STOCHASTIC OUTPUT-VARIABLE FEEDBACK CONTROL

by

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ABSTRACT

In this paper, output-feedback control is applied to a general, linear, time-invariant, stochastic regulator problem. The system of equations defining the feedback gain matrix is developed and put into the form of an algorithm. These equations are then applied to a second-order system to demonstrate how the algorithm works. The results of computer simulation for this system using the constant output-feedback control are compared to results for the same system using a state-variable estimator.

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CHAPTER 1: INTRODUCTION

One of the basic problems in control theory is to find the input signals necessary to cause the desired outputs from a predetermined system. In the ideal system, where it is assumed that the system equations are known exactly and that all of the state variables can be measured exactly, there is a known one-to-one correspondence between a given set of inputs and the resulting state. In this ideal case, there is a unique solution for the input signals needed to produce the desired output signals. In general, the system equations are given as a set of differential equations stated in vector form by:

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.1}$$

where $x(t)$ is the state variable vector and $u(t)$ is the input vector. A block diagram of this ideal system is shown in Figure 1.

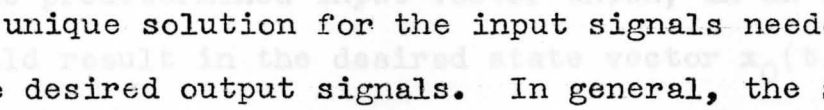


FIGURE 1 : IDEAL SYSTEM [1]

However in practically any physical system, it is not possible to have an exact set of system equations. Usually the system equations are based upon some model of the actual system, and any approximations that are made in arriving at the model will necessarily introduce some uncertainty into these equations. Because of these uncertainties, the state will deviate from the desired values. To overcome this problem it is necessary to introduce some form of feedback control system which can produce a correction signal for the input, based on the deviation of the state. A block diagram of this type of system is shown in Figure 2. In this diagram $u_0(t)$ is the predetermined input vector which, in an ideal system, would result in the desired state vector $x_0(t)$. However, due to errors in the physical system, the resulting physical state vector $x(t)$ differs from the desired state and the deviation is given by $\delta x(t)$. The control system consists of a gain matrix which operates on the state deviation vector $\delta x(t)$ to produce the input correction vector $\delta u(t)$, resulting in the input vector $u(t)$ to the physical system which should cause the physical state vector to more closely approximate the desired state vector, provided the control system has been properly designed.

In order to find the desired control system, it is necessary to find the small-signal, or perturbation, model of the system. This can be done by taking a Taylor series expansion of the system equations about the desired quantities $x_0(t)$ and $u_0(t)$. Since, in general, $\dot{x}(t) = f(x(t),u(t))$

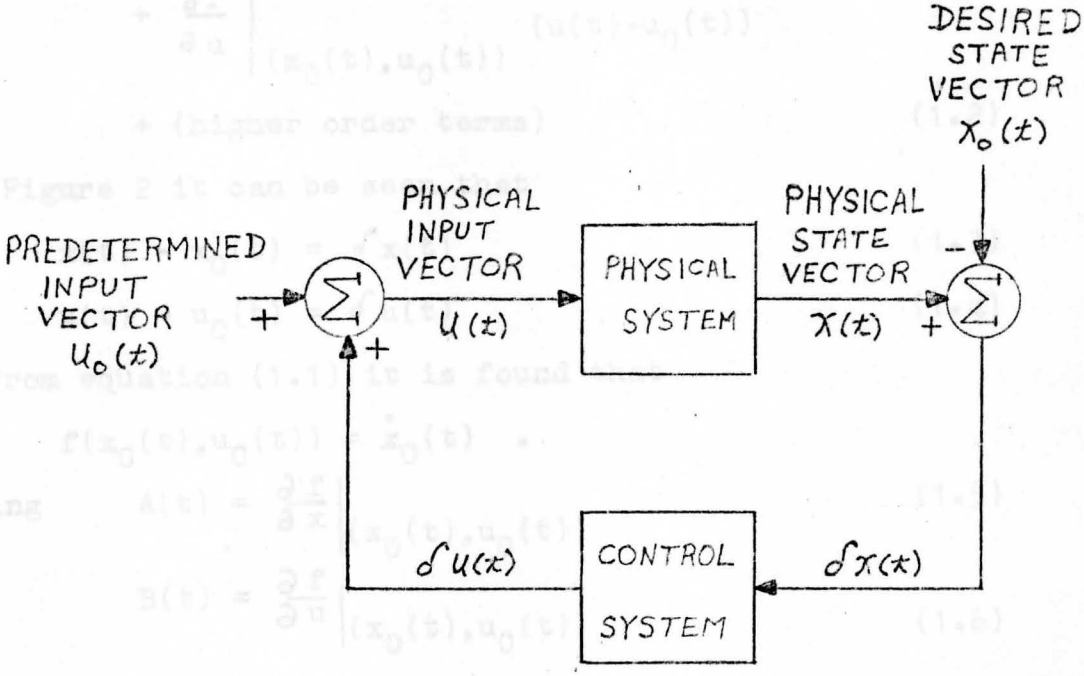


FIGURE 2 : PHYSICAL SYSTEM WITH FEEDBACK.

then a Taylor series expansion about the point $(x_0(t),u_0(t))$ is given by: [1]

$$\dot{x}(t) = f(x_0(t),u_0(t)) + \left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t))} (x(t)-x_0(t))$$

$$+ \left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t))} (u(t)-u_0(t))$$

$$+ \text{(higher order terms)} \qquad (1.2)$$

From Figure 2 it can be seen that

$$x(t) - x_0(t) = \delta x(t) \qquad (1.3)$$

$$u(t) - u_0(t) = \delta u(t) \qquad (1.4)$$

and from equation (1.1) it is found that

$$f(x_0(t),u_0(t)) = \dot{x}_0(t) \, .$$

Letting $$A(t) = \left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t))} \qquad (1.5)$$

and $$B(t) = \left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t))} \qquad (1.6)$$

equation (1.2) becomes

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + \text{(higher order terms)} \qquad (1.7)$$

The "higher order terms" in equation (1.7) contain terms that are at least quadratic in δx and δu. In most cases it is assumed that $\delta x(t)$ and $\delta u(t)$ are small and the "higher order terms" are negligible. This assumption results in the linear perturbation equation:

$$\delta\dot{x}(t) = A(t)\,\delta x(t) + B(t)\,\delta u(t) \qquad (1.8)$$

However, since this equation neglects the "higher order terms", an additional constraint must be added to this perturbation model to ensure that these terms will remain negligible.

Taylor's theorem [2] states that, if $T_n(x)$ is the Taylor series expansion of $f(x)$ in a neighborhood of $x = c$, then there exists a number $\overline{x}$ between x and c such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\overline{x})}{(n+1)!}(x-c)^{n+1} .$$

Applying this theorem to equation (1.2) yields:

$$\text{(higher order terms)} = \tfrac{1}{2}\Big\{(x(t)-x_0(t))' \left.\frac{\partial^2 f}{\partial x^2}\right|_{(\overline{x}(t),\overline{u}(t))} (x(t)-x_0(t))$$
$$+ 2(x(t)-x_0(t))' \left.\frac{\partial f}{\partial x \partial u}\right|_{(\overline{x}(t),\overline{u}(t))} (u(t)-u_0(t))$$
$$+ (u(t)-u_0(t))' \left.\frac{\partial^2 f}{\partial u^2}\right|_{(\overline{x}(t),\overline{u}(t))} (u(t)-u_0(t))\Big\} .$$

where $\overline{x}(t)$ is between $x(t)$ and $x_0(t)$; $\overline{u}(t)$ is between $u(t)$ and $u_0(t)$. As Athans [1] points out, this expression shows that the "higher order terms" are quadratic in $\delta x(t)$ and $\delta u(t)$. Therefore, one way of insuring that the "higher order terms" are negligible is to add the constraint of minimizing the quadratic cost functional:

$$J = \tfrac{1}{2}\int_{t_0}^{T}\left[\delta x'(t)Q(t)\delta x(t) + \delta u'(t)R(t)\delta u(t)\right]dt \qquad (1.9)$$

where $Q(t)$ and $R(t)$ are weighting matrices selected to reflect the desired amount of emphasis to be placed on minimizing $\delta x(t)$ and $\delta u(t)$. Both matrices should be symmetric, with $R(t)$ being positive definite and $Q(t)$ at least positive semidefinite.

Based on this linear perturbation model, the problem of finding the control system now becomes an optimization problem since the optimum control system will be one which keeps $\delta x(t)$ and $\delta u(t)$ small, which in turn, implies keeping the cost functional J small. Therefore the control system problem becomes one of finding the input correction vector $\delta u(t)$ which minimizes the cost function J, subject to the constraint of equation (1.8).

This problem is often referred to as the deterministic linear-quadratic regulator problem. Deterministic refers to the fact that the system has a definite input-output relationship; there are no random changes.

Although the deterministic linear-quadratic regulator problem accounts for uncertainties in the system equations because of approximations in the system model, it is still a rather idealistic approach. It must be realized that in an actual physical system it is not always possible to measure all of the state variables of the system. Furthermore, any measurements that are made should not be considered exact since there is uncertainty associated with any physical measurements. Consider also the possibility of random disturbances that may affect the physical system. A simple block diagram illustrating these additional uncertainties is given in Figure 3.

The measurements shown in Figure 3 are the outputs of the system. This output vector $y(t)$ is usually considered to be a function of the state vector,

$$y(t) = g(x(t)) \quad . \tag{1.10}$$

The linear perturbation equation of $y(t)$ can be found in a similar manner to equations (1.2) thru (1.8) for $x(t)$. The result is:

$$\delta y(t) = C(t)\,\delta x(t) \tag{1.11}$$

where $C(t) = \left.\frac{\partial g}{\partial x}\right|_{x_0(t)}$.

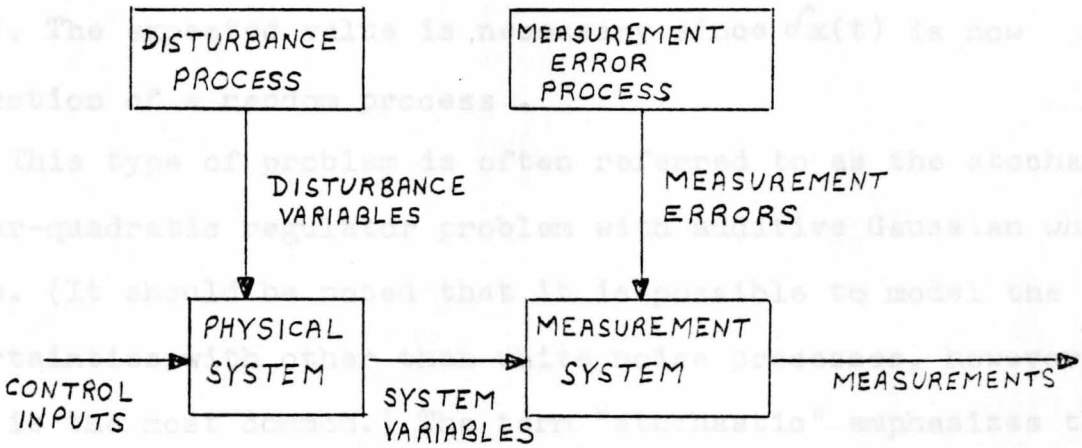


FIGURE 3: EFFECT OF DISTURBANCE AND MEASUREMENT ERROR. [3]

To account for the measurement errors and disturbance variables in the mathematical equations of the system, these uncertainties are usually treated as completely random processes and modeled by a Gaussian white noise process. When these white noise processes are included in the system model, the linear perturbation equations become:

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + \xi(t) \qquad (1.12)$$

$$\delta y(t) = C(t)\delta x(t) + v(t) \qquad (1.13)$$

and subject to the expected value of the cost functional of (1.9). The expected value is necessary since $\delta x(t)$ is now a function of a random process .

This type of problem is often referred to as the stochastic linear-quadratic regulator problem with additive Gaussian white noise. (It should be noted that it is possible to model the uncertainties with other than white noise processes, however this is the most common.) The term "stochastic" emphasizes the fact that some of the variables in the system are random and there is little or no way to control these variables.

The solution of the stochastic linear-quadratic Gaussian regulator problem separates the control system into two parts: (1) a Kalman filter which produces estimates of the system state variables, and (2) the optimal control gains of the deterministic case. It has been shown [3] - [6] that these two parts can be solved independently of each other and then cascaded to give the complete control system. A block diagram of this type of solution is shown in Figure 4.

This approach leads to the optimal control system for the stochastic linear-quadratic problem. However, this system

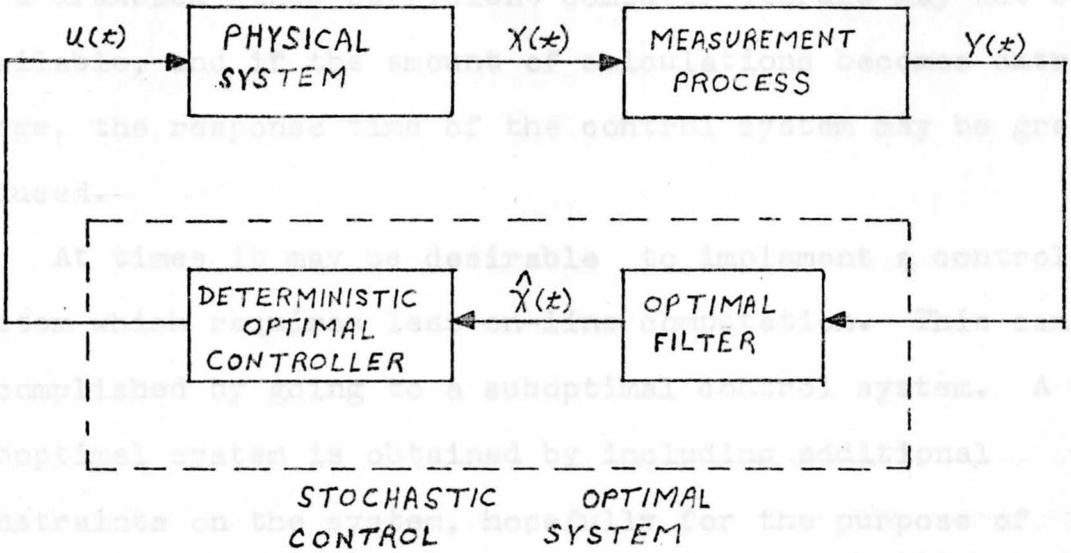


FIGURE 4: OPTIMAL CONTROL SYSTEM.[3]

normally involves an appreciable amount of on-line computation since the optimal filter must calculate a new state estimate vector for each time period based on the present estimates and the present measurements. In addition, a set of gains for the deterministic optimal controller must be precomputed and stored in computer memory. Often times these factors can be a drawback since sufficient computer storage may not be available, and if the amount of calculations becomes extremely large, the response time of the control system may be greatly reduced.

At times it may be desirable to implement a control system which requires less on-line computation. This can be accomplished by going to a suboptimal control system. A suboptimal system is obtained by including additional constraints on the system, hopefully for the purpose of simplifying the system. The suboptimal system sacrifices optimality for some type of reduction in complexity.

One particular type of suboptimal control system is the output-feedback control formulation [7] - [13], in which the control inputs are found by multiplying the system measurements or outputs by an appropriate gain matrix. This eliminates the need of the optimal filter shown in Figure 4 and also changes the formulation of the optimal controller. Elimination of the optimal filter results in an appreciable reduction of on-line computation. However, since the output signals are now used directly rather than being used to find the estimated state variables, the cost function will have a greater value and the system will no longer be "optimal". The formulation is referred to as suboptimal, but it should be noted that the solution found is the optimal solution for the given constraints.

The gain matrix by which the output signals are multiplied to get the control inputs may be found for either the infinite-time or finite-time case. The finite-time formulation assumes that the system operates for a finite period of time and yields a series of gain matrices , one for each interval. The infinite-time formulation assumes that the feedback gain matrix will approach a constant matrix and therefore yields only one gain matrix that is used for the entire time of operation. The infinite-time, or constant, formulation was chosen for this thesis since it requires less on-line computation and storage than the finite-time case and, therefore, would require less hardware in an actual system implementation. Note, however, that the infinite-time formulation is applicable only to systems with time-invariant coefficients, and may yield a solution which is suboptimal compared to the finite-time formulation.

This thesis considers a constant, output-feedback control for a linear, stochastic system having a quadratic cost function.

The general system to be considered is described in Chapter 2, and a solution is derived in the form of a set of matrix equations. Also given is an algorithm which can be used to determine the control matrix from this set of equations.

Chapter 3 describes the application of this control formulation to a second-order system. The system operation with output feedback control is compared to results obtained using estimated state variable feedback.

CHAPTER 2: SOLUTION OF GENERAL SYSTEM

This thesis applies the concept of a constant, output feedback control formulation to a linear, time-invariant, stochastic regulator problem. The mathematical equations for the general system that is considered can be given by: [4]

$$\dot{x}(t) = Ax(t) + Bu(t) + D_0 w(t) + D_1 \rho(t) \qquad (2.1)$$

$$\dot{w}(t) = A_w w(t) + B_w \rho(t) \qquad (2.2)$$

$$y(t) = Cx(t) + C_w w(t) + v(t) \qquad (2.3)$$

for $t_0 < t < t_f$.

where: $x(t)$ is the state vector

$u(t)$ is the control vector

$y(t)$ is the measurement vector

$v(t)$ is a white, gaussian measurement noise vector

and $A, B, D_0, D_1, A_w, B_w, C$, and C_w are time-invariant coefficient matrices.

Also, $w(t)$ is an external disturbance which influences the physical system as shown in equation (2.1), and is mathematically described as the result of a linear dynamic system driven by the white noise vector $\rho(t)$ in equation (2.2). Furthermore, equation (2.3) states that the measurements $y(t)$ can contain some terms dependent on this external disturbance.

The performance criteria for the control system is the quadratic cost function:

$$J = \tfrac{1}{2}E\left\{\int_{t_0}^{t_f}\left[x'(t)Qx(t) + u'(t)Ru(t)\right]dt\right\} \quad (2.4)$$

where Q and R are positive definite, time-invariant weighting matrices.

For the output feedback control formulation, the constraint added to this system of equations is

$$u(t) = Fy(t) \quad (2.5)$$

where F is a constant gain matrix for the control system.

However, it must be noted that the cost functional contains a quadratic term of u(t), and u(t) is a linear function of y(t) which contains a white noise vector. Therefore the control vector u(t) will have infinite variance and the cost functional will be undefined. As Ermer [7] points out, this difficulty is avoided by formulating the problem in discrete time rather than continuous time. In the discrete time case, the measurements will have finite variance.

Equation (2.1) thru (2.4) can be converted to the discrete time form by integrating each equation over the sampling period and then changing the interval of integration to $\left[t_k, t_{k+1}\right]$. The resulting discrete-time equations are: [5]

$$x_{k+1} = \emptyset x_k + T_2 w_k + T_1 u_k + \xi_k \quad (2.6)$$

$$w_{k+1} = \emptyset_w w_k + \eta_k \quad (2.7)$$

$$y_k = Cx_k + C_w w_k + v_k \quad (2.8)$$

$$J = \tfrac{1}{2}E\left\{\sum_{k=0}^{N}(x'_{k+1}\hat{Q}x_{k+1} + 2x'_{k+1}\hat{N}w_{k+1} + 2x'_k\hat{M}u_k + u'_k\hat{R}u_k)\right\} \quad (2.9)$$

and the constraint of equation (2.5) becomes

$$u_k = Fy_k \quad (2.10)$$

where u_k is held constant over the sampling interval $t_k<t<t_{k+1}$. For the finite-time case this set of equations is applicable. But, for the infinite-time case, N goes to infinity and causes the cost function to diverge [7]. Ermer suggests taking the average cost for each sampling interval rather than taking the total cost over the entire interval. When this is done, equation (2.9) becomes:

$$J = \tfrac{1}{2}\lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=0}^{N}(x'_{k+1}\hat{Q}x_{k+1} + 2x'_{k+1}\hat{N}w_{k+1} + 2x'_k\hat{M}u_k + u'_k\hat{R}u_k)\right\} .$$

Therefore the general, discrete-time, stochastic linear regulator problem can be stated as follows. Find the control vector u_k which satisfies the system equations

$$x_{k+1} = \phi x_k + T_2 w_k + T_1 u_k + \xi_k \qquad (2.11)$$

$$w_{k+1} = \phi_w w_k + \eta_k \qquad (2.12)$$

$$y_k = Cx_k + C_w w_k + v_k \qquad (2.13)$$

and minimizes the cost functional

$$J = \lim_{N\to\infty} \frac{1}{2N} E\left\{\sum_{k=0}^{N}(x'_{k+1}\hat{Q}x_{k+1} + 2x'_{k+1}\hat{N}w_{k+1} + 2x'_k\hat{M}u_k + u'_k\hat{R}u_k)\right\} \qquad (2.14)$$

subject to the constraint

$$u_k = Fy_k \quad . \qquad (2.15)$$

The solution for this general problem, given by equations (2.11) thru (2.15), will now be developed.

Substitute equations (2.15) and (2.13) into (2.14):

$$J = \lim_{N\to\infty} \frac{1}{2N} E\Big\{\sum_{k=0}^{N}\big[x'_{k+1}\hat{Q}x_{k+1} + 2x'_{k+1}\hat{N}w_{k+1} + 2x'_k\hat{M}(FCx_k + FC_w w_k + Fv_k) + (x'_kC'F' + w'_kC'_wF' + v'_kF')\hat{R}(FCx_k + FC_w w_k + Fv_k)\big]\Big\} .$$

Applying the fact that $E\{a'Ba\} = tr\{B\ E(aa')\}$,

$$J = \lim_{N\to\infty} \frac{1}{2N}\ tr\sum_{K=0}^{N}\Big\{\hat{Q}E(x_{k+1}x_{k+1}') + 2\hat{N}E(w_{k+1}x_{k+1}') +$$
$$2\hat{M}FCE(x_kx_k') + 2\hat{M}FC_wE(w_kx_k') + 2\hat{M}FE(v_kx_k') +$$
$$C'F'\hat{R}FCE(x_kx_k') + C'F'\hat{R}FC_wE(w_kx_k') + C'F'\hat{R}FE(v_kx_k') +$$
$$C_w'F'\hat{R}FCE(x_kw_k') + C_w'F'\hat{R}FC_wE(w_kw_k') + C_w'F'\hat{R}FE(v_kw_k') +$$
$$F'\hat{R}FCE(x_kv_k') + F'\hat{R}FC_wE(w_kv_k') + F'\hat{R}FE(v_kv_k')\Big\} \qquad (2.16)$$

Since v_k is a white, gaussian noise vector, completely independent of x_k or w_k, it is known that

$$E(x_kv_k') = E(v_kx_k') = 0$$
$$E(w_kv_k') = E(v_kw_k') = 0$$

and $\qquad E(v_kv_k') = V\delta_{ij}$

$$\text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases} \qquad \text{for } i,j = 0,1,2,\ldots$$

Let $E(x_kx_k') = P_k$ and $E(w_kw_k') = \overline{W}_k$.

Equation (2.16) becomes:

$$J = \lim_{N\to\infty} \frac{1}{2N}\ tr\sum_{K=0}^{N}\Big\{\hat{Q}P_{k+1} + 2\hat{N}E(w_{k+1}x_{k+1}') + 2\hat{M}FCP_k +$$
$$2\hat{M}FC_wE(w_kx_k') + C'F'\hat{R}FCP_k + C'F'\hat{R}FC_wE(w_kx_k') +$$
$$C_w'F'\hat{R}FCE(x_kw_k') + C_w'F'\hat{R}FC_w\overline{W}_k + F'\hat{R}FV\Big\} \qquad (2.17)$$

To find $E(x_{k+1}w_{k+1}')$, first substitute equations (2.15) and (2.13) into (2.11),

$$x_{k+1} = (\emptyset+T_1FC)x_k + (T_2+T_1FC_w)w_k + T_1Fv_k + \xi_k \qquad (2.18)$$

Then $E\{x_{k+1}w_{k+1}'\} = E\{((\emptyset+T_1FC)x_k + (T_2+T_1FC_w)w_k +$
$$T_1Fv_k + \xi_k)(\emptyset_w w_k + \eta_k)'\}$$

$$E\{x_{k+1}w_{k+1}'\} = (\emptyset+T_1FC)E\{x_kw_k'\}\emptyset_w' + (T_2+T_1FC_w)E\{w_kw_k'\}\emptyset_w' +$$
$$T_1FE\{v_kw_k'\}\emptyset_w' + E\{\xi_kw_k'\}\emptyset_w' + (\emptyset+T_1FC)E\{x_k\eta_k'\} +$$
$$(T_2+T_1FC_w)E\{w_k\eta_k'\} + T_1FE\{v_k\eta_k'\} + E\{\xi_k\eta_k'\} \qquad (2.19)$$

But since v_k is a white noise vector, independent of w_k or η_k,

$$E\{v_k w_k'\} = 0 \quad \text{and} \quad E\{v_k \eta_k'\} = 0 \quad .$$

Likewise, w_k depends on η_{k-1} but not on η_k, so $E\{w_k \eta_k'\} = 0$,

x_k depends on η_{k-2} but not on η_k, so $E\{x_k \eta_k'\} = 0$,

and w_k is not directly dependent upon ξ_k, so $E\{\xi_k w_k'\} = 0$.

However, $E\{\xi_k \eta_k'\}$ is not zero. Recall from equations (2.1) and (2.2) that both ξ_k and η_k depend on $\rho(t)$. Therefore $E\{\xi_k \eta_k'\}$ can be calculated in terms of $E\{\rho(t)\rho'(t)\}$.

Using the above equalities in equation (2.19),

$$E\{x_{k+1} w_{k+1}'\} = (\emptyset + T_1 FC) E\{x_k w_k'\} \emptyset_w' + (T_2 + T_1 FC_w) E\{w_k w_k'\} \emptyset_w' + E\{\xi_k \eta_k'\}$$

Let $E\{x_k w_k'\} = G_k$ and $E\{\xi_k \eta_k'\} = Z_k$.

$$\text{Then } G_{k+1} = (\emptyset + T_1 FC) G_k \emptyset_w' + (T_2 + T_1 FC_w) \overline{W}_k \emptyset_w' + Z_k \, . \tag{2.20}$$

Using the definition of $E\{x_k w_k'\} = G_k$, equation (2.17) becomes:

$$J = \lim_{N\to\infty} \frac{1}{2N} \operatorname{tr} \sum_{k=0}^{N} \Big\{ \hat{Q} P_{k+1} + 2\hat{N} G_{k+1}' + 2\hat{M} F C P_k + 2\hat{M} F C_w G_k' + C' F' \hat{R} F C P_k + C' F' \hat{R} F C_w G_k' + C_w' F' \hat{R} F C G_k + C_w' F' \hat{R} F C_w \overline{W}_k + F' \hat{R} F V \Big\}$$

Combining terms,

$$J = \lim_{N\to\infty} \frac{1}{2N} \operatorname{tr} \sum_{k=0}^{N} \Big\{ \hat{Q} P_{k+1} + 2\hat{N} G_{k+1}' + (2\hat{M} + C' F' \hat{R}) F (C P_k + C_w G_k') + C_w' F' \hat{R} F (C_w \overline{W}_k + C G_k) + F' \hat{R} F V \Big\} \tag{2.21}$$

By applying the properties of limits and series,

$$J = \tfrac{1}{2} \operatorname{tr} \Big\{ \hat{Q} \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} P_{k+1} + 2\hat{N} \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} G_{k+1}' + (2\hat{M} + C' F' \hat{R}) F \Big(C \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} P_k + C_w \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} G_k' \Big) + C_w' F' \hat{R} F \Big(C_w \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \overline{W}_k + C \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} G_k \Big) + F' \hat{R} F V \Big\} \tag{2.22}$$

For the cost function J to be finite, it is necessary that $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} P_k$, $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} G_k$, and $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \overline{W}_k$ converge.

To find the recursive equation for P_k, recall that $P_k = E\{x_k x_k'\}$. Using equation (2.18),

$$P_{k+1} = E\{x_{k+1} x_{k+1}'\}$$

$$P_{k+1} = E\{((\emptyset + T_1FC)x_k + (T_2 + T_1FC_w)w_k + T_1Fv_k + \xi_k)\cdot (x_k'(\emptyset + T_1FC)' + w_k'(T_2 + T_1FC_w)' + v_k'F'T_1' + \xi_k'))\} \qquad (2.23)$$

Expanding, and using the following equalities,

$$E\{x_k v_k'\} = 0\ ,\quad E\{w_k v_k'\} = 0\ ,\quad E\{x_k \xi_k'\} = 0\ ,$$

$$E\{w_k w_k'\} = \overline{W}_k\ ,\quad E\{v_k v_k'\} = V\ ,\quad E\{\xi_k \xi_k'\} = \mathcal{E}\ ,$$

$$E\{x_k w_k'\} = G_k\ ,\quad E\{w_k \xi_k'\} = 0\ ,\quad E\{v_k \xi_k'\} = 0\ .$$

then equation (2.23) becomes:

$$P_{k+1} = (\emptyset + T_1FC)P_k(\emptyset + T_1FC)' + (\emptyset + T_1FC)G_k(T_2 + T_1FC_w)' + (T_2 + T_1FC_w)G_k'(\emptyset + T_1FC)' + (T_2 + T_1FC_w)\overline{W}_k(T_2 + T_1FC_w)' + T_1FVF'T_1' + \mathcal{E} \qquad (2.24)$$

Likewise, to find the recursive equation for $\overline{W}_k$,

$$\overline{W}_{k+1} = E\{w_{k+1} w_{k+1}'\}$$

$$\overline{W}_{k+1} = E\{(\emptyset_w w_k + \eta_k)(w_K'\emptyset_w' + \eta_k')\}$$

$$\overline{W}_{k+1} = \emptyset_w E\{w_k w_k'\}\emptyset_w' + \emptyset_w E\{w_k \eta_k'\} + E\{\eta_k w_k'\}\emptyset_w' + E\{\eta_k \eta_k'\}$$

Therefore,

$$\overline{W}_{k+1} = \emptyset_w \overline{W}_k \emptyset_w' + \eta_k \qquad (2.25)$$

The limit of $\overline{W}_k$ will converge if $\emptyset_w$ is a stability matrix (i.e. all eigenvalues of the matrix are less than one). It can then be seen from equations (2.20) and (2.24) that any F which makes $(\emptyset + T_1FC)$ a stability matrix will cause the limits of G_k and P_k to converge. [14],[15] Assuming that the necessary F can be found, then $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} P_k = P$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} G_k = G$, and $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \overline{W}_k = \overline{W}$.

Equations (2.22), (2.24), (2.25), and (2.20) will converge to:

$$J = \tfrac{1}{2}\,\mathrm{tr}\left\{ \hat{Q}P + 2\hat{N}G' + (2\hat{M}+C'F'\hat{R})F(CP+C_wG') + C_w'F'\hat{R}F(C_w\overline{W}+CG) + F'\hat{R}FV \right\} \tag{2.26}$$

$$P = (\emptyset+T_1FC)P(\emptyset+T_1FC)' + (\emptyset+T_1FC)G(T_2+T_1FC_w)' + (T_2+T_1FC_w)G'(\emptyset+T_1FC)' + (T_2+T_1FC_w)\overline{W}(T_2+T_1FC_w)' + T_1FVF'T_1' + \mathcal{E} \tag{2.27}$$

$$\overline{W} = \emptyset_w\overline{W}\emptyset_w' + \mathcal{N} \tag{2.28}$$

$$G = (\emptyset+T_1FC)G\,\emptyset_w' + (T_2+T_1FC_w)\overline{W}\emptyset_w' + Z \tag{2.29}$$

To solve this system of equations for the control matrix F, form the Lagrangian: [16]

$$L(F,P,\Lambda,G,\lambda) = \tfrac{1}{2}\,\mathrm{tr}\Big\{\Big[\hat{Q}P + 2\hat{N}G' + (2\hat{M}+C'F'\hat{R})F(CP+C_wG') + C_w'F'\hat{R}F(C_w\overline{W}+CG) + F'\hat{R}FV\Big] + \Lambda\Big[(\emptyset+T_1FC)P(\emptyset+T_1FC)' + (\emptyset+T_1FC)G(T_2+T_1FC_w)' + (T_2+T_1FC_w)G'(\emptyset+T_1FC)' + (T_2+T_1FC_w)\overline{W}(T_2+T_1FC_w)' + T_1FVF'T_1' + \mathcal{E} - P\Big] + \lambda\Big[(\emptyset+T_1FC)G\emptyset_w' + (T_2+T_1FC_w)\overline{W}\emptyset_w' + Z - G\Big]\Big\} \tag{2.30}$$

where Λ and λ are each a matrix of multipliers.

For J to be at its extreme value, it is necessary that:

$$\frac{\partial L}{\partial F} = 0\,,\quad \frac{\partial L}{\partial P} = 0\,,\quad \frac{\partial L}{\partial \Lambda} = 0\,,\quad \frac{\partial L}{\partial G} = 0\,,\quad \frac{\partial L}{\partial \lambda} = 0\,.$$

[Note: For some properties concerning differentiation of matrix equations, see Appendix I.]

(1) $\frac{\partial L}{\partial F} = 0$

$$\frac{\partial}{\partial F}\cdot\tfrac{1}{2}\,\mathrm{tr}\Big\{\Big[(CP+C_wG')(2\hat{M}+C'F'\hat{R}) + (C_w\overline{W}+CG)C_w'F'\hat{R} + VF'\hat{R} + CP(\emptyset+T_1FC)'\Lambda T_1 + CG(T_2+T_1FC_w)'\Lambda T_1 + C_wG'(\emptyset+T_1FC)'\Lambda T_1 + C_w\overline{W}(T_2+T_1FC_w)'\Lambda T_1 + VF'T_1'\Lambda T_1 + CG\emptyset_w'\lambda T_1 + C_w\overline{W}\emptyset_w'\lambda T_1\Big]F + \Big[\hat{R}F(CP+C_wG')C' + \hat{R}F(C_w\overline{W}+CG)C_w' + \hat{R}FV + T_1'\Lambda(\emptyset+T_1FC)PC' + T_1'\Lambda(\emptyset+T_1FC)GC_w' + T_1'\Lambda(T_2+T_1FC_w)G'C' + T_1'\Lambda(T_2+T_1FC_w)\overline{W}C_w' + T_1'\Lambda T_1FV\Big]F'\Big\} = 0$$

$$0 = \hat{R}FCPC' + \hat{R}FC_wG'C' + \hat{R}FC_w\overline{W}C_w' + \hat{R}FCGC_w' + \hat{R}FV + \hat{M}'PC' +$$
$$\hat{M}'GC_w' + \tfrac{1}{2}T_1'(\Lambda + \Lambda')(\phi+T_1FC)(PC'+GC_w') +$$
$$\tfrac{1}{2}T_1'(\Lambda + \Lambda')(T_2+T_1FC_w)(G'C'+\overline{W}C_w') + \tfrac{1}{2}T_1'(\Lambda + \Lambda')T_1FV +$$
$$\tfrac{1}{2}T_1'\lambda'\phi_w(G'C'+\overline{W}C_w')$$

$$0 = (\hat{R}+\tfrac{1}{2}T_1'(\Lambda+\Lambda')T_1)F(CPC' + C_wG'C' + C_w\overline{W}C_w' + CGC_w' + V) +$$
$$\hat{M}'(PC'+GC_w') + \tfrac{1}{2}T_1'(\Lambda+\Lambda')\phi(PC'+GC_w') + \tfrac{1}{2}T_1'(\Lambda+\Lambda')T_2(G'C'+\overline{W}C_w') +$$
$$\tfrac{1}{2}T_1'\lambda'\phi_w(G'C'+\overline{W}C_w')$$

Letting $\Lambda_s = \tfrac{1}{2}(\Lambda+\Lambda')$ and solving for F:

$$F = -(\hat{R}+T_1'\Lambda_sT_1)^{-1}\Big[\hat{M}'(PC'+GC_w') + T_1'\Lambda_s\phi(PC'+GC_w') +$$
$$T_1'\Lambda_sT_2(G'C'+\overline{W}C_w') + \tfrac{1}{2}T_1'\lambda'\phi_w(G'C'+\overline{W}C_w')\Big]$$
$$(CPC' + C_wG'C' + C_w\overline{W}C_w' + CGC_w' + V)^{-1}$$

(2) $\dfrac{\partial L}{\partial \Lambda} = 0$

$$P = (\phi+T_1FC)P(\phi+T_1FC)' + (\phi+T_1FC)G(T_2+T_1FC_w)' +$$
$$(T_2+T_1FC_w)G'(\phi+T_1FC)' + (T_2+T_1FC_w)\overline{W}(T_2+T_1FC_w)' +$$
$$T_1FVF'T_1' + \mathcal{E}$$

(3) $\dfrac{\partial L}{\partial \lambda} = 0$

$$G = (\phi+T_1FC)G\phi_w' + (T_2+T_1FC_w)\overline{W}\phi_w' + Z$$

(4) $\dfrac{\partial L}{\partial P} = 0$

$$0 = \hat{Q} + (2\hat{M}+C'F'\hat{R})FC + (\phi+T_1FC)'\Lambda(\phi+T_1FC) - \Lambda$$
$$\Lambda = (\phi+T_1FC)'\Lambda(\phi+T_1FC) + (2\hat{M}+C'F'\hat{R})FC + \hat{Q}$$

(5) $\dfrac{\partial L}{\partial G} = 0$

$$0 = \frac{\partial}{\partial G}\tfrac{1}{2}\,\mathrm{tr}\Big\{\Big[C_w'F'\hat{R}FC + (T_2+T_1FC_w)'\Lambda(\phi+T_1FC) + \phi_w'\lambda(\phi+T_1FC) -$$
$$\lambda\Big]G + \Big[2\hat{N} + (2\hat{M}+C'F'\hat{R})FC_w + (\phi+T_1FC)'\Lambda(T_2+T_1FC_w)\Big]G'\Big\}$$
$$0 = \tfrac{1}{2}\Big[C'F'\hat{R}FC_w + 2(\phi+T_1FC)'\Lambda_s(T_2+T_1FC_w) + (\phi+T_1FC)'\lambda'\phi_w' -$$
$$\lambda' + 2\hat{N} + (2\hat{M} + C'F'\hat{R})FC_w\Big]$$

$$\lambda' = (\phi+T_1FC)'\lambda'\phi_w' + 2(\phi+T_1FC)'\mathcal{N}_s(T_2+T_1FC_w) + 2(\hat{M}+C'F'\hat{R})FC_w + 2\hat{N}$$

$$\lambda = \phi_w\lambda(\phi+T_1FC) + 2(T_2+T_1FC_w)'\mathcal{N}_s(\phi+T_1FC) + 2C_w'F'(\hat{M}+C'F'\hat{R})' + 2\hat{N}'$$

Also, since $\mathcal{N}_s$ was defined as $\frac{1}{2}(\mathcal{N}+\mathcal{N}')$,

$$\mathcal{N}_s = (\phi+T_1FC)'\mathcal{N}_s(\phi+T_1FC) + C'F'\hat{R}FC + \hat{Q} + \hat{M}FC + C'F'\hat{M}'$$

Therefore, the set of equations to be solved is :

$$F = -(\hat{R}+T_1'\mathcal{N}_sT_1)^{-1}\Big[\hat{M}'(PC'+GC_w') + T_1'\mathcal{N}_s\phi(PC'+GC_w') + T_1'\mathcal{N}_sT_2(G'C'+\overline{W}C_w') + \tfrac{1}{2}T_1'\lambda'\phi_w(G'C'+\overline{W}C_w')\Big](CPC' + C_wG'C' + CGC_w' + C_wWC_w' + V)^{-1} \qquad (2.31)$$

$$P = (\phi+T_1FC)P(\phi+T_1FC)' + (\phi+T_1FC)G(T_2+T_1FC_w)' + (T_2+T_1FC_w)G'(\phi+T_1FC)' + (T_2+T_1FC_w)\overline{W}(T_2+T_1FC_w)' + T_1FVF'T_1' + \mathcal{E} \qquad (2.32)$$

$$\mathcal{N}_s = (\phi+T_1FC)'\mathcal{N}_s(\phi+T_1FC) + C'F'\hat{R}FC + \hat{M}FC + C'F'\hat{M}' + \hat{Q} \qquad (2.33)$$

$$G = (\phi+T_1FC)G\phi_w' + (T_2+T_1FC_w)\overline{W}\phi_w' + Z \qquad (2.34)$$

$$\lambda = \phi_w\lambda(\phi+T_1FC) + 2(T_2+T_1FC_w)'\mathcal{N}_s(\phi+T_1FC) + 2C_w'F'(\hat{M}'+\hat{R}FC) + 2\hat{N}' \qquad (2.35)$$

$$\overline{W} = \phi_w\overline{W}\phi_w' + \mathcal{W} \qquad (2.36)$$

In addition, it has been assumed that the system is stabilizable, so there must exist an F such that the magnitude of each eigenvalue of $(\phi+T_1FC)$ is less than one.

The desired solution for the control system is the matrix F which satisfies all of the above constraints. Notice from equation (2.31) that F is a function of P, $\mathcal{N}_s$, G, λ, and $\overline{W}$, which, in turn, are all dependent on F (except for $\overline{W}$) as shown by equations (2.32) thru (2.36). Because of this interdependence, a direct solution is not possible, and an iterative method must be used. This leads to the following general algorithm for solving the system of equations (2.31) thru (2.36):

1. Assume an initial matrix F which satisfies the constraint that the eigenvalues of $(\phi+T_1FC)$ are less than one so that the system is stable.
2. Solve for $\overline{W}$ in equation (2.36).
3. Solve for Λ_s in equation (2.33) and G in equation (2.34).
4. Using these values of Λ_s and G, solve for λ in equation (2.35) and P in equation (2.32).
5. Using the values of Λ_s and G from step 3 and the values of λ and P from step 4, calculate a new F matrix using equation (2.31).
6. Using this calculated F matrix from step 5, repeat steps 3 thru 5 until the F matrix converges to the solution.

It should be noted that equations (2.32) thru (2.36) are nonlinear, nonseparable, matrix equations. There are no direct solutions for this type of equation. It is therefore necessary to solve these equations by an iterative method also. An algorithm which can be used is one developed by Bartels and Stewart [17] for a matrix equation of the form $AX + XB = C$, where X is the unknown matrix.

When the complete set of equations (2.31) thru (2.36) is solved, the resulting solution will be the matrix F, which is the gain matrix of the output feedback control system. A block diagram of this system is shown in Figure 5.

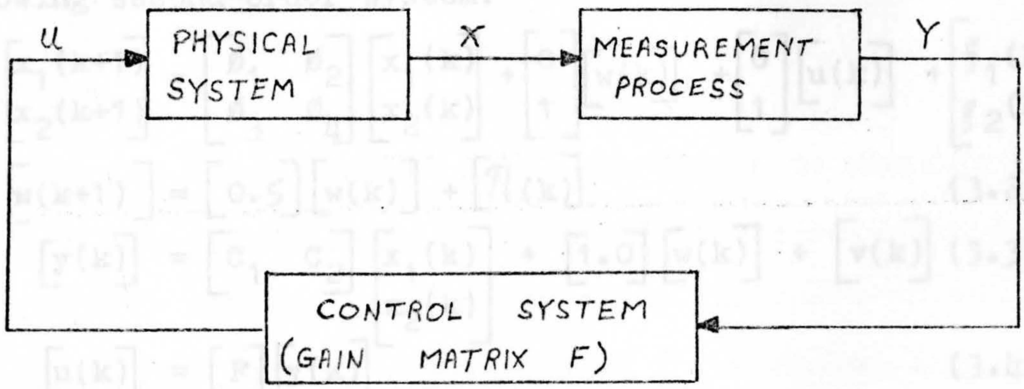


FIGURE 5: OUTPUT FEEDBACK CONTROL SYSTEM.

CHAPTER 3: APPLICATION TO A SECOND ORDER SYSTEM

In order to demonstrate how the algorithm developed in Chapter 2 works it was applied to three variations of the following second-order system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} w(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} u(k) \end{bmatrix} + \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \qquad (3.1)$$

$$\begin{bmatrix} w(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 \end{bmatrix} \begin{bmatrix} w(k) \end{bmatrix} + \begin{bmatrix} \eta(k) \end{bmatrix} \qquad (3.2)$$

$$\begin{bmatrix} y(k) \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.0 \end{bmatrix} \begin{bmatrix} w(k) \end{bmatrix} + \begin{bmatrix} v(k) \end{bmatrix} \qquad (3.3)$$

$$\begin{bmatrix} u(k) \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} y(k) \end{bmatrix} \qquad (3.4)$$

where $\xi(k)$, $\eta(k)$, and $v(k)$ are Gaussian, white noise vectors such that:

$$\Xi = E\{\xi(k)\xi'(k)\} = \begin{bmatrix} 1.0\text{x}10^{-6} & 0.0 \\ 0.0 & 1.0\text{x}10^{-6} \end{bmatrix}$$

$$\eta = E\{\eta(k)\eta'(k)\} = \begin{bmatrix} 1.0\text{x}10^{-6} \end{bmatrix}$$

$$V = E\{v(k)v'(k)\} = \begin{bmatrix} 1.0\text{x}10^{-6} \end{bmatrix}$$

$$Z = E\{\xi(k)\eta'(k)\} = \begin{bmatrix} 1.0\text{x}10^{-6} \\ 1.0\text{x}10^{-6} \end{bmatrix}$$

and the cost function is given by:

$$J = \lim_{N\to\infty} \frac{1}{2N} E\left\{\sum_{k=0}^{N} x'(k+1)\hat{Q}x(k+1) + 2x'(k+1)\hat{N}w(k+1) + 2x'(k)\hat{M}u(k) + u'(k)\hat{R}u(k)\right\}$$

with $\hat{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\hat{R} = \begin{bmatrix} 2 \end{bmatrix}$

$\hat{M} = \hat{N} = \begin{bmatrix} 0 \end{bmatrix}$.

Three variations of this system were obtained by choosing the matrices ϕ and C such that the open loop transfer function, given by

$$GH(z) = C(zI-\phi)^{-1}T_1$$

had the following configurations of poles and zeroes:

Case #1: two real poles, no zeroes.

Case #2: two real poles and one real zero.
(with zero < pole 1 < pole 2).

Case #3: two complex poles and one real zero.

To apply the algorithm of Chapter 2, it is necessary to choose an initial F matrix. In each of the three configurations, the initial F was chosen by considering the shape of the root locus plot. For case #1, the initial F was chosen at the breakaway point of the root locus from the real axis. For case #2 and case #3, the initial F was chosen at the breakin point to the real axis. A computer program was written to perform the algorithm by starting with the initial F matrix and iterating until the F matrix converged.

Having found the desired F matrix, a computer simulation was run for each of the three configurations to see how the control matrix performed in the output-feedback system. In each configuration, the response of the state variables was monitored as a function of time.

As an indication of how well the output-feedback control formulation performed, a comparison was made between the state variable response of the output-feedback formulation to the response obtained by using a feedback system similar to the stochastic optimal control system shown in Figure 4.

The general equation for the control system considered is stated as: [4]

$$u(k) = -H\hat{x}(k) - H_w\hat{w}(k) \quad . \tag{3.5}$$

The vectors $\hat{x}(k)$ and $\hat{w}(k)$ are the estimated values of the state and disturbance vectors and are given by: [5]

$$\hat{x}(k+1) = \emptyset\hat{x}(k) + T_2\hat{w}(k) + T_1\hat{u}(k) + L(y(k)-C\hat{x}(k)-C_w\hat{w}(k)) \tag{3.6}$$

$$\hat{w}(k+1) = \emptyset_w\hat{w}(k) + L_w(y(k)-C\hat{x}(k)-C_w\hat{w}(k)) \tag{3.7}$$

Where L and L_w are the gain matrices of the Kalman filter for the respective vectors. The equations used to compute the control gain matrices (H and H_w) and the estimator matrices (L and L_w) are discussed in Appendix II.

As in the output-feedback control case, a computer simulation was performed on this estimator-control case to determine the response of the state variables. The complete system of equations used for this simulation consisted of equations (3.1) thru (3.3) in addition to equations (3.5) thru (3.7).

Following is a description of each of the three configurations considered and a summary of the results obtained.

Case #1 Two real poles.

The poles were chosen at z = 1.0 and z = 0.9 by selecting

$$\emptyset = \begin{bmatrix} 0.0 & 1.0 \\ -0.9 & 1.9 \end{bmatrix}$$

and $\quad C = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$.

The open loop transfer function is:

$$GH(z) = \frac{K}{(z - 1.0)(z - 0.9)} \quad .$$

The breakaway point of the root locus from the real axis (found by solving $\frac{dK}{dz} = 0$) is at $z = 0.95$,

and the value of the gain at this point is $K = 0.0025$.

By applying the algorithm of Chapter 2 to this system and using an initial $F = \begin{bmatrix}-0.0025\end{bmatrix}$, it is found that the optimal gain for the output feedback control system is $F = \begin{bmatrix}-0.0543\end{bmatrix}$.

A sketch of the root locus for this system is shown in Figure 6(a).

For the state-estimator control system the control gain matrices are found to be:

$$H = \begin{bmatrix}-0.7373 & 1.3386\end{bmatrix}$$

$$H_w = \begin{bmatrix}0.9346\end{bmatrix}$$

and the estimator matrices are:

$$L = \begin{bmatrix}1.0590 \\ 1.3928\end{bmatrix}$$

$$L_w = \begin{bmatrix}0.0462\end{bmatrix} \quad .$$

The state variable response for the output-feedback control system is shown in Figures 7(a) and 7(b), and those for the state-estimator control system are shown in Figure 8.

Case #2 Two real poles and one real zero.

The poles were chosen at $z = 0.8$ and $z = 1.2$, and the zero at $z = 0.5$ by selecting

$$\phi = \begin{bmatrix}0.5 & 1.0 \\ -0.21 & 1.5\end{bmatrix}$$

and $C = \begin{bmatrix}0.0 & 1.0\end{bmatrix}$.

The open loop transfer function is:

$$GH(z) = \frac{K(z - 0.5)}{z^2 - 2z + 0.96} \quad .$$

The breakaway point and breakin point of the root locus at the real axis are at

$$z = 0.96 \text{ , breakaway point}$$

$$z = 0.04 \text{ , breakin point}$$

and the value of the gain at these breakpoints is:

$$K = 0.0835 \text{ , at } z = 0.96$$

$$K = 1.9165 \text{ , at } z = 0.04 \text{ .}$$

By applying the algorithm of Chapter 2 to this system, and using an initial $F = \begin{bmatrix}-1.9165\end{bmatrix}$, it is found that the optimal gain for the output-feedback control system is $F = \begin{bmatrix}-0.93356\end{bmatrix}$.

A sketch of the root locus of this system is shown in Figure 6(b).

For the state-estimator control system the control gain matrices are found to be:

$$H = \begin{bmatrix}-0.1299 & 1.3038\end{bmatrix}$$

$$H_W = \begin{bmatrix}3.2911\end{bmatrix}$$

and the estimator matrices are:

$$L = \begin{bmatrix}0.6358 \\ 1.0159\end{bmatrix}$$

$$L_W = \begin{bmatrix}0.1196\end{bmatrix} \text{ .}$$

The state variable response for both systems is shown in Figure 9.

<u>Case #3</u> Two complex poles and one zero.

The poles were chosen at $z = 0.866 \pm j0.5$ and the zero at $z = 0.9$ by selecting:

$$\emptyset = \begin{bmatrix}0.9 & 1.0 \\ -0.25 & 0.832\end{bmatrix}$$

and $$C = \begin{bmatrix}0.0 & 1.0\end{bmatrix} \text{ .}$$

The open loop transfer function is :

$$GH(z) = \frac{K(z - 0.9)}{z^2 - 1.732z + 0.9988} .$$

The breakin point of the root locus to the real axis is at

$$z = 0.4$$

and the value of the gain at this point is

$$K = 0.932.$$

By applying the algorithm of Chapter 2 to this system, and using an initial $F = \begin{bmatrix}-0.932\end{bmatrix}$, it is found that the optimal gain for the output-feedback control system is $F = \begin{bmatrix}-0.8303\end{bmatrix}$.

A sketch of the root locus for this system is shown in Figure 6(c).

For the state-estimator control system the control gain matrices are found to be:

$$H = \begin{bmatrix}0.1012 & 0.9976\end{bmatrix}$$

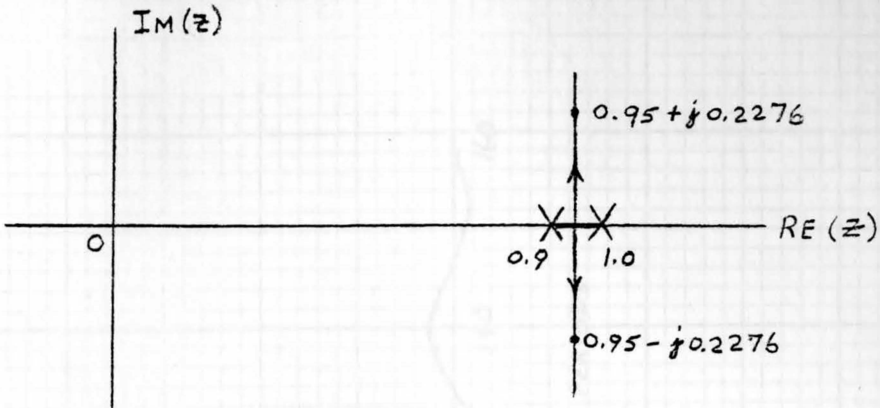
$$H_W = \begin{bmatrix}0.8864\end{bmatrix}$$

and the estimator matrices are:

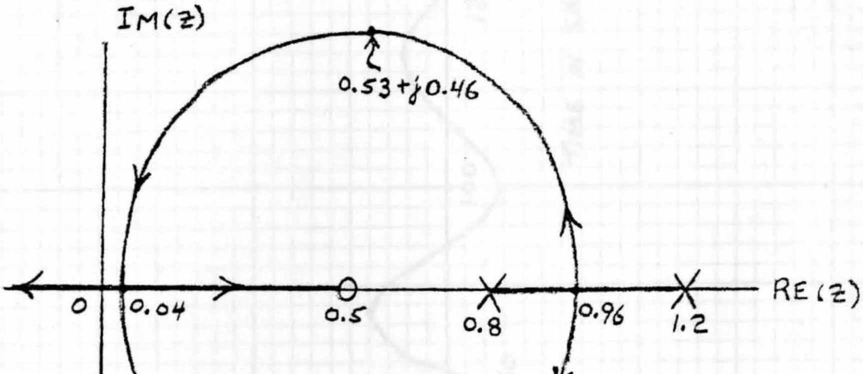
$$L = \begin{bmatrix}0.2988 \\ 0.6988\end{bmatrix}$$

$$L_W = \begin{bmatrix}0.1287\end{bmatrix} .$$

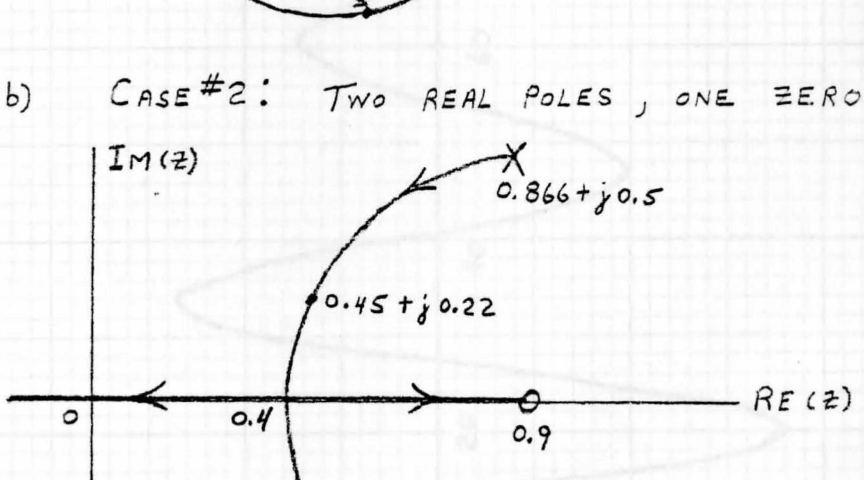
The state variable response for both systems is shown in Figure 10.



(a) CASE #1: TWO REAL POLES.



(b) CASE #2: TWO REAL POLES, ONE ZERO.



(c) CASE #3: TWO COMPLEX POLES, ONE ZERO.

FIGURE 6 ROOT LOCUS SKETCHES

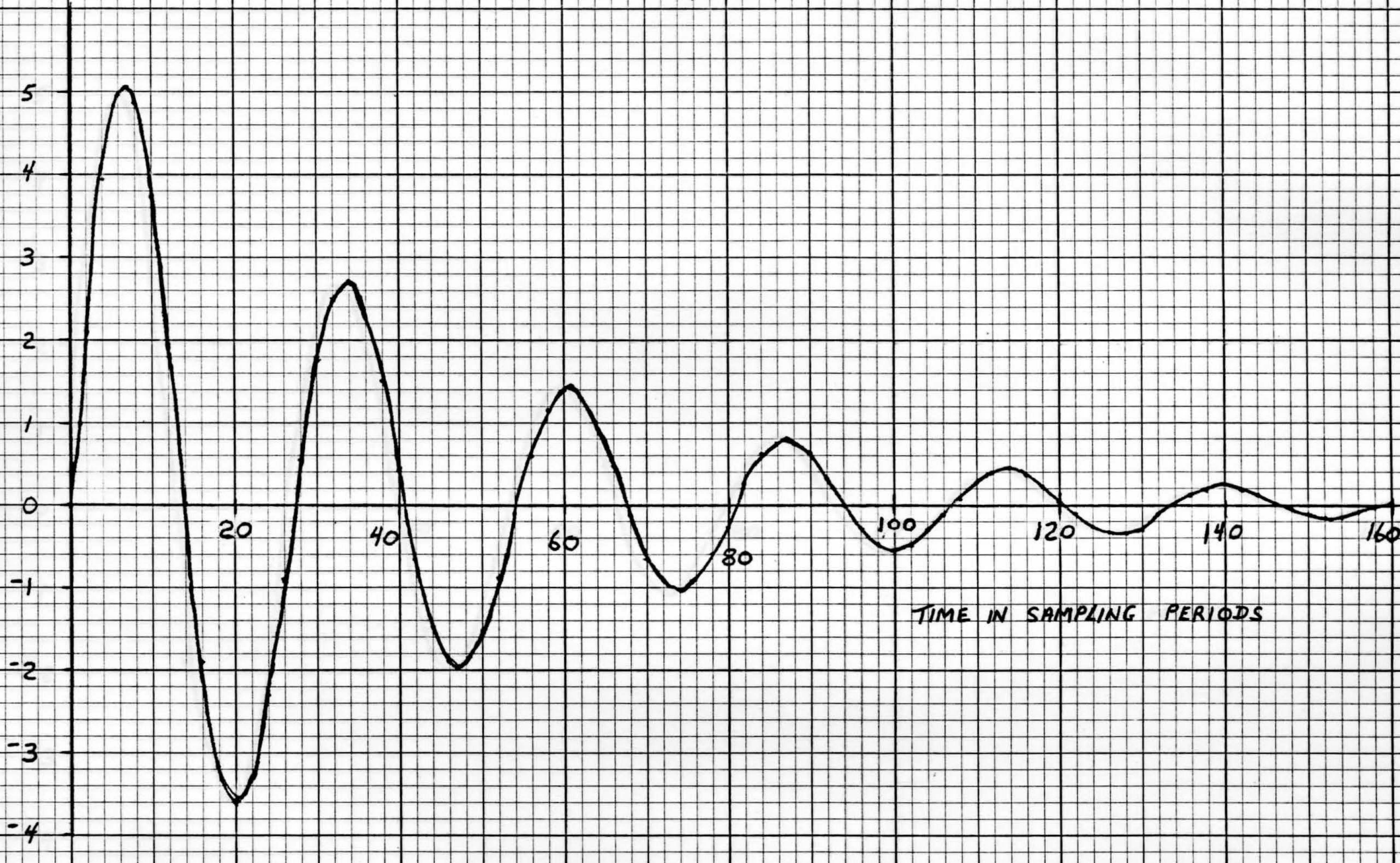


FIGURE 7(a) RESONSE OF X_1 FOR CASE # 1
WITH OUTPUT FEEDBACK

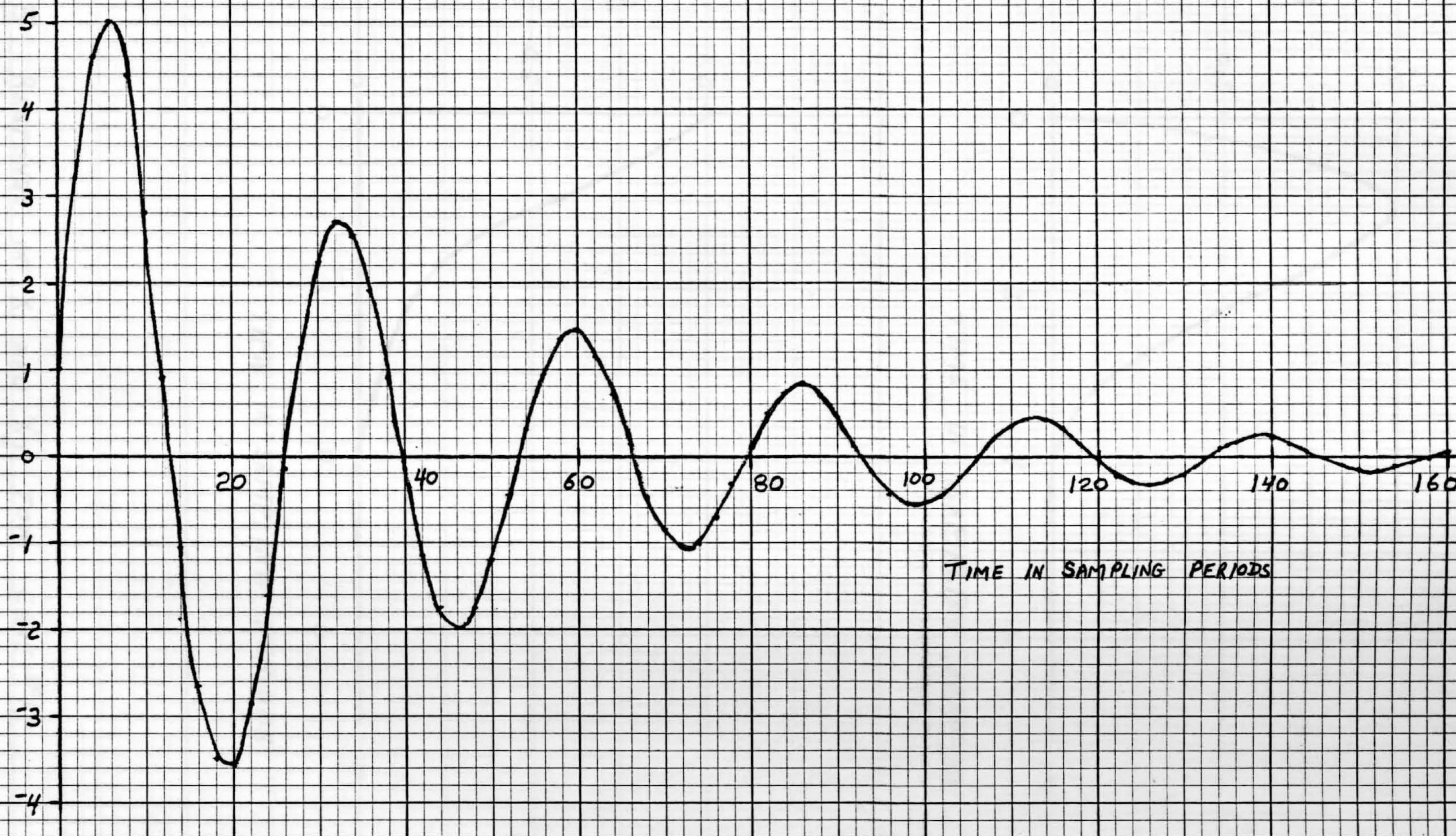
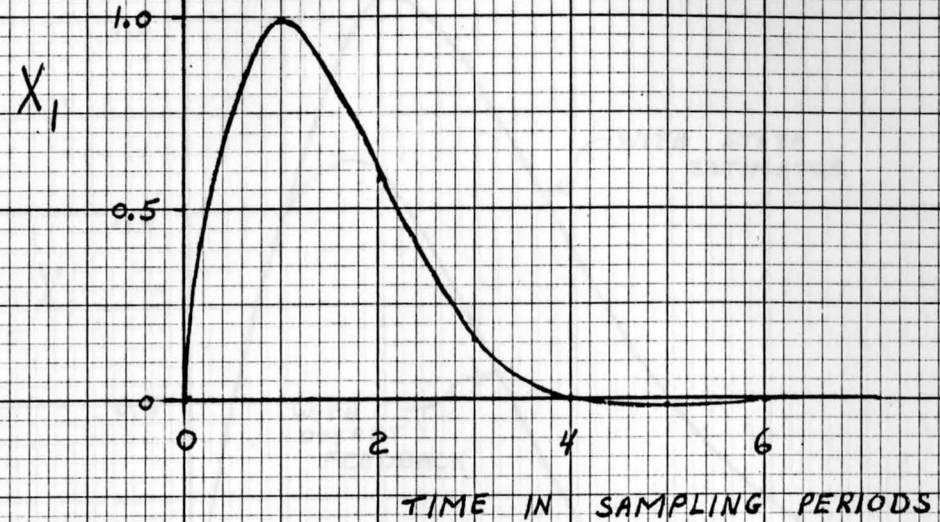


FIGURE 7(b) RESPONSE OF X_2 FOR CASE #1
WITH OUTPUT FEEDBACK



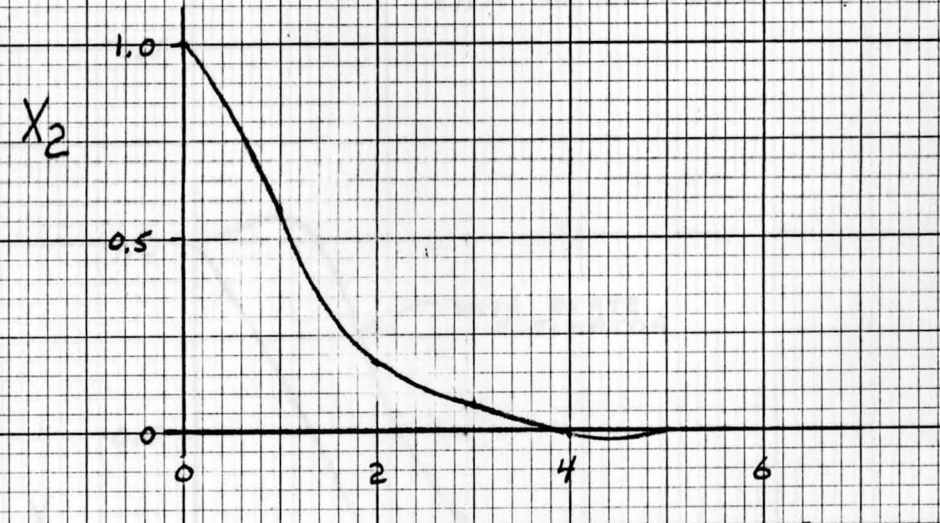


FIGURE 8 RESPONSE OF X_1 AND X_2 FOR CASE #1 WITH STATE ESTIMATOR

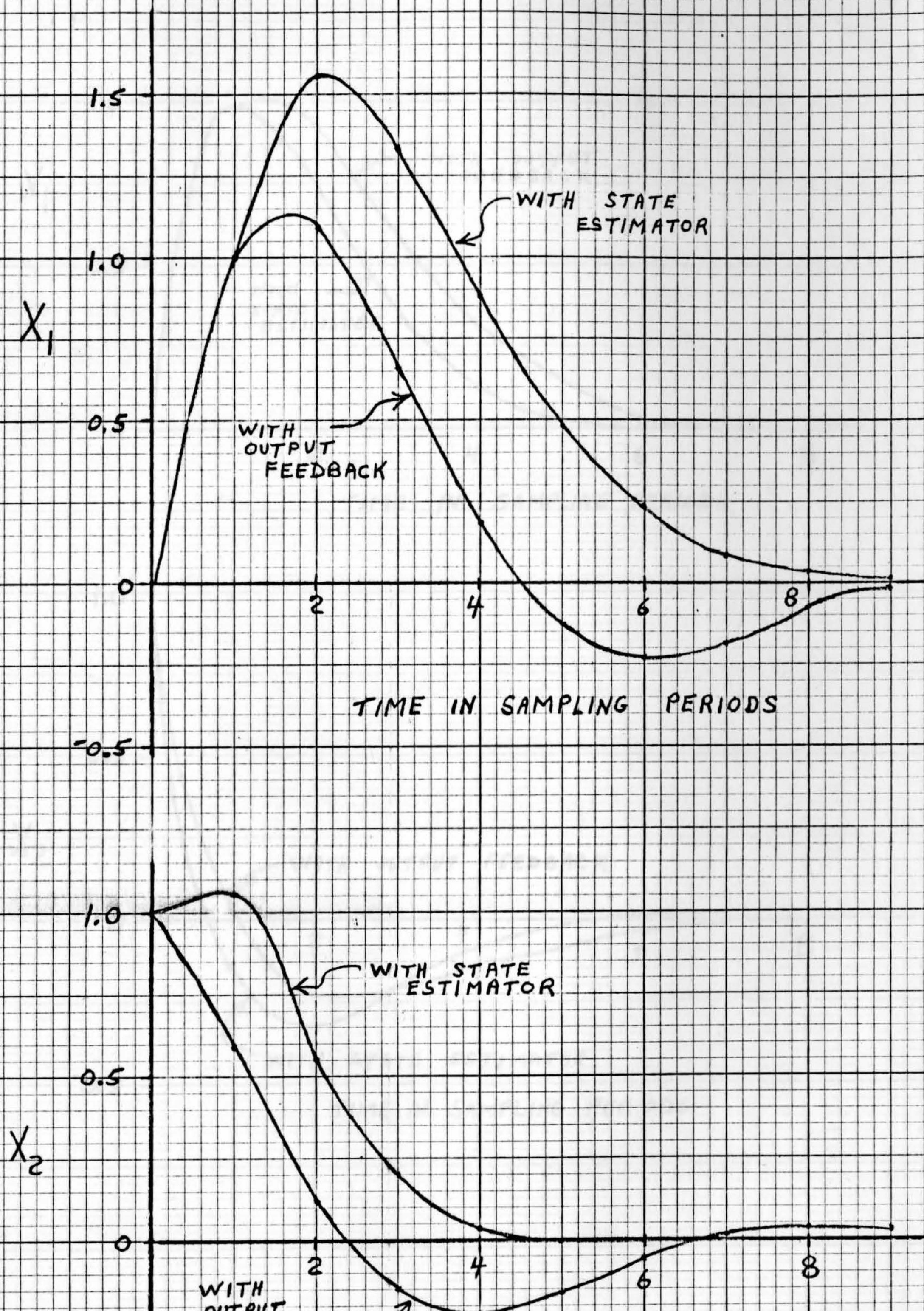


FIGURE 9 RESPONSE OF X_1 AND X_2 FOR CASE #2

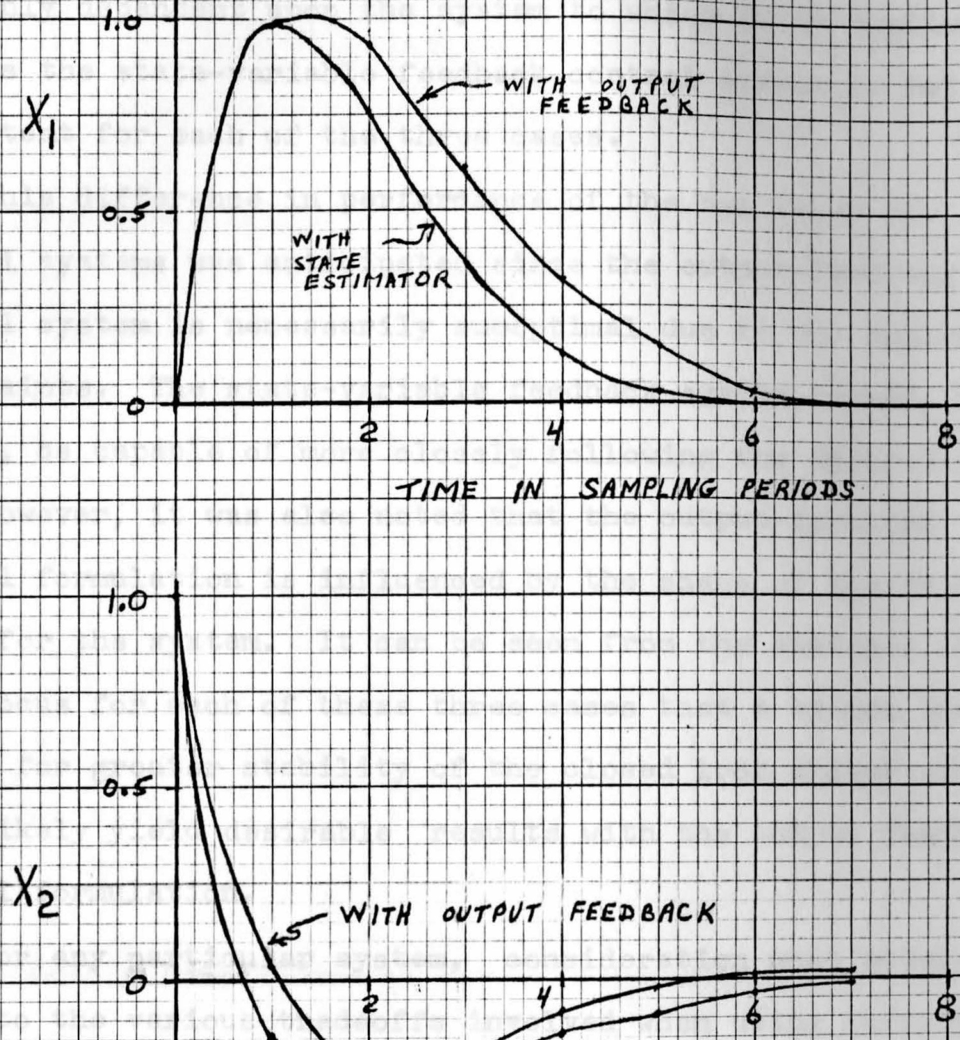


FIGURE 10 RESPONSE OF X_1 AND X_2 FOR CASE #3

Comparing the results of these three cases it is seen that the performance of the output-feedback control formulation is highly dependent upon the system to which it is applied, whereas the state-variable feedback control system is fairly consistent for each of the three cases.

This difference in performance of the two types of control systems was anticipated since the output-feedback control system is necessarily suboptimal due to the additonal constraints. The state-variable feedback system should, by design, be capable of more closely following the desired state.

However, it was also noted that the output-feedback control formulation is influenced by the shape of the root locus for the system. It can be seen from the sketches of the root locus for each of these three cases that a system which allows for greater stability of the closed loop system will most likely yield desirable results with the output feedback control formulation.

For any particular system, consideration must always be given to the various tradeoffs involved when using output-feedback control or any other formulation. Because of the added constraints, output-feedback control is suboptimal when compared to the state-estimator control. However, at times it may be desirable to utilize a suboptimal control system in exchange for some reduction in cost of implementation.

APPENDIX I

DIFFERENTIATION OF MATRIX EQUATIONS

(1) If $f(X)$ is a function of the $(m \times n)$ matrix X, then the derivative of $f(X)$ with respect to X is: [18]

$$\frac{\partial f}{\partial X} = \left[\frac{\partial f}{\partial x_{ij}} \right] .$$

(i.e. $\frac{\partial f}{\partial X}$ is an $(m \times n)$ matrix whose ij^{th} element is $\frac{\partial f}{\partial x_{ij}}$.)

(2) If $f(X) = a'Xb$, where a is an $(m \times 1)$ vector, b is an $(n \times 1)$ vector, and X is an $(m \times n)$ matrix, then [18]

$$\frac{\partial f}{\partial X} = ab' \quad .$$

(3) If $f(X) = AX$, where A is an $(n \times n)$ matrix which is not a function of X, then [7]

$$\frac{\partial}{\partial X}\left[tr\left\{f(X)\right\} \right] = A' \quad .$$

(4) If $f(X) = X'AX$, where A is not a function of X, then

$$\frac{\partial}{\partial X}\left[tr\left\{f(X)\right\} \right] = (A+A')X \quad .$$

If A is a symmetric matrix, then

$$\frac{\partial}{\partial X}\left[tr\left\{f(X)\right\} \right] = 2AX \quad .$$

APPENDIX II

EQUATIONS FOR ESTIMATOR-CONTROL SYSTEM

The control equation (3.5) for the estimator - control system can be written in augmented matrix form as:

$$\left[u(k)\right] = -\left[\begin{array}{c|c} H & H_w \end{array}\right]\left[\begin{array}{c} x(k) \\ \hline w(k) \end{array}\right] .$$

Likewise, the equations for the estimated state and disturbance vectors can be written:

$$\left[\begin{array}{c} \hat{x}(k+1) \\ \hline \hat{w}(k+1) \end{array}\right] = \left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]\left[\begin{array}{c} \hat{x}(k) \\ \hline \hat{w}(k) \end{array}\right] + \left[\begin{array}{c} T_1 \\ \hline 0 \end{array}\right]\left[u(k)\right] + \left[\begin{array}{c} L \\ \hline L_w \end{array}\right]\left(\left[y(k)\right] - \left[\begin{array}{c|c} C & C_w \end{array}\right]\left[\begin{array}{c} \hat{x}(k) \\ \hline \hat{w}(k) \end{array}\right]\right) .$$

As shown by Quaranta, [5] the estimator matrix is given by:

$$\left[\begin{array}{c} L \\ \hline L_w \end{array}\right] = \left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c} C \\ \hline C_w \end{array}\right]\left(\left[\begin{array}{c|c} C & C_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c} C \\ \hline C_w \end{array}\right] + \left[\begin{array}{c} V \\ \hline 0 \end{array}\right]\right)^{-1}$$

and

$$\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right] = \left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]' -$$

$$\left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c} C \\ \hline C_w \end{array}\right]\left(\left[\begin{array}{c|c} C & C_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c} C \\ \hline C_w \end{array}\right] + \left[\begin{array}{c} V \\ \hline 0 \end{array}\right]\right)^{-1} \cdot$$

$$\left[\begin{array}{c|c} C & C_w \end{array}\right]\left[\begin{array}{c|c} \hat{P} & \hat{G} \\ \hline \hat{G}' & \hat{W} \end{array}\right]\left[\begin{array}{c|c} \phi & T_2 \\ \hline 0 & \phi_w \end{array}\right]' + \left[\begin{array}{c|c} \xi & Z \\ \hline Z' & \eta \end{array}\right]$$

where $\hat{P}_k = E\{\hat{x}(k)\hat{x}'(k)\}$, $\hat{W}_k = E\{\hat{w}(k)\hat{w}'(k)\}$, $\hat{G}_k = E\{\hat{x}(k)\hat{w}'(k)\}$.

The equations used to determine the control gain matrices H and H_w are those developed by Halyo and Foulkes: [4]

$$H = \tilde{R}^{-1} G_1$$

$$H_w = \tilde{R}^{-1} G_2$$

where

$$\tilde{R} = \hat{R} + T_1' P_1 T_1$$

$$G_1 = T_1' P_1 \phi + \hat{M}'$$

$$G_2 = T_1'(P_2 \phi_w + P_1 T_2)$$

$$P_1 = \phi' P_1 \phi + \hat{Q} + G_1' \tilde{R}^{-1} G_1$$

$$P_2 = (\phi - T_1 \tilde{R}^{-1} G_1)'(P_2 \phi_w + P_1 T_2) + N' \; .$$

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