset{*}{C}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix} \tag{3-32a}$$

and

$$[\Lambda_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \qquad [B^*] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix} \tag{3-32b}$$

and hence, by Equation (3-31c), it follows that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix} \tag{3-32c}$$

The three invariants of the matrices are

MATRIX	I_1	I_2	I_3
$[C]$	$2a+1$	$a^2+b^2+2a = 2a+1$	1
$[A]$	$2a+1$	a^2+2a	a^2
$[B]$	0	b^2	0

It should be noted that $\mathrm{TR}[A] = \mathrm{TR}[C]$, $I_1 = I_2$ for matrix $[C]$, and the sum of the second invariants for $[A]$ and $[B]$ equals the second invariant of matrix $[C]$.

<u>Numerical Example</u>

Given $[C] = \begin{bmatrix} .6 & .7845 & .1569 \\ .8 & -.5883 & -.1177 \\ 0 & .1961 & -.9806 \end{bmatrix}$, $[C]^T[C] = [I]$

The three invariants of $[C]$ are

$I_1 = -.9689 \quad , \quad I_2 = -.9689 \quad , \quad I_3 = 1$

The characteristic equation becomes

$$\lambda^3 + .9689\lambda^2 - .9689\lambda - 1 = 0$$

with eigenvalues determined as

$$\lambda = 1, -.9845 \pm .1757i$$

$$[A] = \frac{1}{2}\left[[C] + [C]^T\right] = \begin{bmatrix} .6 & .7963 & .0785 \\ .7923 & -.5883 & .0392 \\ .0785 & .0392 & -.9802 \end{bmatrix}$$

The three invariants of [A] are

$I_1 = -.9689 \quad , \quad I_2 = -1 \quad , \quad I_3 = .9689$

The characteristic equation becomes

$$\lambda^3 + .9689\lambda^2 - \lambda - .9689 = 0$$

with eigenvalues determined as

$$\lambda = 1, -.9845, -.9845$$

$$[B] = \frac{1}{2}\left[[C] - [C]^T\right] = \begin{bmatrix} 0 & -.0078 & .0785 \\ .0078 & 0 & -.1569 \\ -.0785 & .1569 & 0 \end{bmatrix}$$

The three invariants of [B] are

$I_1 = 0 \quad , \quad I_2 = .0308 \quad , \quad I_3 = 0$

The characteristic equation becomes

$$\lambda^3 + .0308\lambda = 0$$

with eigenvalues determined as

$$\lambda = 0, \pm .1756 i$$

Using the complex analysis approach the complex eigenvector matrix [W] becomes

$$[W] = \begin{bmatrix} .8936 & .3174 & .3174 \\ .4468 & (-.6286-.0697i) & (-.6286+.0697i) \\ .0441 & (-.0626-.7038i) & (-.0626+.7038i) \end{bmatrix}$$

with

$$[\tilde{W}]^T = \begin{bmatrix} .8936 & .4468 & .0441 \\ .3174 & (-.6286+.0697i) & (-.0626+.7038i) \\ .3174 & (-.6286-.0697i) & (-.0626-.7038i) \end{bmatrix}$$

and

$$[\tilde{W}]^T[W] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I] .$$

Also

$$[\tilde{W}]^T[C][W] = [\Lambda_c] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845+.1757i & 0 \\ 0 & 0 & -.9845-.1757i \end{bmatrix}$$

$$[\tilde{W}]^T[A][W] = [\Lambda_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845 & 0 \\ 0 & 0 & -.9845 \end{bmatrix}$$

$$[\tilde{W}]^T[B][W] = [\Lambda_B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .1757i & 0 \\ 0 & 0 & -.1757i \end{bmatrix}$$

It should be noted that $a = -.9845$, $b = .1757$ and $a^2 + b^2 = 1$. Also, the first eigenvector of the $[W]$ matrix is comprised of the normalized components of the vector associated with the skew symmetric matrix $[B]$.

Using the real analysis approach the real eigenvector matrix $[M]$ is determined as

$$[M] = \begin{bmatrix} .8936 & .0981 & -.4415 \\ .4468 & -.0981 & .8874 \\ .0441 & -.9903 & -.1325 \end{bmatrix}$$

and

$$[M]^T[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I]$$

Also

$$[M]^T[C][M] = [C^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845 & -.1757 \\ 0 & .1757 & -.9845 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}$$

$$[M]^T[A][M] = [\Lambda_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845 & 0 \\ 0 & 0 & -.9845 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$[M]^T[B][M] = [\Lambda_B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -.1757 \\ 0 & .1757 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix}$$

hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845 & -.1757 \\ 0 & .1757 & -.9845 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.9845 & 0 \\ 0 & 0 & -.9845 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -.1757 \\ 0 & .1757 & 0 \end{bmatrix}$$

Let $\{b\}$ be the equivalent vector associated with the skew symmetric matrix $[B]$, or

$$\{b\} = \begin{Bmatrix} .1569 \\ -.0785 \\ .0078 \end{Bmatrix}$$

After normalizing $\{b\}$ to the unit vector $\{b_u\}$ with

$$\{b_u\} = \begin{Bmatrix} .8936 \\ .4468 \\ .0441 \end{Bmatrix},$$

$\{b_u\}$ is equal to the one eigenvector of $[M]$ which is associated with $\lambda_A = 1$, and the single real eigenvector of $[W]$. The magnitude of $\{b\}$ before normalizing is

$$|\{b\}| = \sqrt{.1569^2 + .0785^2 + .0078^2} = .1757 = b$$

which is the remaining component of $[B^*]$.

CHAPTER IV

EIGENVALUE-EIGENVECTOR PROBLEM

4.1 The Multiplication of a Matrix by Its Transpose

The multiplication of a nonsymmetric matrix $[A]$ with real components by its transpose produces a symmetric matrix, that is,

$$[A]^T[A] \quad = \quad [B] \quad = \quad [B]^T \qquad (4\text{-}1a)$$

$$[A][A]^T \quad = \quad [C] \quad = \quad [C]^T \qquad (4\text{-}1b)$$

where both $[B]$ and $[C]$ are distinct but symmetric matrices. Postmultiplying Equation (4-1b) by $[A]$ and noting Equation (4-1a) gives

$$[C][A] \quad = \quad [A][B]$$

or

$$[C] \quad = \quad [A][B][A]^{-1} \qquad (4\text{-}2)$$

Subtracting the quantity $\lambda[I]$ from both sides of Equation (4-2) gives

$$\left[[C]-\lambda[I]\right] \quad = \quad \left[[A][B][A]^{-1}-\lambda[I]\right]$$

Substituting the condition $[A][A]^{-1}=[I]$ into the latter equation and simplifying yields

$$\Big[[C] - \lambda[I]\Big] = \Big[[A][B][A]^{-1} - \lambda[A][A]^{-1}\Big]$$

or

$$\Big[[C] - \lambda[I]\Big] = [A]\Big[[B] - \lambda[I]\Big][A]^{-1}$$

Taking the determinant of both sides of the previous equation gives

$$\Big|\big[[C] - \lambda[I]\big]\Big| = \big|[A]\big|\,\Big|\big[[B]-\lambda[I]\big]\Big|\,\big|[A]^{-1}\big|$$

Noting that $\big|[A][A]^{-1}\big| = \big|[A]\big|\big|[A]^{-1}\big| = 1$, the latter equation reduces to

$$\Big|\big[[C] - \lambda[I]\big]\Big| = \Big|\big[[B] - \lambda[I]\big]\Big| \qquad (4\text{-}3a)$$

Hence, $[B]$ and $[C]$ have the same characteristic equation and identical eigenvalues, that is,

$$[\Lambda_B] = [\Lambda_C] = [\hat{\Lambda}] \qquad (4\text{-}3b)$$

In general $[B]$ and $[C]$ have different sets of orthogonal eigenvectors.

4.2 General Case, $[B] \neq [C]$

In general matrix $[B]$ is not related to matrix $[C]$. Let $[X]$ and $[Y]$ be the orthogonal matrices which transform $[B]$ and $[C]$ into the diagonal matrix. Also, let $[U]$ and $[V]$ be the biorthogonal transformation matrices that transform $[A]$ and $[A]^T$ into $[\Lambda]$. Noting Equations (3-21b) and (3-22a), one obtains

$$[A] = [U][\Lambda][V]^T \qquad (4\text{-}4a)$$

$$[A]^T = [V][\Lambda][U]^T \tag{4-4b}$$

and

$$[\hat{\Lambda}] = [X]^T[B][X] = [Y]^T[C][Y] \tag{4-5}$$

Substituting Equations (4-4a) and (4-4b) into Equations (4-1a) and (4-1b) gives

$$[B] = [V][\Lambda][U]^T[U][\Lambda][V]^T \tag{4-6a}$$

and

$$[C] = [U][\Lambda][V]^T[V][\Lambda][U]^T \tag{4-6b}$$

Substituting Equations (4-6a) and (4-6b) into Equation (4-5) gives

$$[\hat{\Lambda}] = [X]^T[V][\Lambda][U]^T[U][\Lambda][V]^T[X] = [Y]^T[U][\Lambda][V]^T[V][\Lambda][U]^T[Y] \tag{4-7}$$

Equation (4-7) represents the general relationship between the eigenvalues of the compound matrices [B] and [C] and the eigenvalues of the single matrices [A] and $[A]^T$.

4.3 Special Case, [B] = [C]

A special case arises when matrices [B] and [C] commute in the form

$$[B][C] = [C][B] \tag{4-8}$$

Matrices [B] and [C] have the same principal axes, thus

$$[X] = [Y] \tag{4-9}$$

Noting Equations (4-5) and (4-9), one obtains

$$[B] = [C] = [X][\hat{\Lambda}][X]^T = [Y][\hat{\Lambda}][Y]^T \qquad (4\text{-}10)$$

Equation (4-7) reduces to

$$[V][\Lambda][U]^T[U][\Lambda][V]^T = [U][\Lambda][V]^T[V][\Lambda][U]^T \qquad (4\text{-}11)$$

This special case occurs for three important properties of matrix $[A]$

1) $[A]$ is orthogonal, $[A]^{-1} = [A]^T$, then

$$[B] = [C] = [I] \qquad (4\text{-}12a)$$

2) $[A]$ is skew symmetric, $[A] = -[A]^T$, then

$$[B] = [C] = -[A]^2 = -\left[[A]^T\right]^2 . \qquad (4\text{-}12b)$$

3) $[A]$ is symmetric, $[A] = [A]^T$, then

$$[B] = [C] = [A]^2 \qquad (4\text{-}12c)$$

4.4 $[A]$ Is Orthogonal.

Consider the case when $[A]$ is a nonsymmetric orthogonal matrix. Matrices $[A]$, $[B]$, $[C]$, $[\hat{\Lambda}]$, $[X]$, and $[Y]$ have all real components but matrices $[U]$, $[V]$ and $[\Lambda]$ may possess certain components which are complex numbers. Since $[A]$ is a real matrix it equals its complex conjugate (i.e. $[A] = [\tilde{A}]$). Equation (4-4a) is written as

$$[A] = [U][\Lambda][V]^T = [\tilde{U}][\tilde{\Lambda}][\tilde{V}]^T$$

Noting the latter equation and Equation (4-12a), Equation

(4-6a) becomes

$$[B] = [V][\Lambda][U]^T[\tilde{U}][\tilde{\Lambda}][\tilde{V}]^T = [I]$$

Noting that $[V]^T[U] = [U]^T[V] = [I]$, the previous equation is written as

$$[\tilde{U}][\tilde{\Lambda}][\tilde{V}]^T = [V][\Lambda]^{-1}[U]^T$$

or

$$[\Lambda]^{-1} = [U]^T[\tilde{U}][\tilde{\Lambda}][\tilde{V}]^T[V] \qquad (4\text{-}13)$$

Since $[A]$ is orthogonal the absolute value of each eigenvalue of $[A]$ must be one. Thus

$$[\Lambda][\tilde{\Lambda}] = [I] \qquad (4\text{-}14a)$$

or

$$[\tilde{\Lambda}] = [\Lambda]^{-1} \qquad (4\text{-}14b)$$

Comparing Equation (4-13), one obtains

$$[U]^T[\tilde{U}] = [\tilde{V}]^T[V] = [I]$$

hence

$$[V] = [\tilde{U}] \qquad (4\text{-}15)$$

It should be noted that if $[A]$ is an odd ordered matrix, at least one eigenvalue and eigenvector must be real. Also, the complex eigenvalues and eigenvectors must occur in complex conjugate pairs.

4.5 [A] Is Skew Symmetric

For the case when [A] is a skew symmetric matrix, the matrices [A], [B], [C], $[\hat{\Lambda}]$, [X] and [Y] all have real components; the matrices [U], [V] and $[\Lambda]$ may have some components which are complex numbers. Squaring Equations (4-4a) and (4-4b) one obtains, respectively,

$$[A]^2 = [U][\Lambda]^2[V]^T \qquad (4\text{-}16a)$$

and

$$\left[[A]^T\right]^2 = [V][\Lambda]^2[U]^T \qquad (4\text{-}16b)$$

Substituting Equation (4-16b) into Equation (4-12b) gives

$$[B] = -[V][\Lambda]^2[U]^T$$

Taking the complex conjugate of the latter equation and noting that real numbers are the conjugate of themselves, yields

$$[B] = -[\tilde{V}][\tilde{\Lambda}^2][\tilde{U}]^T \qquad (4\text{-}16c)$$

Noting Equations (4-12b), (4-16a) and (4-16c), it follows that

$$[\tilde{V}][\Lambda]^2[\tilde{U}]^T = [U][\Lambda]^2[V]^T$$

Noting that $[V]^T[U] = [I] = [\tilde{V}]^T[\tilde{U}]$ the latter equationis rewritten as

$$\left[[V]^T[\tilde{V}]\right][\Lambda]^2 = [\Lambda]^2\left[[V]^T[\tilde{V}]\right]$$

Since $[[V]^T[\tilde{V}]]$ and $[\Lambda]^2$ commute, they have the same principal axes, and $[\Lambda]^2$ is a diagonal matrix. The matrix $[[V]^T[\tilde{V}]]$ is also a diagonal matrix. But each eigenvector associated with $[V]$ is normalized to a unit vector, thus

$$[V]^T[\tilde{V}] = [I]$$

It follows that

$$[U] = [\tilde{V}] \tag{4-17}$$

Noting that $[A] = -[A]^T$, equating Equation (4-4a) to the complex conjugate of equation (4-4b) gives

$$[U][\Lambda][V]^T = -[\tilde{V}][\tilde{\Lambda}][\tilde{U}]^T$$

Substituting Equation (4-17) into the latter equation, one obtains

$$[U][\Lambda][V]^T = -[U][\tilde{\Lambda}][V]^T$$

hence

$$[\Lambda] = -[\tilde{\Lambda}] \tag{4-18a}$$

or

$$[\Lambda] + [\tilde{\Lambda}] = [0] \tag{4-18b}$$

Thus, the eigenvalues of the skew symmetric matrix have zero real part. For matrices of odd order at least one eigenvalue is zero. The complex eigenvalues have only imaginary parts and occur in complex conjugate pairs.

4.6 $[A]$ Is Symmetric

All matrices involved in this case have all real components. Squaring Equation (3-22a) gives

$$[\Lambda]^2 = [U]^T[A][U][U]^T[A][U]$$

or

$$[\Lambda]^2 = [U]^T[A]^2[U]$$

Noting Equation (4-12c) the latter equation becomes

$$[\Lambda]^2 = [U]^T[B][U] \tag{4-19}$$

Comparing Equation (4-5) to Equation (4-19) one obtains

$$[\hat{\Lambda}] = [\Lambda]^2 \tag{4-20a}$$

and

$$[U] = [X] \tag{4-20b}$$

It should be noted that all eigenvalues and eigenvectors of the symmetric matrix are real. If the matrix is both symmetric and orthogonal the eigenvalues are only ± 1.

4.7 Summary of Results

For general case $[B] \neq [C]$

$$[A] = [U][\Lambda][V]^T \quad ; \quad [A]^T = [V][\Lambda][U]^T$$

$$[B] = [X][\hat{\Lambda}][X]^T \quad ; \quad [C] = [Y][\hat{\Lambda}][Y]^T$$

$$[\hat{\Lambda}] = [\Lambda_B] = [\Lambda_C]$$

$$[\hat{\Lambda}] = [X]^T[V][\Lambda][U]^T[U][\Lambda][V]^T[X] = [Y]^T[U][\Lambda][V]^T[V][\Lambda][U]^T[Y]$$

For special case when $[B][C] = [C][B]$, i.e. $[B] = [C]$

$$[X] = [Y]$$

$$[U][\Lambda][V]^T[V][\Lambda][U]^T = [V][\Lambda][U]^T[U][\Lambda][V]^T$$

1) $[A]^T[A] = [B] = [C] = [I]$

$$[V] = [\tilde{U}] \quad ; \quad [V]^T[\tilde{V}] = [I]$$

$$[\Lambda]^{-1} = [\tilde{\Lambda}] \quad ; \quad [\hat{\Lambda}] = [I]$$

2) $[A] = -[A]^T$

$$[V] = [\tilde{U}] \quad ; \quad [V]^T[\tilde{V}] = [I]$$

$$[\Lambda] = -[\tilde{\Lambda}] \quad ; \quad [\hat{\Lambda}] = -[\Lambda]^2$$

3) $[A] = [A]^T$

$$[V] = [U] = [X] = [Y]$$

$$[\hat{\Lambda}] = [\Lambda]^2$$

CHAPTER V

THE SUPER MATRIX

5.1 Super Matrix Formulation

Any nonsymmetric matrix of order $(n \times n)$ may be utilized to construct a partitioned symmetric matrix of order $(2n \times 2n)$. Given a nonsymmetric matrix $[A]$ one constructs a super matrix $[A_s]$ in the partitioned form

$$[A_s] = [A_s]^T = \left[\begin{array}{c|c} [0] & [A] \\ \hline [A]^T & [0] \end{array}\right] \qquad (5\text{-}1a)$$

where $[A_s]$ is a symmetric matrix of order $(2n \times 2n)$, twice that of matrix $[A]$. In addition the square of $[A_s]$ becomes

$$[A_s]^2 = \left[\begin{array}{c|c} [A][A]^T & [0] \\ \hline [0] & [A]^T[A] \end{array}\right] \qquad (5\text{-}1b)$$

The eigenvalues of $[A_s]^2$ are equal to those of $\left[[A]^T[A]\right]$ and $\left[[A][A]^T\right]$ which are always positive or zero. Let α^2 be an eigenvalue of $[A_s]^2$, the eigenvalues of $[A_s]$ become $\pm\alpha$. Since $[A_s]$ is symmetric the eigenvectors of $[A_s]$ and $[A_s]^2$ are identical. Let $\{w\}$ be an eigenvector of $[A_s]$ and $[A_s]^2$ written in the partitioned form as $\left\{\begin{matrix} m \\ \hline n \end{matrix}\right\}$. Hence,

$$\Big[[A_s] - \alpha[I]\Big]\{W\} = \{0\} \tag{5-2a}$$

or

$$\left[\begin{array}{c|c} -\alpha[I] & [A] \\ \hline [A]^T & -\alpha[I] \end{array}\right]\left\{\begin{array}{c} \{m\} \\ \hline \{n\} \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \{0\} \end{array}\right\} \tag{5-2b}$$

Noting Equation (5-2b), one has

$$[A]\{n\} = \alpha\{m\} \tag{5-3a}$$

and

$$[A]^T\{m\} = \alpha\{n\} \tag{5-3b}$$

which is a set of linearly coupled algebraic equations. For the squared form of matrix $[A_s]$, one obtains

$$\Big[[A_s]^2 - \alpha^2[I]\Big]\{W\} = \{0\} \tag{5-4a}$$

or

$$\left[\begin{array}{c|c} [A][A]^T - \alpha^2[I] & [0] \\ \hline [0] & [A]^T[A] - \alpha^2[I] \end{array}\right]\left\{\begin{array}{c} \{m\} \\ \hline \{n\} \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \{0\} \end{array}\right\} \tag{5-4b}$$

Equation (5-4b) yields the following two uncoupled equations:

$$[A][A]^T\{m\} = \alpha^2\{m\} \tag{5-5a}$$

and

$$[A][A]^T\{n\} = \alpha^2\{n\} \qquad (5\text{-}5b)$$

From the theory developed in Chapter III, one obtains

$$[A]^T[A]\{x\} = \alpha^2\{x\}$$

and

$$[A][A]^T\{y\} = \alpha^2\{y\}$$

with the condition

$$\{m\} = \{y\}, \qquad \{n\} = \{x\},$$

and hence,

$$\{W\} = \begin{Bmatrix} \{y\} \\ \hline \{x\} \end{Bmatrix}.$$

Since one normalizes $\{y\}$ and $\{x\}$ to unit vectors the magnitude of $\{W\}$ becomes $\sqrt{1+1} = \sqrt{2}$. Let $\{W_n\}$ be the normalized vector of $\{W\}$; it follows that

$$\{W_n\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} \{y\} \\ \hline \{x\} \end{Bmatrix}. \qquad (5\text{-}6a)$$

Let $[W]$ be the matrix which transforms $[A_s]$ and $[A_s]^2$ into $[\Lambda_{A_s}]$ and $[\Lambda_{A_s^2}]$, respectively. The matrix $[W]$ which is orthogonal is written as

$$[W] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} [Y] & [Y] \\ \hline [X] & -[X] \end{array}\right] \tag{5-6b}$$

Thus,

$$[W]^T[A_s][W] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} [Y]^T & [X]^T \\ \hline [Y]^T & -[X]^T \end{array}\right] \left[\begin{array}{c|c} [O] & [A] \\ \hline [A]^T & [O] \end{array}\right] \left[\begin{array}{c|c} [Y] & [Y] \\ \hline [X] & -[X] \end{array}\right] \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2} \left[\begin{array}{c|c} [Y]^T[A][X] + [X]^T[A]^T[Y] & [X]^T[A]^T[Y] \quad [Y]^T[A][X] \\ \hline [Y]^T[A][X] - [X]^T[A]^T[Y] & [Y]^T[A]^T[X] + [X]^T[A]^T[Y] \end{array}\right] \tag{5-6c}$$

Noting Equations (5-3a) and (5-3b) it follows that

$$\left.\begin{aligned} [Y]^T[A][X] &= [\alpha] \\ [X]^T[A]^T[X] &= [\alpha] \end{aligned}\right\} \tag{5-6d}$$

Substituting the latter equations into Equation (5-6c) gives

$$[W]^T[A_s][W] = \left[\begin{array}{c|c} \alpha[I] & [O] \\ \hline [O] & -\alpha[I] \end{array}\right] \tag{5-7a}$$

In a similar manner it may be shown that

$$[W]^T[A_s]^2[W] = \left[\begin{array}{c|c} \alpha^2[I] & [O] \\ \hline [O] & \alpha^2[I] \end{array}\right] \tag{5-7b}$$

Numerical Example

Given $$[A] = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$[A][A]^T = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} , \quad [A]^T[A] = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

$$[A_s] = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}\right] , \quad [A_s]^2 = \left[\begin{array}{c|c} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \end{array}\right]$$

The eigenvalues of $[A_s]^2$ are 2 , 8 , 2 and 8 and those of $[A_s]$ are $\sqrt{2}$, $\sqrt{8}$, $-\sqrt{2}$ and $-\sqrt{8}$. The eigenvector matrices $[X]$ and $[Y]$ are written as

$$[Y] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} , \quad [X] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $[W]$ becomes

$$[W] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} & \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}\right]$$

Then,

$$[W]^T[A_s][W] = \left[\begin{array}{c|c} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{8} \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} [\alpha] & [0] \\ \hline [0] & -[\alpha] \end{array}\right]$$

and

$$[W]^T[A_s]^2[W] = \left[\begin{array}{c|c} \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} [\alpha]^2 & [0] \\ \hline [0] & [\alpha]^2 \end{array}\right]$$

5.2 Relationship Between $[\Lambda]$ and $[\alpha]$

Let $[U]$ and $[V]$ be the biorthogonal matrices which transform $[A]$ and $[A]^T$ into $[\Lambda]$. Hence, referring to Chapter III, one obtains

$$[A] = [U][\Lambda][V]^T \tag{5-8a}$$

and

$$[A]^T = [V][\Lambda][U]^T \tag{5-8b}$$

Noting Equations (5-6d), (5-8a) and (5-8b), one obtains

$$[A] = [Y][\alpha][X]^T = [U][\Lambda][V]^T \tag{5-9a}$$

and

$$[A]^T = [X][\alpha][Y]^T = [V][\Lambda][U]^T \tag{5-9b}$$

It follows that

$$[\alpha] = [Y]^T[U][\Lambda][V]^T[X] = [X]^T[V][\Lambda][U]^T[Y] \tag{5-10a}$$

Equation (5-10a) is the relationship between $[\Lambda]$ and $[\alpha]$

where $[X]^T[X] = [Y]^T[Y] = [V]^T[U] = [I]$

It should be noted that although $[X]$ and $[Y]$ are individually orthogonal matrices, matrices $[U]$ and $[V]$ form a biorthogonal set of vectors.

CHAPTER VI

DISCUSSION AND CONCLUSION

6.1 Discussion

A typical asymmetric matrix in the field of linear elasticity as shown in Chapter II is the fundamental matrix which defines the shape of the deformed body in curvilinear coordinate form. The matrix specifies how straight lines originally parallel to the coordinate axes deform into curvilinear shapes in the deformed equilibrium state. If this matrix is orthogonal the curvilinear axes of the deformed body remain orthogonal as in the problems of uniaxial extension of long slender rod, a long slender rod hanging under its own weight, pure bending of long slender rod, and plane stress analysis of a beam in pure bending.

A nonsymmetric matrix is usually defined with respect to a skew angular reference system. In this case it is necessary to formulate a dual set of skew angular vectors, defined as the adjoint system, in order to operate mathematically. This process leads to the notion of biorthogonal coordinates. The transformation of a nonsymmetric matrix into the purely diagonal matrix of eigenvalues relies upon the concept of a biorthogonal transformation.

The replacement of nonsymmetric matrix by a symmetric and skew symmetric matrix is applied to the replacement of the Jacobian matrix by the linear strain matrix and a rotation matrix. When the Jacobain matrix is orthogonal all eigenvectors of these three matrices are the same and they are transformed into purely diagonal matrices of their eigenvalues by the same transformation matrix which is usually in complex form. In an alternate way one may find the real orthogonal matrix which transforms the symmetric part of the Jacobian matrix into diagonal matrix. This matrix transforms the Jacobian matrix into a classical orthogonal matrix and the rotation matrix into a skew symmetric matrix with only one real component.

The eigenvalue-eigenvector problems of nonsymmetric matrices in general deal with complex numbers as in the case of orthogonal and skew symmetric matrices. When the eigenvalues are complex numbers they occur in complex conjugate pairs. The complex eigenvectors of the first basis set and the second basis set are complex conjugate to each other and the real eigenvectors are coincident. The relationship between the eigenvalues and eigenvectors of the nonsymmetric matrix and those of the symmetric matrix formed by the product of the nonsymmetric matrix and its transpose is generated. It is complicated by the fact that the two sets of eigenvalues are not indepently interrelated and the two sets of eigenvectors are not interrelated. However, a general relationship interrelating the two sets of eigenvalues and two sets of eigenvectors is obtained.

The development of the super matrix and its solution leads to no new information connecting eigenvalues and eigenvectors. However, it does lead to an efficient compact form of the solution to the $[A]^T[A]$ and the $[A][A]^T$ problem as described in Chapter IV.

6.2 Conclusion

If the asymmetric matrix which defines the change in shape of a deformed body in curvilinear form is orthogonal, it follows that, there is no shear stress present in the body; the linear strain matrix is a diagonal matrix as in problems of unixial extension of a long slender rod, a long slender rod under its own weight, pure bending of a beam, and plane stress analysis of beam in pure bending

When shear stress occurs in the body the curvilinear coordinate axes are not orthogonal as in the case of beams bending with constant shear as well as beams bending with linearly varying shear. In this case the first fundamental nonsymmetric matrix is not sufficient to specify how planes originally parallel to the coordinate planes deform into the curvilinear shapes in the deformed equilibrium states. The second set of vectors which forms a biorthogonal set with the first set of vectors associated with the fundamental nonsymmetric matrices must be constructed.

The first set of vectors form unit vectors tangent to the curvilinear axes of the deformed body and the second set of vectors form the unit vectors normal to the planes parallel to the curvilinear coordinate planes of the defomed body. This second set of vectors is formed by the cross product of pairs of the first set of vectors in the cyclic system. The matrix associated with the second set of vectors ia a nonsymmetric matrix which is biorthogonal to the fundamental nonsymmetric matrix. This concept is shown in Appendix I with the example of a long cylindrical circular bar under torsion.

The equation of a quadratic surface which is defined with respect to the basis set of skew angular coordinate must rely on the biorthogonal basis. The matrix associated with this equation in always a nonsymmetric matrix. If the equation of the same quadratic surface is defined with respect to an orthogonal coordinate the matrix associated to this equation is symmetric (See Appendix II). Thus, any nonsymmetric matrix may be transformed to a symmetric matrices by the transformation of skew angular coordinate frame to the orthogonal coordinate frame. There is no exact solution for converting a nonsymmetric matrix to the symmetric matrix. The usual classical solution is obtained by utilizing the Gram-Schmidt orthogonalization procedure.

APPENDIX I

Biorthogonal Curvilinear Axes

From Equation (1-3d)

$$\{\dot{\tilde{\iota}}^{*}\} = [A]^{T}\{\dot{\iota}\} \tag{A-1}$$

where $\dot{\tilde{\iota}}_i^{*}$ is the unit vector tangent to the curved line in the deformed body which is originally a straight line parallel to X_i axis in the undeformed body. The matrix $[A]$ describes the curvilinear shape of the deformed body. In order to determine whether planes parallel to the coordinate planes before deformation remain planes in the deformed body, one must define the unit vectors $\tilde{J}_1^{*}$, $\tilde{J}_2^{*}$, $\tilde{J}_3^{*}$ normal to the curvilinear planes $\tilde{X}_2^{*}\tilde{X}_3^{*}$, $\tilde{X}_3^{*}\tilde{X}_1^{*}$ and $\tilde{X}_1^{*}\tilde{X}_2^{*}$, respectively. The vectors $\tilde{J}_1^{*}$, $\tilde{J}_2^{*}$ and $\tilde{J}_3^{*}$ are defined as

$$\left.\begin{aligned}
\tilde{J}_1^{*} &= (\dot{\tilde{\iota}}_2^{*}\times\dot{\tilde{\iota}}_3^{*})/|(\dot{\tilde{\iota}}_2^{*}\times\dot{\tilde{\iota}}_3^{*})| \\
\tilde{J}_2^{*} &= (\dot{\tilde{\iota}}_3^{*}\times\dot{\tilde{\iota}}_1^{*})/|(\dot{\tilde{\iota}}_3^{*}\times\dot{\tilde{\iota}}_1^{*})| \\
\tilde{J}_3^{*} &= (\dot{\tilde{\iota}}_1^{*}\times\dot{\tilde{\iota}}_2^{*})/|(\dot{\tilde{\iota}}_1^{*}\times\dot{\tilde{\iota}}_2^{*})|
\end{aligned}\right\} \tag{A-2a}$$

where from Equation (1-3c)

$$\begin{Bmatrix} \dot{\tilde{\iota}}_1^{*} \\ \dot{\tilde{\iota}}_2^{*} \\ \dot{\tilde{\iota}}_3^{*} \end{Bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & 1 & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & 1 \end{bmatrix} \begin{Bmatrix} \dot{\iota}_1 \\ \dot{\iota}_2 \\ \dot{\iota}_3 \end{Bmatrix} \tag{A-2b}$$

Substituting and combining yields

$$\tilde{J}_1^* = \dot{i}_1 - \frac{\partial u_1}{\partial x_2}\dot{i}_2 - \frac{\partial u_1}{\partial x_3}\dot{i}_3$$

$$\tilde{J}_2^* = -\frac{\partial u_2}{\partial x_1}\dot{i}_1 + \dot{i}_2 - \frac{\partial u_2}{\partial x_3}\dot{i}_3 .$$

$$\tilde{J}_3^* = -\frac{\partial u_3}{\partial x_1}\dot{i}_1 - \frac{\partial u_3}{\partial x_2} + \dot{i}_3$$

or in matrix form

$$\begin{Bmatrix} \tilde{J}_1^* \\ \tilde{J}_2^* \\ \tilde{J}_3^* \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{\partial u_1}{\partial x_2} & -\frac{\partial u_1}{\partial x_3} \\ -\frac{\partial u_2}{\partial x_1} & 1 & -\frac{\partial u_2}{\partial x_3} \\ -\frac{\partial u_3}{\partial x_1} & -\frac{\partial u_3}{\partial x_2} & 1 \end{bmatrix} \begin{Bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \end{Bmatrix} \tag{A-3a}$$

and symbolically

$$\{\tilde{J}^*\} = [B]^T\{\dot{i}\} \tag{A-3b}$$

If $\tilde{J}_i^*$ is constant vector when X_i is constant, this means that plane normals to the X_i direction before deformation remains plane after deformation with a normal vector $\tilde{J}_i^*$.

In linear elasticity we neglect the higher order terms in comparison to unity which leads to the condition

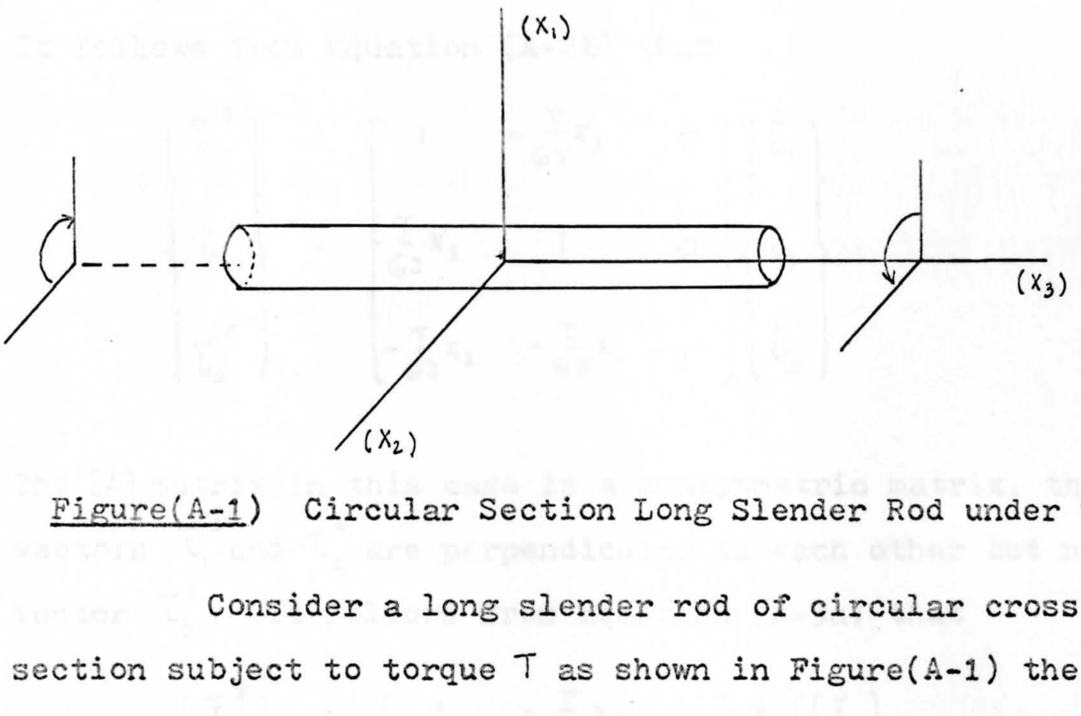
$$[B]^T[A] = [A]^T[B] = [I] \tag{A-4}$$

Comparing the form of matrices $[A]$ and $[B]$ it may be easily shown that

$$[B] = [\mathrm{COF}[A]] \tag{A-5}$$

This is exactly the concept of biorthogonal coordinates where the unit vectors $\tilde{\iota}_i^*$ tangent to the deformed lines form a biorthogonal set with the unit vectors $\tilde{J}_i^*$ normal to the curvilinear coordinate planes. For the special case of orthogonal curvilinear coordinates $[A]$ is an orthogonal matrix and $[B]=[A]$, that is, the tangent vectors and the planar normal vectors are identical.

Sample Example



Figure(A-1) Circular Section Long Slender Rod under Torsion

Consider a long slender rod of circular cross section subject to torque T as shown in Figure(A-1) the stress state within the rod is

$$\tau_{13} = \frac{T}{J} X_2$$

$$\tau_{23} = -\frac{T}{J} X_1$$

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$$

Satisfying the three equations of stress equilibrium, the six equations of Hooke's Law for a linear elastic material, the Cauchy equations, and the six linear strain displacement equations, it follows that the displacement field for the elastic body becomes

$$u_1 = \frac{T}{GJ} x_2 x_3$$

$$u_2 = -\frac{T}{GJ} x_1 x_3$$

$$u_3 = 0$$

It follows from Equation (A-2b) that

$$\begin{Bmatrix} \dot{\tilde{\iota}}_1^{*} \\ \dot{\tilde{\iota}}_2^{*} \\ \dot{\tilde{\iota}}_3^{*} \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{T}{GJ}x_3 & 0 \\ \frac{T}{GJ}x_3 & 1 & 0 \\ -\frac{T}{GJ}x_2 & -\frac{T}{GJ}x_1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\iota}_1 \\ \dot{\iota}_2 \\ \dot{\iota}_3 \end{Bmatrix}$$

The [A] matrix in this case is a nonsymmetric matrix, the vectors $\dot{\tilde{\iota}}_1^{*}$ and $\dot{\tilde{\iota}}_2^{*}$ are perpendicular to each other but not to vector $\dot{\tilde{\iota}}_3^{*}$. It follows from Equation (A-3a) that

$$\begin{Bmatrix} \tilde{J}_1^{*} \\ \tilde{J}_2^{*} \\ \tilde{J}_3^{*} \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{T}{GJ}x_3 & -\frac{T}{GJ}x_2 \\ \frac{T}{GJ}x_3 & 1 & \frac{T}{GJ}x_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\iota}_1 \\ \dot{\iota}_2 \\ \dot{\iota}_3 \end{Bmatrix}$$

The [B] matrix in this case is also a nonsymmetric matrix and the vector $\tilde{J}_3^{*}$ is a constant vector simultaneously perpendicular to the plane of $\tilde{\dot{\iota}}_1^*$ and $\tilde{\dot{\iota}}_2^*$ which lies in a plane parallel to the X_1X_2 plane. Hence, planes parallel to X_1X_2 plane before deformation remain planes parallel to X_1X_2 plane after deformation.

The coordinates of a point in the deformed body are

$$X_1^* = X_1 + \frac{T}{GJ} X_2 X_3$$

$$X_2^* = X_2 - \frac{T}{GJ} X_1 X_3$$

$$X_3^* = X_3$$

For the plane $X_3 = \text{constant} = \hat{X}_3$ (i.e. a plane parallel to the X_1X_2 plane), the equation of the plane passing through the point (X_1^*, X_2^*, $\hat{X}_3$) is $X_3^* = \hat{X}_3$ and having the unit normal vector $\{0 \quad 0 \quad 1\}$ equal to $\tilde{\jmath}_3^*$.

APPENDIX II

Equation of the Quadratic Surface

The matrix equation of the quadratic surface is given as

$$\{x\}^T[A]\{x\} = 1 \tag{B-1a}$$

with

$$[A]^T = [A] \tag{B-1b}$$

where the equation is defined with respect to an orthogonal axes set.

If a situation exists where the basis set is a skew system y_1, y_2, y_3, the notion of a biorthogonal basis is introduced where the second axes set is defined as z_1, z_2, z_3. It follows from Chapter III that

$$\{y\} = [\hat{U}]^T\{x\} \tag{B-2a}$$

and

$$\{z\} = [\hat{V}]^T\{x\} \tag{B-2b}$$

with

$$[\hat{V}]^T[\hat{U}] = [I] \tag{B-2c}$$

where vectors $\{y\}$ and $\{z\}$ from a biorthogonal set. Substituting the results into Equation (B-1a) gives

$$\{y\}^T[\hat{V}]^T[A][\hat{U}]\{z\} = 1 \tag{C-3}$$

Denoting

$$[\hat{V}]^T[A][\hat{U}] = [A^*] \qquad (C-4)$$

it follows that, in general, the matrix $[A^*]$ is nonsymmetric. Thus, the equation of a quadratic surface is associated with a particular class of matrices depending upon the properties of the basis axes.

BIBLIOGRAPHY

1. Shames, I. H., "Mechanics of Deformable Solids", Printice-Hall, Englewood Cliffs, N. J., 1964.

2. Pines, L. A., "Matrix Methods for Engineering", Printice-Hall, Englewood Cliffs, N. J., 1963.

3. Borg, S. F., "Matrix-Tensor Methods in Continuum Mechanics", Van Nostrand, Princeton, N. J., 1963.

4. Kuntakom, B., "Matrix Mathods in the Nonlinear Theory of Elasticity", Unpublished Master Thesis, Youngstown State University, June 1977.

5. Lanczos, C., "Applied Analysis", Prentice-Hall, Englewood Cliffs, N. J., 1956.

6. Lanczos, C., "Linear Differential Operators", Prentice-Hall, Englewood Cliffs, N. J., 1961.

7. Bellini, P.; Lecture Notes, CE. 941, Structural Mechanics, Youngstown State University, Fall 1976.

8. Kardestuncer, H., "Elementary Matrix Analysis of Structures", McGraw-Hill, New York, 1974.