THE MATRIX ANALYSIS

OF NONORTHOGONAL FRAMES

by

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ABSTRACT

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The purpose of this thesis is to investigate the characteristics of the forces and displacements associated with the statical stiffness matrix analysis of orthogonal and nonorthogonal portal frames. The frames are subject to the effects of axial force as well as transverse static forces.

A variety of problems are analyzed using matrix analysis techniques. Problems of orthogonal frames with different transverse load conditions, including or excluding axial force are considered initially for analytical purposes. Next, problems of nonorthogonal frames with different loading conditions, both axial and transverse, are considered. The effect of member slope on the matrix analysis techniques is investigated and calculated.

Finally, a symmetric nonorthogonal frame with both members inclined is considered using an approximate method as well as exact method. The nonlinear load-deflection characteristics of the frame are determined.

In general, it is found that the inclusion of axial stiffness for orthogonal frames is not necessary for solving the analytical

problem. It must, however, be present in the stiffness matrix definition for nonorthogonal frames if reasonable numerical results are desired.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Cross-sectional area of member
E	Young's modulus of Elasticity
I	Moment of inertia
$[K]$	Exact stiffness matrix
$[K]$	Approximate stiffness matrix
$[K]_E$	Elastic stiffness matrix
$[K]_G$	Geometrical stiffness matrix
$[K]_M$	Mass stiffness matrix
L	Length of member
M	Bending moment
P	Axial force
$\{P\}$	Member-end forces matrix
$\{Q\}$	The nodal forces matrix
U	Horizontal shear force
V	Vertical shear force
u	Horizontal displacement
v	Vertical displacement
θ_B	Rotation at the node B
θ	Angle of the inclined member
Σ	Summation of the mathematical terms that follow
Ω	Natural frequency of free vibration of the beam-column
$\{\Delta\}$	The displacement matrix
ϕ	Square of slenderness ratio $= (L/r)^2$

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CHAPTER I

INTRODUCTION

1.1 Background

Since the introduction of the digital computer in early 1950, matrix methods of structural analysis have been widely utilized. Matrix structural methods include: the flexibility method, where forces are unknowns, and the stiffness method, where displacements are chosen as unknowns.

In the flexibility method, equilibrium conditions are sufficient to obtain the solution for problems in determinate structural systems. For indeterminate cases, the compatibility conditions are necessary to develop the additional equations to solve the problem. The coefficients in the flexibility method are much more complicated than in the stiffness method, especially when the number of degrees of freedom increases.

In the stiffness method, the compatibility conditions between node displacements and member deformations are initially established. The unknown node displacements are computed by the solution of a system of linear or non-linear equations obtained through the application of the equilibrium conditions between internal and external forces at the node. Since one-to-one correspondence exists between joint displacements and equilibrium conditions, the stiffness method can equally be applied to determinate and indeterminate structural analysis problems. The stiffness method is used in finite element analysis. Experience[(1)*] has

*Number in parenthesis refers to literature cited in the Bibliography.

shown that the stiffness method is more desirable since its formulation is simpler for the majority of structural analysis problems. Therefore, the analysis in this thesis utilizes stiffness method.

1.2 Definitions

The sign convention used throughout this work is first defined. The direction of displacement (force) and rotation (moment) as applied is shown on a general beam element (see Fig. 1-1) in the positive directions.

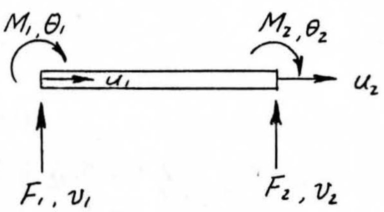


Fig. 1-1 General Beam Element

The relationships between joint forces and joint displacements of finite element stiffness equation are linear algebraic equtions in the form:

$$\{F\} = [K]\{\Delta\}$$

The matrix $[K]$ is the element stiffness matrix, and $\{F\}$ and $\{\Delta\}$ are element force and displacement vectors. The symbol $[\;]$ is used to designate a square matrix.

1.3 Formulation of the Exact Stiffness Matrix

1.3.1 Exact stiffness matrix of an axial force element

To illustrate the procedure of formulating the stiffness matrix, we choose two simple elements: an axial member and a beam element. Consider first the axial member (see Fig. 1-2).

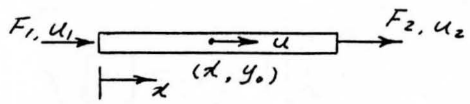


Fig. 1-2 Axial Member

By a polynomial representation, the one-dimensional variation of u, the arbitrary axial displacement at point (x, y_0) is taken as:

$$u(x, y_0) = a_1 + a_2 x$$
$$= \{1, x\}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \qquad (1\text{-}1a)$$

or

$$\{\Delta\} = [P]^T\{a\} \qquad (1\text{-}1b)$$

Applying the boundary conditions

(1) $x = 0$ $\quad u = u_1$

(2) $x = L$ $\quad u = u_2$

yields the matrix equation:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \qquad (1\text{-}2a)$$

or in matrix symbolic form

$$\{\Delta\} = [B]\{u\} \qquad (1\text{-}2b)$$

Thus,

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (1\text{-}3a)$$

or

$$\{a\} = [B]^{-1}\{\Delta\} \qquad (1\text{-}3b)$$

It follows from Equation (1-1b) and (1-3b) that

$$u = [P]^T\{a\} = [P]^T[B]\{\Delta\}$$

$$u = [N][\Delta] \qquad (1\text{-}4a)$$

where

$$[N] = \{1, x\}\begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$$
$$[N] = [(1 - x/L), x/L]$$
$$[N] = [(1-\xi), \xi] \qquad (1\text{-}4b)$$

and

$$\xi = x/L$$

The element stiffness matrix $[K]^{(2)}$ is given as

$$[K] \simeq \int_V \{b\}^T [\chi] \{b\} dV \tag{1-5}$$

where $\{b\}$ represents a matrix of the exact strain due to unit displacement, $[\chi]$ is a square matrix of constant terms relating stress, and strain components expressed in terms of Young's modulus E and Poisson's Ratio v .

The displacement u along the longitudinal axis of the member is given by Equation (1-4a) as

$$u = (1 - \xi) u_1 + (\xi) u_2$$

or

$$u = u_1 + (u_2 - u_1) \cdot \frac{x}{L} \tag{1-6}$$

Noting that the longitudinal strain component $e_{xx} = \partial u / \partial x$, it follows from Equation (1-6) that

$$e_{xx} = \frac{1}{L}\{ u_2 - u_1 \} = \frac{1}{L}\{ -1 , 1 \} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{1-7}$$

In this case the strain distribution given by Equation (1-7) is not only compatible but also exact, hence,

$$\{b\} = \frac{1}{L}\{ 1 , -1 \} \tag{1-8}$$

Since the bar in Fig. 1-2 is a one-dimensional element for which

$$[\chi] = [E] \tag{1-9}$$

the stiffness matrix $[K]$, determined from Equation (1-5), becomes

$$[K] = \int_0^L \frac{1}{L^2} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [E] \{-1, 1\} A \, dx$$

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1-10}$$

where A represents the cross-section area of the bar which is assumed to be constant.

1.3.2 Exact stiffness matrix of beam-column elements

In nonlinear analysis the equation of motion of the beam-column element as given by

$$EIy^{iv} + Py'' = 0 \tag{1-11}$$

Assuming

$$y(x) = A_1 \cos kx + A_2 \sin kx + A_3 x + A_4 \tag{1-12a}$$

where

$$k^2 = \frac{P}{EI} ,$$

it follows that,

$$y'(x) = -kA_1 \sin kx + kA_2 \cos kx + A_3 \tag{1-12b}$$

$$y''(x) = -k^2 A_1 \cos kx - k^2 A_2 \sin kx \tag{1-12c}$$

$$y'''(x) = k^3 A_1 \sin kx - k^3 A_2 \cos kx \tag{1-12d}$$

The boundary conditions on y and y' (see Fig. 1-3a) are as follow:

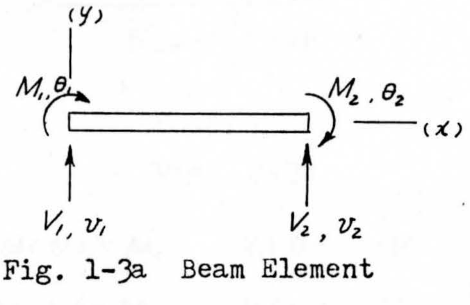


Fig. 1-3a Beam Element

$$y(0) = v_1 \qquad y'(0) = -\theta_1$$

$$y(L) = v_2 \qquad y'(L) = -\theta_2$$

Substituting the latter equations into Equation (1-12a) and (1-12b) gives:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -k & -1 & 0 \\ \cos kL, & \sin kL & L & 1 \\ k\sin kL, & -k\cos kL, & -1, & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{1-13a}$$

or

$$[B]\{a\} = \{U\} \tag{1-13b}$$

It follows that:

$$[B]^{-1} = \frac{1}{2k\cos kL - 2k + k^2 L\sin kL} \begin{bmatrix} (k\cos kL - k), & (\sin kL - kL\cos kL), & (k - k\cos kL), & (kL - \sin kL) \\ (k\sin kL), & (-\cos kL - kL\sin kL + 1), & (-k\sin kL), & (-1 + \cos kL) \\ (-k^2\sin kL), & (k - k\cos kL), & (k^2\sin kL), & (k - k\cos kL) \\ \begin{pmatrix} k^2 L\sin kL - k \\ + k\cos kL \end{pmatrix}, & (-\sin kL + kL\cos kL), & (-k + k\cos kL), & (-kL + \sin kL) \end{bmatrix}$$

The additional boundary conditions from Fig. 1-3a and 1-3b on end moments and shears are:

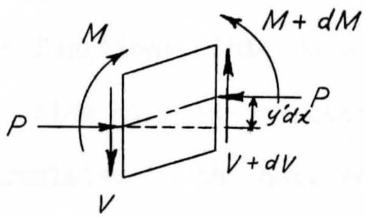


Fig. 1-3b

$$\left.\begin{aligned} M(0) &= M_1 \qquad & V(0) &= -V_1 \\ M(L) &= M_2 \qquad & V(L) &= V_2 \end{aligned}\right\} \tag{1-14a}$$

$$\left.\begin{aligned} EIy''(x) &= M(x) \\ EIy'''(x) + Py'(x) &= -V(x) \end{aligned}\right\} \tag{1-14b}$$

Substituting Equations (1-14a) and (1-14b) into Equations(1-12b,c,d) yields

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = EI \begin{bmatrix} 0 & 0 & k^2 & 0 \\ -k^2 & 0 & 0 & 0 \\ 0 & 0 & -k^2 & 0 \\ k^2\cos kL, & k^2\sin kL, & 0 & 0 \end{bmatrix} \begin{bmatrix} B \end{bmatrix}^{-1} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

that is,

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{EI}{2\cos kL - 2 + kL\sin kL} \begin{bmatrix} -k^3\sin kL, & k^2(1-\cos kL), & k^3\sin kL & k^2(1-\cos kL) \\ -k^2(\cos kL - 1), & k(\sin kL - kL\cos kL), & -k^2(1-\cos kL), & -k(\cos kL - 1) \\ k^3\sin kL, & k^2(\cos kL - 1), & -k^3\sin kL, & -k^2(1-\cos kL) \\ k^2(1-\cos kL), & k(\sin kL - kL), & k^2(\cos kL - 1), & k(kL\cos kL - \sin kL) \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

(1-15a)

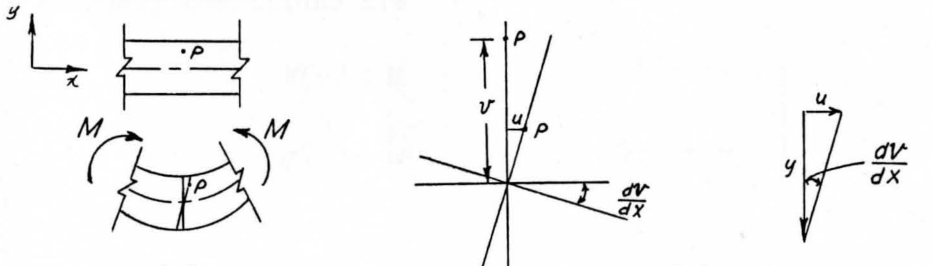
or

$$\{P\} = [K]_{ex}\{U\} \tag{1-15b}$$

where $[K]_{ex}$ is the exact stiffness matrix.

The components of the above exact stiffness matrix contain terms which are trigonometric functions. This form is too complicated for solving actual problem using digital computer techniques. An approximate stiffness matrix is formulated in the next section to simplify matters from a computation standpoint.

1.4 Approximate Stiffness Matrix of Beam-Column Elements



(a) (b)

Fig. 1-4 Beam Element Displacement Geometry

Considering the beam-column element in Fig. 1-3a, the strain field varies within the element. Consistent with the theory of flexure which disregards transverse shear deformation, we have to define both transverse displacements (v_1 and v_2) at the end points and angular displacements (θ_1 and θ_2). The latter are equal to the negative of the slope of the neutral axis since a positive (clockwise) rotation induces a negative end slope for y positive upward, or

$$\theta_1 = -\left.\frac{dV}{dx}\right|_{x=0} \qquad \theta_2 = -\left.\frac{dV}{dx}\right|_{x=L}$$

Thus,
$$\{\Delta\} = [\, v_1 , \theta_1 , v_2 , \theta_2 \,]^T \tag{1-16}$$

Four degrees of freedom are present in this case, so that a cubic polynomial is chosen in order to produce four arbitrary constants in the form

$$V(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4 \tag{1-17a}$$

or

$$V = [\, x^3,\ x^2,\ x,\ 1 \,] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{1-17b}$$

and
$$\{v\} = [P]^T\{a\} \tag{1-17c}$$

also,

$$v'(x) = 3a_1 x^2 + 2a_2 x + a_3 \tag{1-18}$$

The boundary conditions are

$$\left.\begin{aligned} v(0) &= v_1 & \qquad v(L) &= v_2 \\ v'(0) &= -\theta_1 & \qquad v'(L) &= -\theta_2 \end{aligned}\right\} \tag{1-19}$$

Substituting Equation (1-19) into Equations (1-18) and (1-17), gives the matrix form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ L^3 & L^2 & L & 1 \\ -3L^2 & -2L^2 & -1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{1-20}$$

for which the inverse form becomes

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 2/L^3 & -1/L^2 & -2/L^3 & -1/L^2 \\ -3/L^2 & 2/L & 3/L^2 & 1/L \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{1-21a}$$

or

$$\{a\} = [B]^{-1}\{\Delta\} \tag{1-21b}$$

Combining Equations (1-17b) and (1-21b) yields

$$\begin{aligned} \{v\} &= [P]^T \{a\} \\ \{v\} &= [P]^T [B]^{-1} \{\Delta\} \\ \{v\} &= [N]^T \{\Delta\} \end{aligned} \tag{1-22}$$

where

$$[N]^T = [N_1 , N_2 , N_3 , N_4]$$

and

$$\left.\begin{aligned} N_1 &= 2\cdot\frac{x}{L} - 3\frac{x^2}{L^2} + 1 = 1 + 2\xi - 3\xi^2 \\ N_2 &= -\frac{x^3}{L^3} + 2\frac{x^2}{L^2} - x = L(-\xi^3 + 2\xi^2 - \xi) \\ N_3 &= -2\frac{x^3}{L^3} + 3\frac{x^2}{L^2} = -2\xi^3 + 3\xi^2 \\ N_4 &= -\frac{x^3}{L^2} + \frac{x^2}{L} = L(\xi^2 - \xi^3) \end{aligned}\right\} \tag{1-23}$$

with

$$\xi = x/L$$

The horizontal displacement u , due to flexure (see Figs. 1-4a and 1-4b) is given in terms of the vertical displacement as

$$u = -\frac{dv}{dx} \cdot y \tag{1-24}$$

It follows

$$u = [N']^T \{\Delta\} \tag{1-25a}$$

from Equation (1-23) that

$$[N']^T = [N_1', N_2', N_3', N_4'] \tag{1-25b}$$

and

$$\left.\begin{aligned}
N_1' &= -\frac{d}{dx}(2\, x^3/L^3 - 3\, x^2/L^2 + 1)\cdot y = 6\,(x/L^2 - x^2/L^3)\cdot y = 6(\xi - \xi^2)\cdot\eta \\
N_2' &= -\frac{d}{dx}(-x^3/L^2 + 2\, x^2/L - x)\cdot y = (1 - 4\, x/L + 3\, x^2/L^2)\cdot y = (1 - 4\xi + 3\xi^2)L\cdot\eta \\
N_3' &= -\frac{d}{dx}(-2\, x^3/L^3 + 3\, x^2/L^2)\cdot y = (-6\, x^2/L^3 + 6\, x/L^2)\cdot y = 6(-\xi + \xi^2)\cdot\eta \\
N_4' &= -\frac{d}{dx}(-x^3/L^2 + x^2/L)\cdot y = -(-3\, x^2/L^2 + 2\, x/L)\cdot y = (-2\xi + 3\xi^2)\cdot\eta
\end{aligned}\right\} \tag{1-25c}$$

with $\eta = y/L$, $\xi = x/L$

Combining the horizontal and vertical displacements (u and v) into a single matrix equation gives

$$[U] = [\bar{N}]^T \{\Delta\} \tag{1-26a}$$

where

$$[\bar{N}]^T = \begin{bmatrix} (1-\xi), & 6(\xi - \xi^2)\eta, & (1-4\xi+3\xi^2)L\cdot\eta, & \xi, & 6(-\xi+\xi^2)\eta, & (-2\xi+3\xi^2)L\cdot\eta \\ 0, & (1-3\xi^2+2\xi^3), & (-\xi+2\xi^2-\xi^3)\cdot L, & 0, & (3\xi^2-2\xi^3), & (\xi^2-\xi^3)\cdot L \end{bmatrix} \tag{1-26b}$$

with

$$\xi = x/L \qquad \eta = y/L$$

and

$$\{U\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [\bar{N}]^T \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{1-26c}$$

where u_1, v_1, ... θ_2 are the element displacements shown in Fig. 1-5

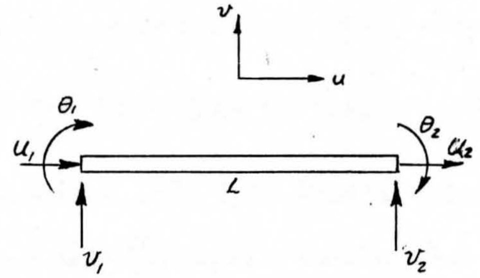


Fig. 1-5 Positive Direction of Displacements on a Beam Element

In calculating the strain energy U , we neglect the contributions from the shearing strains, and include only the normal strains ϵ_{xx} . This strain for large deflections of a beam in bending is given as

$$\epsilon_{xx} = \frac{\partial u_0}{\partial x} - \frac{\partial^2 v}{\partial x^2} \cdot y + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \tag{1-27a}$$

where y is the vertical distance of an element from the neutral axis of the beam. The parameter u_0 denotes the displacement u at $y = 0$.

$$U = \frac{E}{2}\int_V \epsilon_{xx}^2 \, dx$$

$$U = \frac{E}{2}\int_V \left[\frac{\partial u_0}{\partial x} - \frac{\partial^2 v}{\partial x^2} y + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] dV$$

$$U = \frac{E}{2}\int_{x=0}^{x=L}\int_A \left[\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right) \cdot y^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^4 - 2\frac{\partial u_0}{\partial x}\frac{\partial^2 v}{\partial x^2} \cdot y - \frac{\partial^2 v}{\partial x^2}\left(\frac{\partial v}{\partial x}\right)^2 y + \frac{\partial u_0}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2\right] dx \cdot dA \tag{1-27b}$$

Neglecting the higher-order term $\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^4$, and noting all integrals of the form $\int_{-\frac{h}{2}}^{\frac{h}{2}} y^n dA$, n = an odd number, must vanish, one obtains

$$U \simeq \frac{EA}{2}\int_0^L \left(\frac{\partial u_0}{\partial x}\right)^2 dx + \frac{EA}{2}\int_0^L \frac{\partial u_0}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{EA}{2}\int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \tag{1-28}$$

where I denotes the moment of inertia of the cross section with respect to the neutral axis. From Equation 1-26b, we have

$$
\left.\begin{aligned}
\frac{\partial u_0}{\partial x} &= \frac{1}{L}(-u_1+u_2) \\
\frac{\partial v}{\partial x} &= \frac{1}{L}\left[6(-\xi+\xi^2)u_2+(-1+4\xi-3\xi^2)L\cdot u_3+6(\xi-\xi^2)u_5+(2\xi-3\xi^2)L\cdot u_6\right] \\
\frac{\partial^2 v}{\partial x^2} &= \frac{1}{L^2}\left[6(-1+2\xi)u_2+2(2-3\xi)L\cdot u_3+6(1-2\xi)u_5+2(1-3\xi)L\cdot u_6\right]
\end{aligned}\right\} \qquad (1\text{-}29)
$$

Substituting of Equation 1-29 into Equation 1-28 and integrating yields

$$
\begin{aligned}
U = {} & \frac{EA}{2L}(u_1^2-2u_1u_4+u_4^2)+\frac{2EI}{L^3}(3u_2^2+L^2u_3^2+3u_5^2+L^2u_6^2+3Lu_2u_3-6u_2u_5+3Lu_2u_6-3Lu_3u_5 \\
& +L^2u_3u_6-3Lu_5u_6)+\frac{EA}{L^2}(u_4-u_1)(\frac{3}{5}u_2^2+\frac{1}{15}L^2u_3^2+\frac{3}{5}u_5^2+\frac{1}{15}L^2u_6^2+\frac{1}{10}Lu_2u_3 \\
& -\frac{6}{5}u_2u_5+\frac{1}{10}Lu_2u_6-\frac{1}{10}Lu_3u_5-\frac{1}{30}L^2u_3u_6-\frac{1}{10}Lu_5u_6
\end{aligned}
\qquad (1\text{-}30)
$$

As in the case of pin-jointed bar, it follows that

$$p = \frac{EA}{L}(u_4-u_1) \simeq \text{const.}$$

Applying Castigliano's theorem (part I) to the strain energy expression Equation 1-30 results in the following element force-displacement equations:

$$
\begin{Bmatrix} U_1 \\ V_1 \\ M_1 \\ U_2 \\ V_2 \\ M_2 \end{Bmatrix} = EI
\begin{bmatrix}
\frac{AE}{L} & & & & & \\
0, & \frac{12}{L^3} & & \text{Symmetric} & & \\
0, & \frac{-6}{L^2}, & \frac{4}{L} & & & \\
\frac{-AE}{L}, & 0, & 0, & \frac{AE}{L} & & \\
0, & \frac{-12}{L^3}, & \frac{6}{L^2}, & 0, & \frac{12}{L^3} & \\
0, & \frac{-6}{L^2}, & \frac{2}{L}, & 0, & \frac{6}{L^2}, & \frac{4}{L}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
- p
\begin{bmatrix}
0 & & & & & \\
0, & \frac{6}{5L} & & \text{Symmetric} & & \\
0, & \frac{-1}{10}, & \frac{2L}{15} & & & \\
0, & 0, & 0, & 0 & & \\
0, & \frac{-6}{5L}, & \frac{1}{10}, & 0, & \frac{6}{5L} & \\
0, & \frac{1}{10}, & \frac{-L}{30}, & 0, & \frac{1}{10}, & \frac{2L}{15}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
\qquad (1\text{-}31a)
$$

which is written symbolically as

$$\{P\} = \left[[\tilde{K}]_E - p[\tilde{K}]_G\right]\{u\} = [\tilde{K}]\{u\} \qquad (1\text{-}31b)$$

where

$[K]_E$ is the elastic stiffness matrix

and

$[K]_G$ is the geometrical stiffness matrix

1.5 Transformation to Global Coordinates

The stiffness matrix is based on an $x-y$ coordinate system in which the x-axis is the direction along the member. The transformation of the stiffness matrix to a global $X-Y$ system of accomplished as follows:

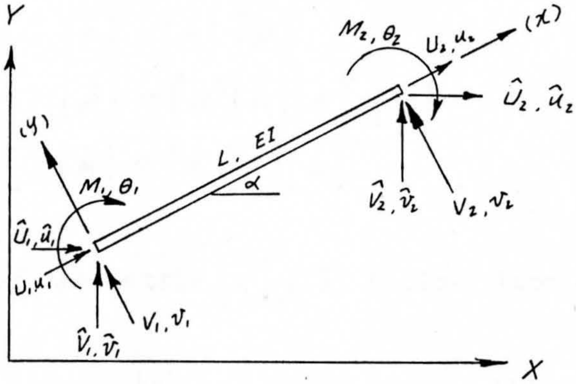


Fig. 1-6 Coordinate Transformation

The relation between $\{\hat{P}\}$, the applied force vector due in the global $X-Y$ coordinates, and $\{P\}$, the applied force vector in the $x-y$ coordinates is written as (see Fig. 1-6)

$$\begin{Bmatrix} \hat{U}_1 \\ \hat{V}_1 \\ M_1 \\ \hat{U}_2 \\ \hat{V}_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha, & -\sin\alpha, & 0, & 0, & 0, & 0 \\ \sin\alpha, & \cos\alpha, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & \cos\alpha, & -\sin\alpha, & 0 \\ 0 & 0 & 0 & \sin\alpha, & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ M_1 \\ U_2 \\ V_2 \\ M_2 \end{Bmatrix} \tag{1-32a}$$

or

$$\{\hat{P}\} = [R_1]\{P\} \tag{1-32b}$$

The displacement vectors $\{\hat{U}\}$ and $\{\hat{V}\}$ are related in a similar form as:

$$\{\hat{U}\} = [R_1]\{U\} \tag{1-33}$$

where

$[R_1]$ is an orthogonal matrix

Combining Equation 1-32b, 1-32b and 1-33 into a single matrix equation gives:

$$\{\hat{P}\} = [R_1]\{P\}$$

$$\{\hat{P}\} = [R_1][K][R_1]^T\{\hat{U}\} \tag{1-34a}$$

$$[\hat{K}] = [R_1][K][R_1]^T \tag{1-34b}$$

In the case of the matrix $[\hat{K}]_E$, it follows from Equation 1-34b that

$$[K]_E = EI \begin{bmatrix} \cos\alpha, & -\sin\alpha, & 0, & 0 & 0 & 0 \\ \sin\alpha, & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha, & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha, & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{A}{LI} & 0 & 0 & \frac{-A}{LI} & 0 & 0 \\ 0 & \frac{12}{L^3} & \frac{-6}{L^2} & 0 & \frac{-12}{L^3} & \frac{-6}{L^2} \\ 0 & \frac{-6}{L^2} & \frac{4}{L} & 0 & \frac{6}{L^2} & \frac{2}{L} \\ \frac{-A}{LI} & 0 & 0 & \frac{A}{LI} & 0 & 0 \\ 0 & \frac{-12}{L^3} & \frac{6}{L^2} & 0 & \frac{12}{L^3} & \frac{6}{L^2} \\ 0 & \frac{-6}{L^2} & \frac{2}{L} & 0 & \frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{bmatrix} \cos\alpha, & \sin\alpha, & 0 & 0 & 0 & 0 \\ -\sin\alpha, & \cos\alpha. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha, & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha, & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$[K]_E = EI \begin{bmatrix} \left(\frac{AC}{IL} + \frac{12S^2}{L^3}\right), & \left(\frac{Asc}{IL} - \frac{12cs}{L^3}\right), & \left(\frac{6S}{L^2}\right), & \left(\frac{-Ac^2}{IL} - \frac{12S^3}{L^3}\right), & \left(\frac{-ACS}{IL} + \frac{12cs}{L^3}\right), & \left(\frac{6S}{L^2}\right) \\ \left(\frac{Asc}{IL} - \frac{12cs}{L^3}\right), & \left(\frac{AS^2}{IL} + \frac{12C^2}{L^3}\right), & \left(\frac{-6c}{L^2}\right), & \left(\frac{-ACS}{IL} + \frac{12cs}{L^3}\right), & \left(\frac{-AS^2}{IL} - \frac{12C^2}{L^3}\right) & \left(\frac{-6c}{L^2}\right) \\ \frac{6S}{L^2} & -\frac{6c}{L^2} & \frac{4}{L} & -\frac{6S}{L^2} & \frac{6c}{L^2} & \frac{2}{L} \\ \left(\frac{-Ac^2}{IL} - \frac{12S^2}{L^3}\right) & \left(-\frac{Asc}{IL} + \frac{12cs}{L^3}\right) & \left(-\frac{6S}{L^2}\right). & \left(\frac{Ac^2}{IL} + \frac{12S^2}{L^3}\right), & \left(\frac{Acs}{IL} - \frac{12cs}{L^3}\right), & \frac{-6S}{L^2} \\ \left(\frac{-Asc}{IL} + \frac{12cs}{L^3}\right) & \left(\frac{-AS^2}{IL} - \frac{12c^2}{L^3}\right) & \left(\frac{6c}{L^2}\right) & \left(\frac{Acs}{IL} - \frac{12cs}{L^3}\right) & \left(\frac{AS^2}{IL} + \frac{12c^2}{L^3}\right) & \frac{6c}{L^2} \\ \frac{6S}{L^2} & \frac{-6c}{L^2} & \frac{2}{L} & \frac{-6S}{L^2} & \frac{6c}{L^2} & \frac{4}{L} \end{bmatrix} \tag{1-35}$$

where $s = \sin\alpha$, $c = \cos\alpha$

In nonlinear analysis, $[\hat{K}] = [\tilde{K}]_E - P[K]_G$, using the same procedure, one obtains $[\hat{K}]$ matrix

$$[\hat{K}] = \begin{bmatrix} \left(\frac{AC}{IL}+\frac{12S^2}{L^3}\right)EI-\frac{6S^2P}{5L} & & & & & \text{Symmetric} \\ \left(\frac{ASC}{IL}-\frac{12CS}{L^3}\right)EI+\frac{6SCP}{5L} & \left(\frac{AS^2}{IL}+\frac{12C^2}{L^3}\right)EI-\frac{6C^2P}{5L} & & & & \\ \frac{6SEI}{L^2}-\frac{PS}{10} & \frac{-6CEI}{L^2}+\frac{CP}{10} & \frac{4EI}{L}-\frac{2PL}{15} & & & \\ \left(-\frac{AC^2}{IL}-\frac{12S^2}{L^3}\right)EI+\frac{6S^2P}{5L} & \left(\frac{-ASC}{IL}+\frac{12CS}{L^3}\right)EI-\frac{6CSP}{5L} & \frac{-6SEI}{L^2}+\frac{SP}{10} & \left(\frac{AC^2}{IL}+\frac{12S^2}{L^3}\right)EI-\frac{6S^2P}{5L} & & \\ \left(\frac{-ACS}{IL}+\frac{12CS}{L^3}\right)EI-\frac{6CSP}{5L} & \left(\frac{-AS^2}{IL}-\frac{12C^2}{L^3}\right)EI+\frac{6C^2P}{5L} & \frac{-6CEI}{L^2}-\frac{CP}{10} & \left(\frac{ACS}{IL}-\frac{12CS}{L^3}\right)EI+\frac{6CSP}{5L} & \left(\frac{AS^2}{IL}-\frac{12C^2}{L^3}\right)EI-\frac{6C^2P}{5L} & \\ \frac{6SEI}{L^2}-\frac{SP}{10} & \frac{-6CEI}{L^2}+\frac{CP}{10} & \frac{2EI}{L}+\frac{PL}{30} & \frac{-6SEI}{L^2}+\frac{SP}{10} & \frac{6CEI}{L^2}-\frac{CP}{10} & \frac{4EI}{L}-\frac{2PL}{15} \end{bmatrix}$$

(1-36)

For the exact stiffness matrix $[\hat{K}]_{ex}$, one obtains

$$[\hat{K}]_{ex} = \begin{bmatrix} \left(\frac{Ac^2}{IL}-\frac{k^4 s^2 \tilde{s}}{D}\right) & \left(\frac{Acs}{IL}+\frac{k^4 c\cdot s\cdot\tilde{s}}{D}\right) & \left(\frac{k^3 s(\tilde{c}-1)}{D}\right) & \left(\frac{-Ac^2}{IL}+\frac{k^4 s\tilde{s}}{D}\right) & \left(\frac{-Acs}{IL}-\frac{k^4\tilde{s}\cdot s\cdot c}{D}\right) & \left(\frac{k^3(\tilde{c}-1)\cdot s}{D}\right) \\ & \left(\frac{As^2}{IL}-\frac{k^4\tilde{s}c^2}{D}\right) & \left(\frac{-k^3 c(\tilde{c}-1)}{D}\right) & \left(\frac{-Asc}{IL}-\frac{k^4 c\cdot s\cdot\tilde{s}}{D}\right) & \left(\frac{-As^2}{IL}+\frac{k^4 c^2\cdot\tilde{s}}{D}\right) & \left(\frac{-k^3 c(\tilde{c}-1)}{D}\right) \\ & & \left(\frac{-k^2(\tilde{s}-kL\tilde{c})}{D}\right) & \left(\frac{-k^3 s(\tilde{c}-1)}{D}\right) & \left(\frac{k^3 c(\tilde{c}-1)}{D}\right) & \left(\frac{k^3(\tilde{s}-kL)}{D}\right) \\ & & & \left(\frac{Ac^2}{IL}-\frac{k^4\tilde{s}\cdot s^2}{D}\right) & \left(\frac{Acs}{IL}+\frac{k^4\tilde{s}\cdot s\cdot c}{D}\right) & \left(\frac{-k^3 s(\tilde{c}-1)}{D}\right) \\ & & & & \left(\frac{As^2}{IL}-\frac{k^4\tilde{s}\cdot c^2}{D}\right) & \left(\frac{k^3 c(\tilde{c}-1)}{D}\right) \\ & \text{Symmetric} & & & & \frac{-k^2(\tilde{s}-kL\bar{c})}{D} \end{bmatrix} EI$$

(1-37)

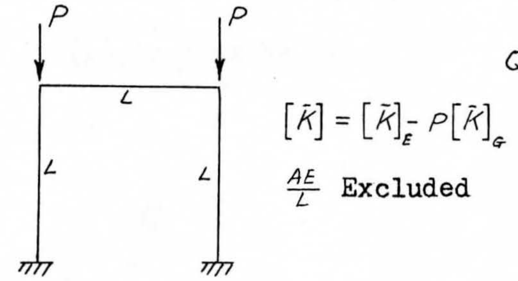
where

$$\tilde{s} = \sin kL \quad , \quad \tilde{c} = \cos kL$$

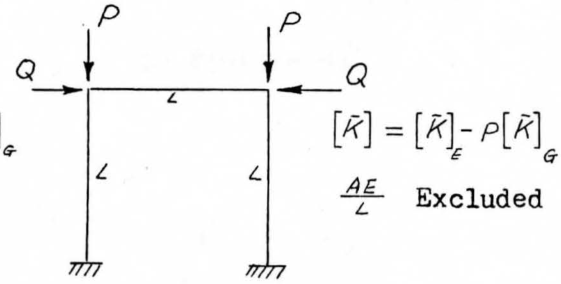
$$D = 2k(\cos kL - 1) + k^2 L \sin kL$$

1.6 List Of Problems

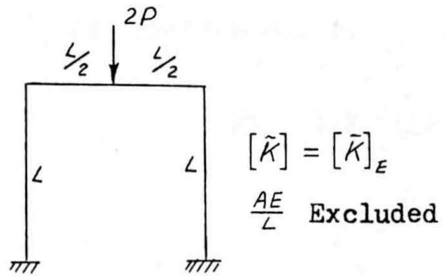
In order to illustrate the basic differences in analysis between orthogonal and nonorthogonal frames, the following geometric frames with constant E I, as listed in Fig. 1-7 are analyzed in this work:



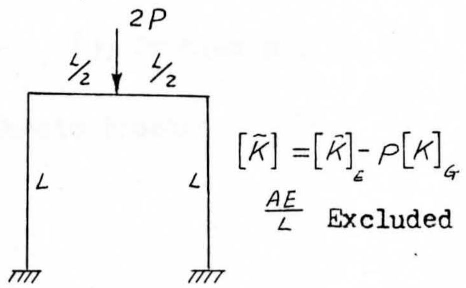
(a) Problem No. 1



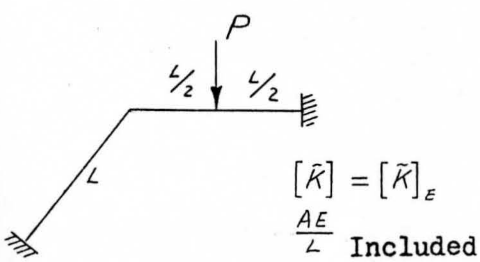
(b) Problem No. 2



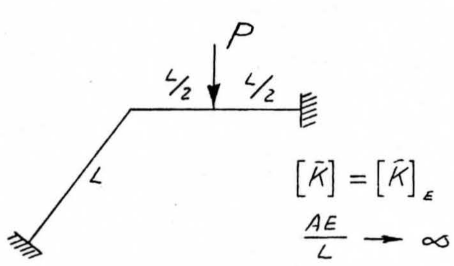
(c) Problem No. 3



(d) Problem No. 4



(e) Problem No. 5



(f) Problem No. 6

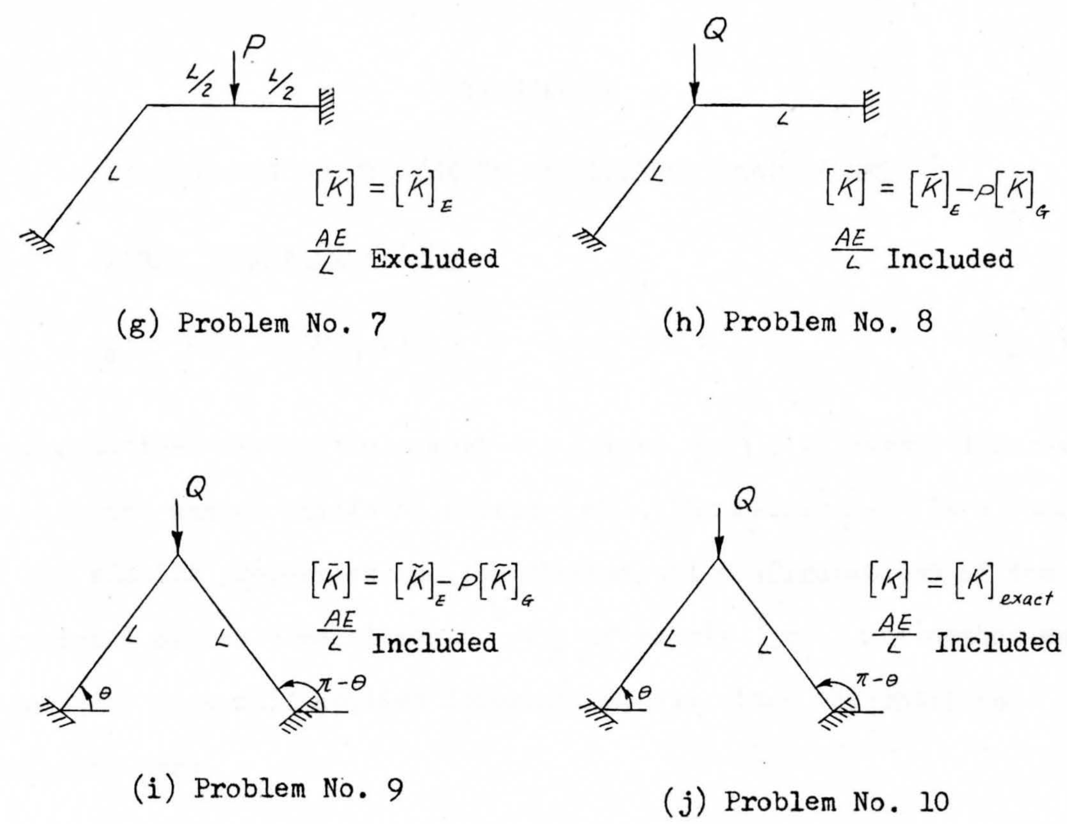


Fig. 1-7 List of Thesis Problems

CHAPTER II

DEVELOPMENT OF THE THEORY OF STIFFNESS MATRIX METHOD

2.1 Basic Matrix Equations

a) $\{P_1\} = [K]\{U\}$ (2-1)

This equation relates the member-end forces $\{P_1\}$, to member deformation $\{U\}$ by the member stiffness matrix $[K]$. The matrix $[K]$ is a function of the elastic properties and the dimensional configurations of the structural system. The elements k_{ij} of matrix $[K]$, are member-end forces due to a unit applied deformation, all other deformations remaining zero.

b) $\{P\} = \{P_0\} + \{P_1\}$ (2-2)

The member-end forces $\{P\}$ are expressed as the summation of two values: $\{P_0\}$, member-end forces due to actual external loads when all displacements at the nodes are held at zero, and $\{P_1\}$, member-end forces due to the joint displacements.

c) $\{U\} = [A]\{D\}$ (2-3)

The member deformations $\{U\}$ are related to node displacements $\{D\}$ by a displacement-deformation $[A]$, which is a function of the geometry of the structure.

d) $\{Q\} = [A]^T\{P\}$ (2-4)

This equation defines the relation between the node-load matrix $\{Q\}$, and member forces matrix $\{P\}$. The vector $\{Q\}$ contains known components. The form of Equation (2-4) is determined using the principle of virtual work. Equating the virtual work of the node forces to that of the member forces gives

$$\{Q\}^T\{\delta D\} = \{P\}^T\{\delta U\}$$

where $\{\delta D\}$ is an arbitrary virtual displacement.

$\{\delta U\}$ is the corresponding virtual deformation of the ends of members.

letting $$\{Q\} = [H]\{P\}$$ then,

$$\{P\}^T[H]^T\{\delta D\} = \{P\}^T\{\delta U\}$$

or noting Equation (2-3)

$$\{P\}^T[H]^T\{\delta D\} = \{P\}^T[A]\{\delta D\}$$

from which it follows that,

$$[H] = [A]^T$$

2.2 The Kinematically Determinate System

The four basic matrix equations which are relevant to the analysis are Equations (2-1),(2-2),(2-3), and (2-4). They are rewritten to illustrate the substitution to the Equations (2-5a), and (2-5b).

$$\{P\} = [K]\{U\}$$

$$\{P\} = \{P_o\} + \{P_1\}$$

$$\{P\} = [K]\{U\} + \{P_o\} \tag{2-5a}$$

$$\{Q\} = [A]^T\{P\}$$

$$\{Q\} = [A]^T[K]\{U\} + [A]^T\{P\}$$

$$\{U\} = [A]\{D\}$$

$$\{Q\} = [A]^T[K][A]\{D\} + [A]^T\{P_o\} \tag{2-5b}$$

$$[S] = [A]^T[K][A] \tag{2-6}$$

The matrix $[S]$ is called the structure stiffness matrix.
Equation (2-5b) is rewritten as

$$\{Q\} = [S]\{D\} + [A]^T\{P_o\} \tag{2-7a}$$

or

$$\{D\} = [S]^{-1}\{Q\} - [S]^{-1}[A]^T\{P_o\} \tag{2-7b}$$

Combining Equations (2-3) and (2-7b) yields

$$\{U\} = [A][S]^{-1}\{Q\} - [A][S]^{-1}[A]^T\{P_o\} \tag{2-8}$$

where $\{U\}$ is the vector of member displacements, $\{Q\}$ is the vector of known node forces, and $\{P_o\}$ is the vector of known fixed-end moments.

Noting Equations (2-1), (2-2), and (2-8), it follows that

$$\{P\} = [K]\,[A]\,[S]^{-1}\{Q\} + \{P_0\} - [K][A][S]^{-1}[A]^T\{P_0\} \qquad (2\text{-}9a)$$

or

$$\{P\} = [K]\,[A]\,[S]^{-1}\{Q\} + \Big[[I]-[K][A][S]^{-1}[A]^T\Big]\{P_0\} \qquad (2\text{-}9b)$$

where $\{P\}$ is the vector of member-end forces

$\{P_0\}$ is the vector of member forces due to actual external loads.

When all loads are applied only at the nodes, the fix-end forces $\{P_0\}$ are all zero. The Equation (2-9b) becomes

$$\{P\} = [K]\,[A]\,[S]^{-1}\{Q\} \qquad (2\text{-}10)$$

2.3 Kinematically Indeterminate System

2.3.1 Method I

The member end deformation $\{U\}$ is expressed in two parts:

$$\{U\} = \{U_0\} + \{U_1\} \qquad (2\text{-}11)$$

The deformations $\{U_0\}$ resulting from the displacements $\{d\}$ of the load points, are determined by Equation (2-3) as

$$\{U_0\} = [A]\,\{d\}$$

The deformations $\{U_1\}$ resulting from the redundant displacements $\{\Delta\}$, which are those displacements that are not defined in terms of transformation matrix $[A]$, are written as follows:

$$\{U_r\} = [r]\{\Delta\} \tag{2-12}$$

Accordingly, the element r_{ij} is defined as the deformation at i produced by a unit redundant displacement Δ_j .

From Equation (2-11), the total member end deformations are given by

$$\{U\} = [A]\{d\} + [r]\{\Delta\} \tag{2-13a}$$

or

$$\{U\} = \left[A \,\vdots\, r\right]\begin{Bmatrix} d \\ \hdashline \Delta \end{Bmatrix} \tag{2-13b}$$

It is seen that if $\{d\}$ represents the displacement of all load points, $\{\Delta\}$ must include all of the displacements of the unloaded nodes that have an influence upon the desired member deformations.

From Equations (2-5b) and (2-13b), and since forces applied on the nodes $\{P_o\}$ are all zero, one obtains

$$\{Q\} = \begin{bmatrix} A^T \\ \hdashline r^T \end{bmatrix}[K]\left[A \,\vdots\, r\right]\begin{Bmatrix} d \\ \hdashline \Delta \end{Bmatrix} \tag{2-14a}$$

or

$$\{Q\} = \begin{bmatrix} [A]^T[K][A], & [A]^T[K][r] \\ [r]^T[K][A], & [r]^T[K][r] \end{bmatrix}\begin{Bmatrix} d \\ \hdashline \Delta \end{Bmatrix} \tag{2-14b}$$

with

$$\{Q\} = \begin{Bmatrix} q \\ \hdashline x \end{Bmatrix} \tag{2-14c}$$

where $\{q\}$ is the load matrix applied in the direction of the displacements $\{d\}$

$\{\chi\}$ is the virtual load matrix applied in the direction of the displacements $\{\Delta\}$

Equations (2-14b) and (2-14c) are separated into two parts:

$$\{q\} = [A]^T[K][A]\{d\} + [A]^T[K][r]\{\Delta\} \tag{2-15a}$$

$$\{\chi\} = [r]^T[K][A]\{d\} + [r]^T[K][r]\{\Delta\} \tag{2-15b}$$

The definition of the stiffness matrix $[S]$ is given from Equation (2-14b) as follows:

$$[S] = \begin{bmatrix} [A]^T[K][A], & [A]^T[K][r] \\ [r]^T[K][A], & [r]^T[K][r] \end{bmatrix} \tag{2-16}$$

The stiffness matrix is square and may be inverted. Then, combining Equations (2-14b) and (2-14c), yields

$$\left\{\begin{array}{c} d \\ \hdashline \Delta \end{array}\right\} = [S]^{-1}\left\{\begin{array}{c} q \\ \hdashline \chi \end{array}\right\} \tag{2-17}$$

From Equations (2-13b) and (2-17), the member deformations and forces are written as follows:

$$\{U\} = [A \,\vdots\, r]\,[S]^{-1}\left\{\begin{array}{c} q \\ \hdashline \chi \end{array}\right\} \tag{2-18}$$

Since

$$\{P\} = [K]\{U\}$$

the member force $\{P\}$ becomes

$$\{P\} = [K][A \vdots r][S]^{-1}\left\{\begin{array}{c} q \\ \hline x \end{array}\right\} \tag{2-19}$$

2.3.2 Method II

When redundant displacements $\{\Delta\}$ for a structure loaded at points other than the nodes are required, the displacements $\{\Delta\}$ are solved by setting virtual force $\{x\} = 0$ in Equation (2-15b), thus

$$[r]^T[K][A]\{d\} + [r]^T[K][r][\Delta] = 0 \tag{2-20}$$

The form of Equation (2-20) is inefficient since the displacements $\{d\}$ are also unknown.

When $\{x\} = 0$, the member deformation $\{U_1\}$ do not exist. From Equations (2-2) and (2-11)

$$\{U\} = \{U_0\} + \{U_1\}$$

$$[K]\{U\} = [K]\{U_0\} + [K]\{U_1\}$$

$$\{P\} = \{P_0\} + \{P_1\}$$

When $\{U_1\} = 0$, Equation (2-1) yields

$$\{P_1\} = [K]\{U_1\} = \{0\}$$

so,

$$\{P\} = \{P_0\} = [K]\{U_0\} = [K]\{U\}$$

From Equation (2-3) and (2-1), one obtains:

$$[r]^T[K][A]\{d\} = [r]^T[K]\{U_o\}$$
$$= [r]^T\{P_o\} \qquad (2\text{-}21)$$

Substituting in Equation (2-20), yields

$$[r]^T\{P_o\} + [r]^T[K][r]\{\Delta\} = 0$$

Since $[r]^T[K][r]$ is a square matrix which is invertable, one obtains:

$$\{\Delta\} = -([r]\,[K][r])^{-1}[r]^T\{P_o\} \qquad (2\text{-}22)$$

From Equation (2-2) the final member forces are as follows:

$$\{P\} = \{P_o\} + \{P_1\}$$
$$= \{P_o\} + [K]\{U_1\}$$

Where $\{U_1\}$ are the member deformations produced by the redundant displacements and $\{P_o\}$ are the end forces produced by the given loads with all redundant displacements held to zero. By Equation (2-12), it follows that

$$\{P\} = [K][r]\{\Delta\} + \{P_o\} \qquad (2\text{-}23)$$

Substituting Equation (2-22) into Equation (2-23) yields

$$\{P\} = -[K][r]([r]^T[K][r])^{-1}[r]^T\{P_o\} + \{P_o\} \qquad (2\text{-}24)$$

Equation (2-24) does not contain the displacements and may be used to solve for the member forces.

CHAPTER III

ANALYSIS OF ORTHOGONAL FRAMES

3.1 Problem No. 1 Axial-Loaded Columns, Node Load

The load analysis of the rigid frame shown in Fig. 3-1 is preformed for each member having the common parameters L, E, and I as constant.

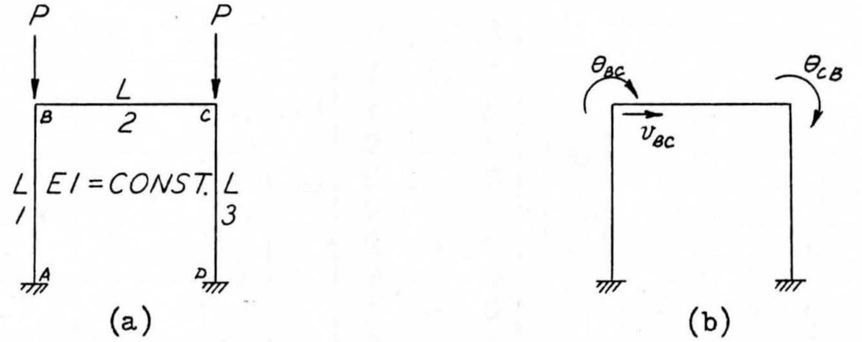


Fig. 3-1 Problem No.1- Frame Loads& Node Displacements

The frame is statically indeterminate to the third degree and kinematically indeterminate to the third degree, also. The redundant displacements are shown in Fig. 3-1b.

The positive nodal deformations and forces are defined in Fig. 3-2 as follow

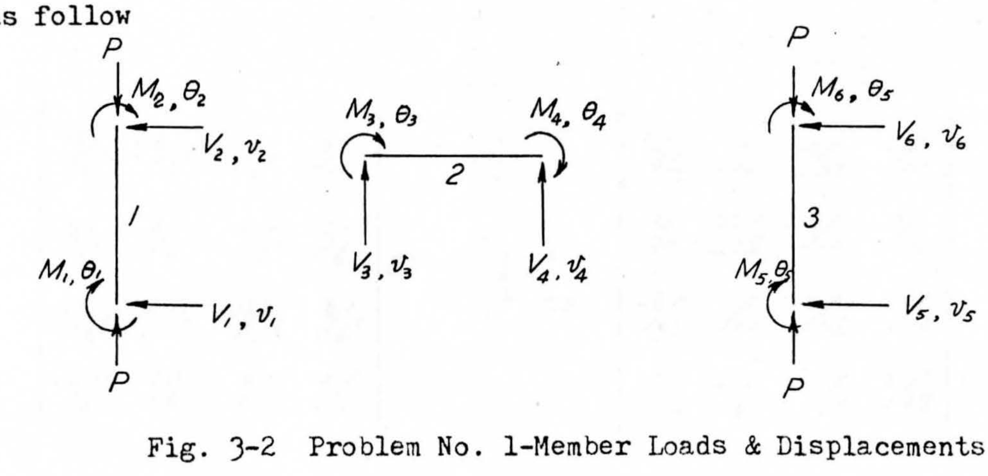


Fig. 3-2 Problem No. 1-Member Loads & Displacements

The displacement matrix $\{\Delta\}$ is written as follows:

$$\{\Delta\} = \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix}$$

Applying the boundary conditions:

$$\theta_1 = 0 \qquad \theta_2 = \theta_{BC} \qquad \theta_3 = \theta_{BC}$$

$$\theta_4 = \theta_{CB} \qquad \theta_5 = 0 \qquad \theta_6 = \theta_{CB}$$

$$v_2 = v_{BC} \qquad v_6 = v_{BC}$$

one obtains:

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ v_5 \\ \theta_5 \\ v_6 \\ \theta_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix} \qquad (3\text{-}1)$$

where $[r]$ is of order (10×3).

The stiffness matrix $[\tilde{K}]$ for the beam-column problem includes elastic bending stiffness matrix $[\hat{K}]_E$, and the geometric stiffness matrix $[\hat{K}]_G$, that is, $[\hat{K}] = [\hat{K}]_E - P[\hat{K}]_G$. For members 1 and 3 , these are defined as:

$$[\hat{K}]_E = \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} EI \qquad [\hat{K}]_G = \begin{bmatrix} 6/5L & -1/10 & -6/5L & -1/10 \\ -1/10 & 2L/15 & 1/10 & -L/30 \\ -6/5L & 1/10 & 6/5L & 1/10 \\ -1/10 & -L/30 & 1/10 & 2L/15 \end{bmatrix}$$

For member 2, there is no axial force applied, there is no transverse displacements V_3 and V_4, hence the bending stiffness matrix becomes

$$[\hat{K}]_E = \begin{bmatrix} 4/L & 2/L \\ 2/L & 4/L \end{bmatrix} EI$$

where $[\hat{K}]_G = [0]$

The global stiffness matrix of the frame is:

$$[\bar{K}] = \begin{bmatrix}
(\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{-12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & (\frac{4EI}{L}-\frac{2PL}{15}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & (\frac{4EI}{L}-\frac{2PL}{15}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \frac{4EI}{L} & \frac{2EI}{L} & 0 & 0 & 0 & 0 \\
 & & & & & \frac{4EI}{L} & 0 & 0 & 0 & 0 \\
 & & & & & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{-12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) \\
 & & & & & & & (\frac{4EI}{L}-\frac{2PL}{15}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) \\
 & \text{Symmetric} & & & & & & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) \\
 & & & & & & & & & (\frac{4EI}{L}-\frac{2PL}{15})
\end{bmatrix} \tag{3-2}$$

The structural stiffness matrix becomes

$$[r]^T[\bar{K}][r] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} [\bar{K}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix}
(\frac{-12EI}{L^3}+\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & 0 & 0 & (\frac{-12EI}{L^3}+\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) \\
(\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{4EI}{L}-\frac{2PL}{15}) & \frac{4EI}{L} & \frac{2EI}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2EI}{L} & \frac{4EI}{L} & (\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{4EI}{L}-\frac{2PL}{15})
\end{bmatrix} [r]$$

$$= \begin{bmatrix}
(\frac{24EI}{L^3}-\frac{12P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{6EI}{L^2}-\frac{P}{10}) \\
(\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{8EI}{L}-\frac{2PL}{15}) & (\frac{2EI}{L}) \\
(\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}) & (\frac{8EI}{L}-\frac{2PL}{15})
\end{bmatrix}$$

For the critical buckling load condition, the node displacements are arbitrary. Hence the following determinant equations hold:

$$\left| [r]^T [K] [r] \right| = 0 \tag{3-3a}$$

or

$$\hat{P}^3 - \frac{383}{3}\hat{P}^2 + 4280\hat{P} - 25200 = 0 \tag{3-3b}$$

where

$$\hat{P} = \frac{PL^2}{EI}$$

Equation (3-3b) is factored into the following product form:

$$(\hat{P} - 45)(\hat{P}^2 - \frac{248}{3}\hat{P} + 560) = 0 \tag{3-3c}$$

The three roots of Equation (3-3c) become:

$$\hat{P}_1 = 7.4$$
$$\hat{P}_2 = 45.0$$
$$\hat{P}_3 = 75.2$$

The lowest critical buckling load corresponds to a sidesway mode pattern, shown in Fig. 3-3:

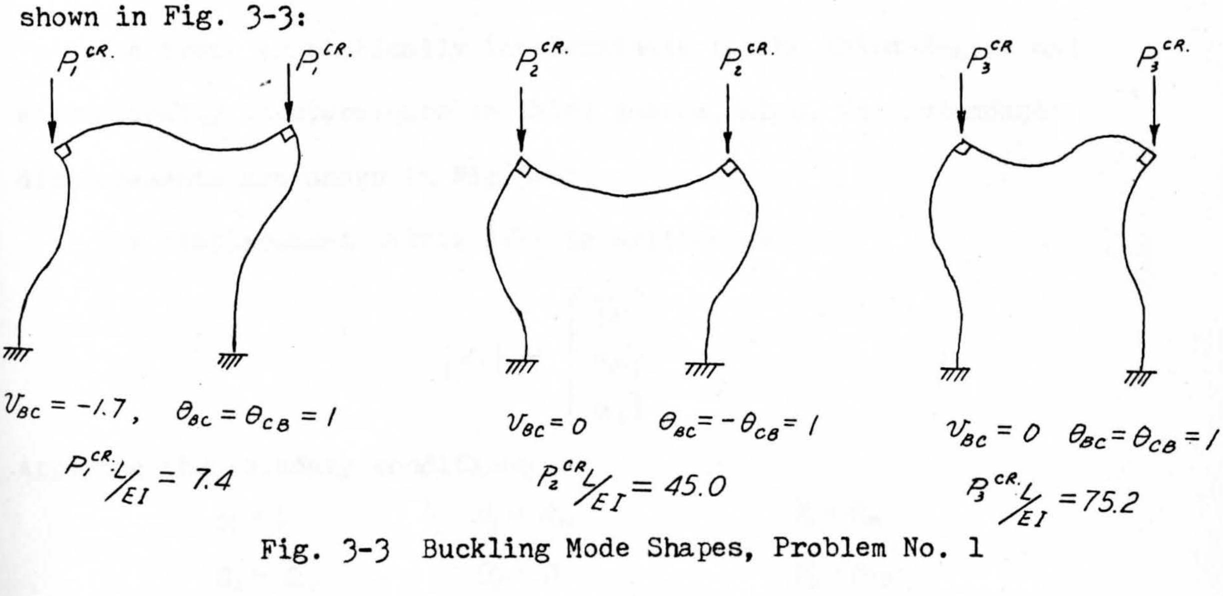


Fig. 3-3 Buckling Mode Shapes, Problem No. 1

3.2 Problem No. 2 Axial-Loaded Columns and Beam, Node Load

The load analysis of the rigid frame shown in Fig. 3-4 is performed for each member having the common parameters L,E, and I as constant.

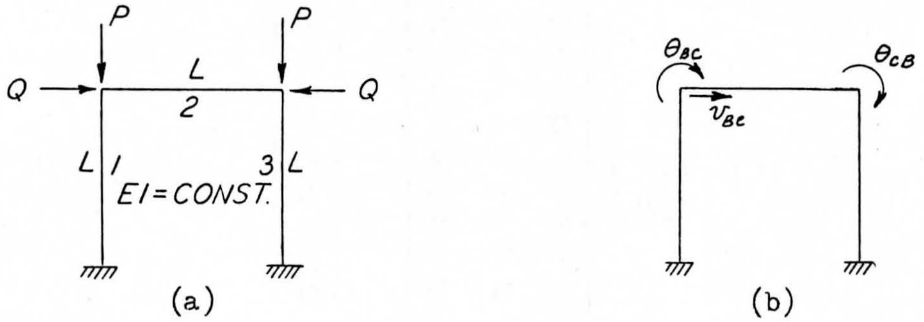


Fig. 3-4 Problem No. 2- Frame Loads & Node Displacements

The positive nodal deformations and forces are defined in Fig. 3-5 as follows:

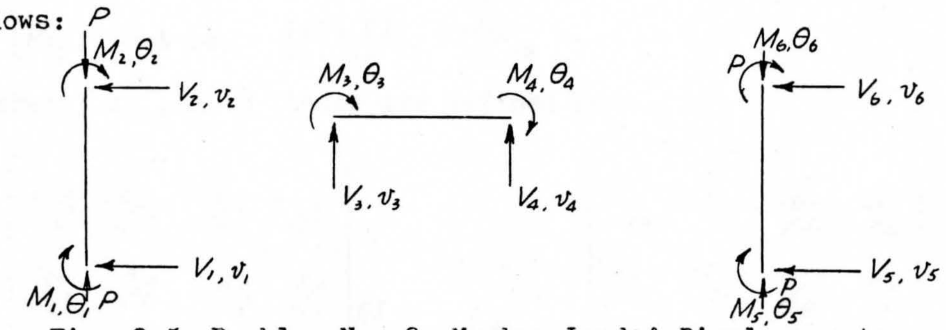


Fig. 3-5 Problem No. 2- Member Loads & Displacements

The frame is statically indeterminate to the third degree and kinematically indeterminate to third degree, also. The redundant displacements are shown in Fig 3-4b.

The displacement matrix $\{\Delta\}$ is written as

$$\{\Delta\} = \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix}$$

Applying the boundary conditions

$$\theta_1 = 0 \qquad \theta_2 = \theta_{BC} \qquad \theta_3 = \theta_{BC}$$

$$\theta_4 = \theta_{CB} \qquad \theta_5 = 0 \qquad \theta_6 = \theta_{CB}$$

$$v_2 = v_{BC} \qquad v_3 = v_{BC}$$

one obtains,

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ v_5 \\ \theta_5 \\ v_6 \\ \theta_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix} \tag{3-4}$$

The stiffness matrix $\left[\tilde{K}\right]$ for the beam-column problem includes elastic bending stiffness matrix $\left[\tilde{K}\right]_E$ and the geometric stiffness matrix $\left[\tilde{K}\right]_G$, that is, $\left[\tilde{K}\right] = \left[\tilde{K}\right]_E - P\left[\tilde{K}\right]_G$

For members 1 , & 3 these are defined as

$$\left[K\right]_E = \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} EI \qquad \left[K\right]_G = \begin{bmatrix} 6/5L & -1/10 & -6/5L & -1/10 \\ -1/10 & 2L/15 & 1/10 & -L/30 \\ -6/5L & 1/10 & 6/5L & 1/10 \\ -1/10 & -L/30 & 1/10 & 2L/15 \end{bmatrix}$$

For member 2 , the two matrices reduce to the form

$$\left[\tilde{K}\right]_E = \begin{bmatrix} 4/L & 2/L \\ 2/L & 4/L \end{bmatrix} EI \qquad \left[\tilde{K}\right]_G = \begin{bmatrix} 2L/15 & -L/30 \\ -L/30 & 2L/15 \end{bmatrix}$$

with the axial force on member BC set equal to the scalar parameter Q.

The global stiffness matrix of the frame is:

$$
[\tilde{k}] = \begin{bmatrix}
(\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{-12EI}{L^3}+\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & (\frac{4EI}{L}-\frac{2PL}{15}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & (\frac{4EI}{L}-\frac{2PL}{15}) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & (\frac{4EI}{L}-\frac{2QL}{15}) & \frac{2EI}{L}+\frac{QL}{30} & 0 & 0 & 0 & 0 \\
 & & & & & (\frac{4EI}{L}-\frac{2QL}{15}) & 0 & 0 & 0 & 0 \\
 & & & & & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) & (\frac{-12EI}{L^3}+\frac{6P}{5L}) & (\frac{-6EI}{L^2}+\frac{P}{10}) \\
 & \text{Symmetric} & & & & & & (\frac{4EI}{L}-\frac{2PL}{15}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}+\frac{PL}{30}) \\
 & & & & & & & & (\frac{12EI}{L^3}-\frac{6P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) \\
 & & & & & & & & & (\frac{4EI}{L}-\frac{2PL}{15})
\end{bmatrix} \tag{3-5}
$$

The structural stiffness matrix becomes:

$$
[r]^T[\tilde{k}][r] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} [\tilde{k}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

or

$$
[r]^T[\tilde{k}][r] = \begin{bmatrix}
(\frac{24EI}{L^3}-\frac{12P}{5L}) & (\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{6EI}{L^2}-\frac{P}{10}) \\
(\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{8EI}{L}-\frac{2L(Q+P)}{15}) & (\frac{2EI}{L}+\frac{QL}{30}) \\
(\frac{6EI}{L^2}-\frac{P}{10}) & (\frac{2EI}{L}+\frac{QL}{30}) & (\frac{8EI}{L}-\frac{2L(Q+P)}{15})
\end{bmatrix}
$$

For arbitrary solutions of the node displacements it follows that

$$
\left| [r]^T[k][r] \right| = 0
$$

Hence, we have

$$-\frac{1}{1500}(60\hat{P}^3+123\hat{P}^2\hat{Q}+60\hat{P}\hat{Q}^2)+\frac{1}{75}(383\hat{P}^2+442\hat{P}\hat{Q}+30\hat{Q}^2)$$
$$-\frac{1}{5}(856\hat{P}+212\hat{Q})+1008=0 \qquad (3\text{-}6)$$

where $\hat{P}=PL^2/EI$, $\hat{Q}=QL^2/EI$

When the horizontal force $Q=0$, we get the same equation as Equation 3-3b, where $P_1=7.4$, $P_2=45.0$, and $P_3=75.2$, and when the vertical force $P=0$, we have:

$$\frac{6}{15}\hat{Q}^2-\frac{212}{5}\hat{Q}+1008=0$$

then

$$\hat{Q}_1=36.3 \quad , \quad \hat{Q}_2=69.7$$

The relation of $\hat{P}$ and $\hat{Q}$ shows in Fig. 3-6 as follows:

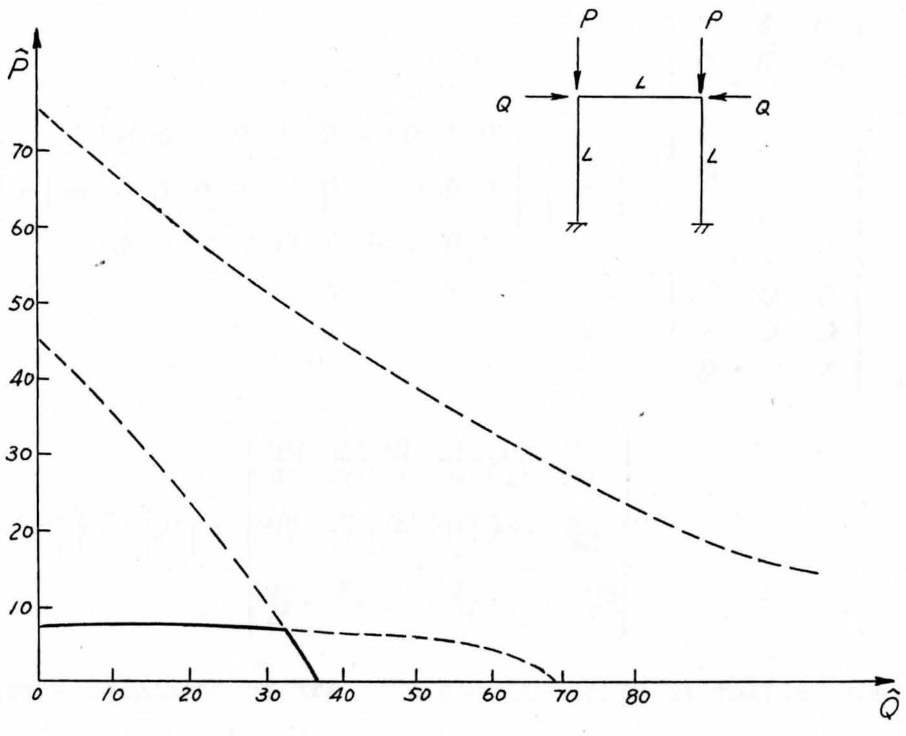


Fig. 3-6 Axial Load Analysis,
$\hat{P}$ & $\hat{Q}$ Relationship for Problem No. 2

The presence of the additional horizontal axial force Q reduces the critical buckling load below the classical value given in Problem No. 1. In addition, Fig. 3-6 also shows that both nonsymmetric sidesway mode as well as the first symmetric nonsidesway mode control the reduced value of the critical buckling load. More details about the relationship between critical buckling load P and Q are given in Appendix 1.

3.3 Problem No. 3 Member-Loaded Frame, Axial Loads Excluded

The load analysis of the rigid frame shown in Fig. 3-6 is performed for each member having the common parameters L,E, and I constant.

Fig. 3-7 Problem No. 3 - Frame Load & Node Displacements

The positive nodal deformations and forces are defined in Fig. 3-8.

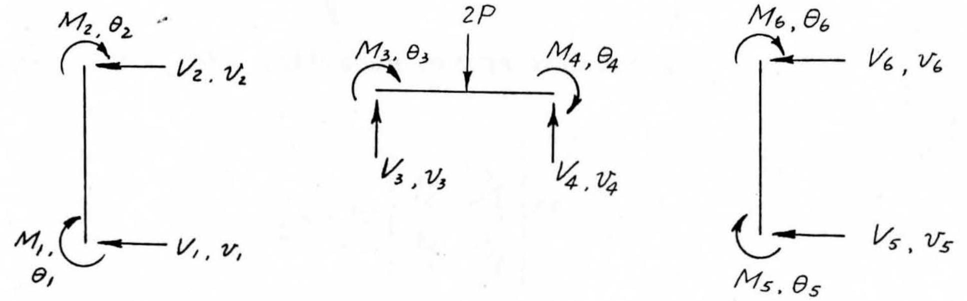


Fig. 3-8 Problem No. 3 - Member Loads & Displacements

Fixed end moments on member 2 are:

$$M_3 = \frac{PL}{4} \qquad M_4 = \frac{-PL}{4}$$

The displacement matrix $\{\Delta\}$ is written as follows:

$$\{\Delta\} = \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix}$$

Applying boundary conditions:

$$\begin{array}{ll} \theta_1 = 0 & , \quad v_1 = 0 \\ \theta_2 = \theta_{BC} & , \quad v_2 = v_{BC} \\ \theta_3 = \theta_{BC} & , \quad v_3 = 0 \\ \theta_4 = \theta_{CB} & , \quad v_4 = 0 \\ \theta_5 = 0 & , \quad v_5 = 0 \\ \theta_6 = \theta_{CB} & , \quad v_6 = v_{BC} \end{array} \qquad \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ v_5 \\ \theta_5 \\ v_6 \\ \theta_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix} \qquad (3\text{-}7)$$

In the bending beam theory, we only consider the bending stiffness matrix $[\hat{K}]_E$. For member 1 and 3 , the stiffness matrix is defined as follows:

$$[\hat{K}]_E = \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ & 4/L & 6/L^2 & 2/L \\ & & 12/L^3 & 6/L^2 \\ & \text{Symmetric} & & 4/L \end{bmatrix} EI$$

For member 2 , the stiffness matrix reduces to:

$$[\hat{K}]_E = \begin{bmatrix} 4/L & 2/L \\ 2/L & 4/L \end{bmatrix} EI$$

The global stiffness matrix of the frame is:

$$[\hat{k}] = \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 4/L & 6/L^2 & 2/L & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 12/L^3 & 6/L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 4/L & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 4/L & 2/L & 0 & 0 & 0 & 0 \\ & & & & & 4/L & 0 & 0 & 0 & 0 \\ & & & & & & 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ & & & & & & & 4/L & 6/L^2 & 2/L \\ & & & & & & & & 12/L^3 & 6/L^2 \\ & \text{Symmetric} & & & & & & & & 4/L \end{bmatrix} EI \qquad (3\text{-}8)$$

The structural stiffness matrix becomes:

$$[r]^T[\hat{k}][r] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} [\hat{k}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 24/L^3 & 6/L^2 & 6/L^2 \\ 6/L^2 & 8/L & 2/L \\ 6/L^2 & 2/L & 8/L \end{bmatrix}$$

Then

$$\left[[r]^T[\hat{k}][r]\right]^{-1} = \begin{bmatrix} 1/16.8 & -1/28L & -1/28L \\ -1/28L & 13/84L^2 & -1/84L^2 \\ -1/28L & -1/84L^2 & 13/84L^2 \end{bmatrix} \frac{L^3}{EI}$$

The nodal forces matrix is:

$$\{Q\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ PL/4 \\ -PL/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ PL/4 \\ -PL/4 \end{Bmatrix}$$

The nodal displacements $\{\Delta\}$ are computed as:

$$\{\Delta\} = -\left[[r]^T[\hat{k}][r]\right]^{-1}\{Q\} = \begin{Bmatrix} 0 \\ -\frac{PL^2}{24EI} \\ \frac{PL^2}{24EI} \end{Bmatrix}$$

The values of the member forces $\{P\}$ are obtained by Equation (2-23)

$$\{P\} = \{P_o\} + [\hat{k}]\,[r]\,\{\Delta\}$$

$$\{P\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{PL}{4} \\ -\frac{PL}{4} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \left[\hat{k}\right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{PL^2}{24} \\ \frac{PL^2}{24} \end{Bmatrix} \frac{-1}{EI}$$

$$\{P\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{PL}{4} \\ -\frac{PL}{4} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -\frac{P}{4} \\ \frac{PL}{12} \\ \frac{P}{4} \\ \frac{PL}{6} \\ \frac{PL}{12} \\ -\frac{PL}{12} \\ \frac{P}{4} \\ -\frac{PL}{12} \\ -\frac{P}{4} \\ -\frac{PL}{6} \end{Bmatrix} = \begin{Bmatrix} \frac{P}{4} \\ -\frac{PL}{12} \\ -\frac{P}{4} \\ -\frac{PL}{6} \\ \frac{PL}{6} \\ -\frac{PL}{6} \\ -\frac{P}{4} \\ \frac{PL}{12} \\ \frac{P}{4} \\ \frac{PL}{6} \end{Bmatrix}$$

The forces on each member are shown below:

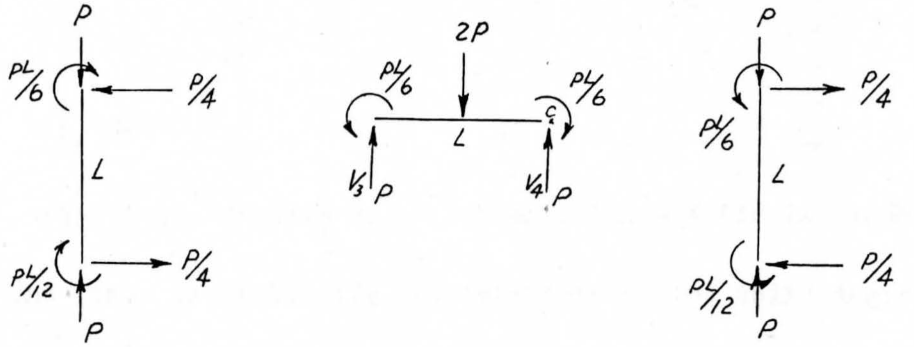


Fig. 3-9 Calculated Member Forces for Problem No. 3

$$\Sigma M_c = 0$$

$$-PL/6 + PL/6 - 2P \cdot L/2 + V_3 = 0$$

$$V_3 = P \qquad V_4 = P$$

The final free body diagram of the frame is shown in Fig. 3-9:

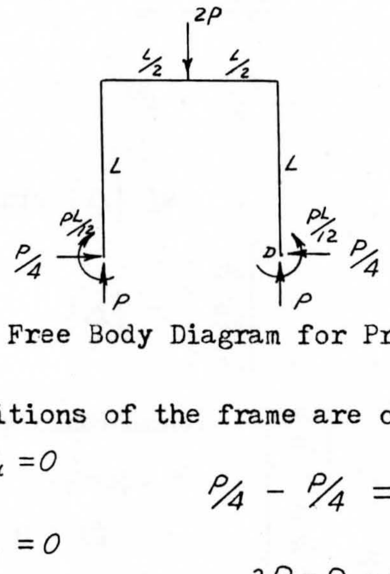


Fig. 3-10 Free Body Diagram for Problem No. 3

The equilibrium conditions of the frame are checked as follows:

(1) $\Sigma F_x = 0$

$$P/4 - P/4 = 0$$

(2) $\Sigma F_y = 0$

$$2P - P - P = 0$$

(3) $\Sigma M_D = 0$

$$PL/12 + PL - 2P\ L/2 - PL/12 = 0$$

O.K.

3.4 Problem No. 4 Member-Loaded Frame, Axial Loads Included

The rigid frame shown in Fig. 3-11 (a) is loaded on member 2 only. The flexural properties L, E, and I are the same for all members.

Fig. 3-11 Problem No. 4 - Member Loads & Displacements

The frame is statically indetermiante to the third degree and kinematically indeterminate to the third degree. The nodal displacemants are shown in Fig. 3-11 (b).

The positive nodal deformations and forces are the same as Problem No. 1.

The fixed-end moments on member 2 are calculated as follows:

$$M_3 = \frac{PL}{4}$$

$$M_4 = \frac{-PL}{4}$$

The displacement matrix $\{\Delta\}$ is

$$\{\Delta\} = \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix}$$

Applying the boundary conditions:

$$\theta_1 = \theta_5 = 0 \qquad v_1 = v_5 = 0$$
$$\theta_2 = \theta_{BC} \qquad v_2 = v_{BC}$$
$$\theta_3 = \theta_{BC} \qquad v_3 = 0$$
$$\theta_4 = \theta_{CB} \qquad v_4 = 0$$
$$\theta_6 = \theta_{CB} \qquad v_6 = v_{BC}$$

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ v_5 \\ \theta_5 \\ v_6 \\ \theta_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix}$$

The stiffness matrix $[\hat{K}]$ for the beam-column problem includes elastic bending stiffness matrix $[\hat{K}]_E$ and the geometric stiffness matrix $[\hat{K}]_G$, which is $[\hat{K}] = [\hat{K}]_E - P[\hat{K}]_G$

For member 1 and 3 , these are defined as:

$$[\hat{K}]_E = \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} EI \qquad [\hat{K}]_G = \begin{bmatrix} 6/5L & -1/10 & -6/5L & -1/10 \\ -1/10 & 2L/15 & 1/10 & -L/30 \\ -6/5L & 1/10 & 6/5L & 1/10 \\ -1/10 & -L/30 & 1/10 & 2L/15 \end{bmatrix}$$

For member 2 , only the bending stiffness matrix $[\hat{K}]_E$ is present due to the condition of no axial force, hence

$$[\hat{K}]_E = \begin{bmatrix} 4/L & 2/L \\ 2/L & 4/L \end{bmatrix} EI$$

The global stiffness matrix of the frame is:

$$[\hat{K}] = \left[\begin{array}{c:c:c} [\hat{K}]_1 & [0] & [0] \\ \hdashline & [\hat{K}]_2 & [0] \\ \hdashline \multicolumn{2}{c:}{\text{Symmetric}} & [\hat{K}]_3 \end{array}\right]$$

The structural stiffness matrix becomes:

$$[r]^T[\hat{K}][r] = EI \begin{bmatrix} 24/L^3 & 6/L^2 & 6/L^2 \\ 6/L^2 & 8/L & 2/L \\ 6/L^2 & 2/L & 8/L \end{bmatrix}$$

The nodal forces matrix is obtained as follows:

$$\{Q\} = [r]^T \{P_o\} = \begin{Bmatrix} 0 \\ PL/4 \\ -PL/4 \end{Bmatrix}$$

$$\{P\} = \begin{Bmatrix} V_1 \\ M_1 \\ \hdashline V_2 \\ M_2 \\ \hdashline M_3 \\ M_4 \\ \hdashline V_5 \\ M_5 \\ \hdashline V_6 \\ M_6 \end{Bmatrix} = \begin{Bmatrix} 15P(\hat{P}-60)/80B \\ PL(\hat{P}+60)/16B \\ \hdashline -15P(\hat{P}-60)/80B \\ -PL(\hat{P}-30)/4B \\ \hdashline P(L/4 + 15L/4B) \\ -P(L/4 + 15L/4B) \\ \hdashline -15P(\hat{P}-60)/80B \\ -PL(\hat{P}+60)/16B \\ \hdashline 15P(\hat{P}-60)/80B \\ PL(\hat{P}-30)/4B \end{Bmatrix}$$

where

$$B = \hat{P} - 45$$

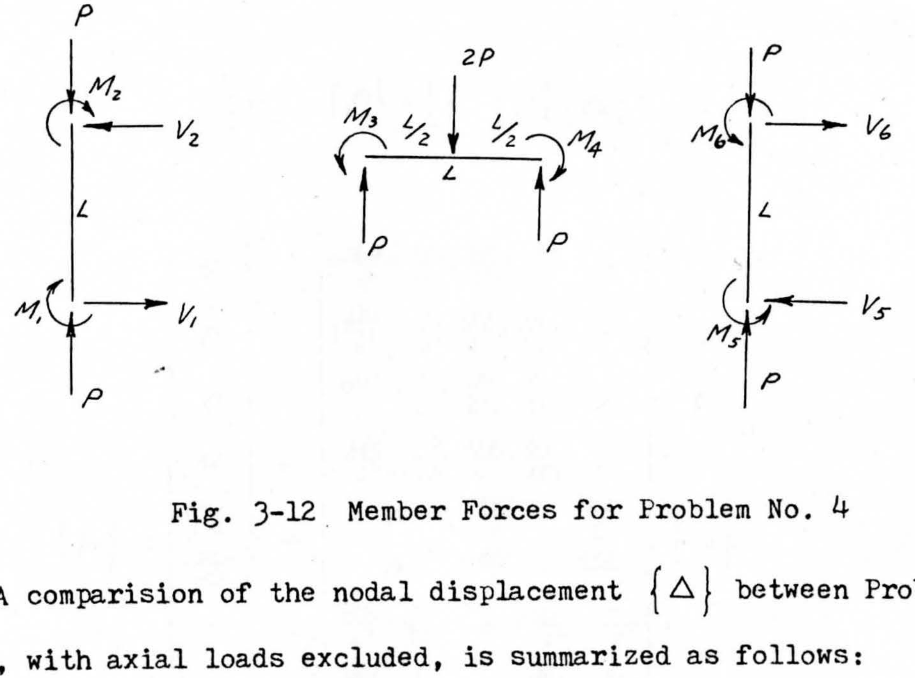


Fig. 3-12 Member Forces for Problem No. 4

A comparision of the nodal displacement $\{\Delta\}$ between Problem No. 3, with axial loads excluded, is summarized as follows:

$$\{\Delta\}_{pro.\#3} = \begin{Bmatrix} v_{BC} \\ \theta_{BC} \\ \theta_{CB} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\hat{P}/24 \\ \hat{P}/24 \end{Bmatrix}$$

The nodal displacements $\{\Delta\}$ take the form

$$\{\Delta\} = -\Big[[r]^T[\hat{k}][r]\Big]^{-1}\{Q\}$$

$$\{\Delta\} = \frac{25}{(\hat{P}-45)(\hat{P}^2 - \frac{248}{3}\hat{P} + 560)} \begin{bmatrix} 0 \\ 42\hat{P} - 62\hat{P}^2 + 3\hat{P}^3/40 \\ -42\hat{P} + 62\hat{P}^2 - 3\hat{P}^3/40 \end{bmatrix}$$

where

$$\hat{P} = PL^2/EI$$

or

$$\{\Delta\} = \begin{Bmatrix} 0 \\ \dfrac{15\hat{P}}{8(\hat{P}-45)} \\ \dfrac{-15\hat{P}}{8(\hat{P}-45)} \end{Bmatrix}$$

The member-ended forces are:

$$\{P\} = \{P_o\} + [\hat{k}][r]\{\Delta\}$$

$$\{P\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ PL/4 \\ -PL/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} (-\frac{12EI}{L^3} + \frac{6P}{L}) & (\frac{-6EI}{L^2} + \frac{P}{10}) & 0 \\ (\frac{6EI}{L^2} - \frac{P}{10}) & (\frac{2EI}{L} + \frac{PL}{30}) & 0 \\ (\frac{12EI}{L^3} - \frac{6P}{5L}) & (\frac{6EI}{L^2} - \frac{P}{10}) & 0 \\ (\frac{6EI}{L^2} - \frac{P}{10}) & (\frac{4EI}{L} - \frac{2PL}{15}) & 0 \\ 0 & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & \frac{2EI}{L} & \frac{4EI}{L} \\ (\frac{-12EI}{L^3} + \frac{6P}{5L}) & 0 & (\frac{-6EI}{L^2} + \frac{P}{10}) \\ (\frac{6EI}{L^2} - \frac{P}{10}) & 0 & (\frac{2EI}{L} + \frac{PL}{30}) \\ (\frac{12EI}{L^3} - \frac{6P}{5L}) & 0 & (\frac{6EI}{L^2} - \frac{P}{10}) \\ (\frac{6EI}{L^2} - \frac{P}{10}) & 0 & (\frac{4EI}{L} - \frac{2PL}{15}) \end{bmatrix} \{\Delta\}$$

and

$$\{\Delta\}_{Pro.\#4} = \begin{Bmatrix} 0 \\ \frac{-\hat{P}}{24}\left(\frac{1}{1-\hat{P}/45}\right) \\ \frac{-\hat{P}}{24}\left(\frac{1}{1-\hat{P}/45}\right) \end{Bmatrix}$$

The results are plotted for the parameter θ_{CB} in Fig. 3-13.

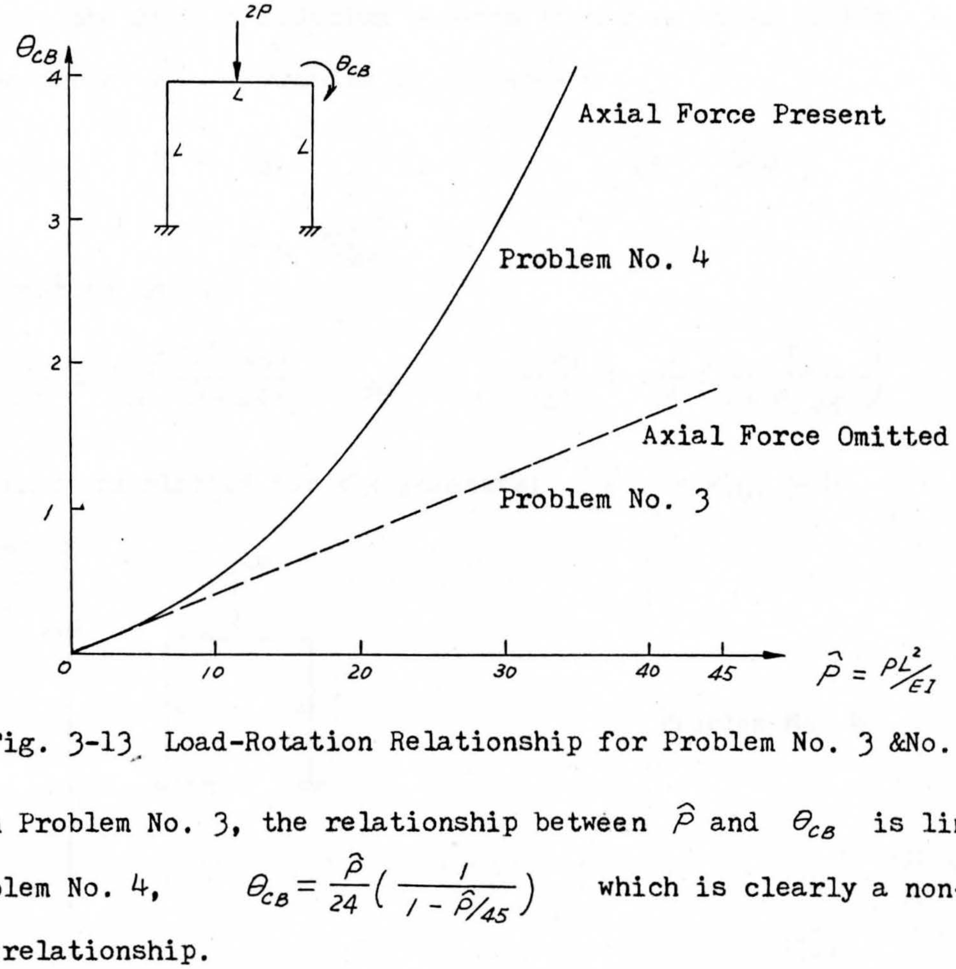


Fig. 3-13 Load-Rotation Relationship for Problem No. 3 &No. 4

In Problem No. 3, the relationship between $\hat{P}$ and θ_{CB} is linear. In problem No. 4, $\theta_{CB} = \frac{\hat{P}}{24}\left(\frac{1}{1-\hat{P}/45}\right)$ which is clearly a non-linear relationship.

In addition,

$$\frac{d\,\theta_{CB}}{d\,\hat{P}} = \frac{(40\hat{P}-1800)\cdot(75) - 7\hat{P}\cdot(40)}{(40\hat{P}-1800)^2}$$

and for $\hat{P} = 0$

$$\frac{d\theta_{CB}}{d\hat{P}} = \frac{-1}{24}$$

Thus, the tangent of the curve of Problem No. 4 is the equation of the line of problem No. 3, which means the addition axial loads produces a nonlinear solution to the load-rotation analysis. If the axial load is neglected, the problem solution becomes linear as shown in Fig. 3-13.

The force V_1 in Problem No. 3 becomes

$$V_1 = P/4 \qquad \text{or} \qquad V_1 EI/L^2 = \hat{P}/4$$

where

$$\hat{P} = PL^2/EI$$

and in Problem No. 4

$$\frac{V_1 EI}{L^2} = \frac{15P(\hat{P}-60)}{80(\hat{P}-45)} \qquad \text{or} \qquad \frac{V_1 EI}{L^2} = \frac{\hat{P}}{4}\left(\frac{1-\hat{P}/60}{1-\hat{P}/45}\right)$$

The results are plotted for the parameter $\frac{V_1 EI}{L^2}$ in Fig. 3-14.

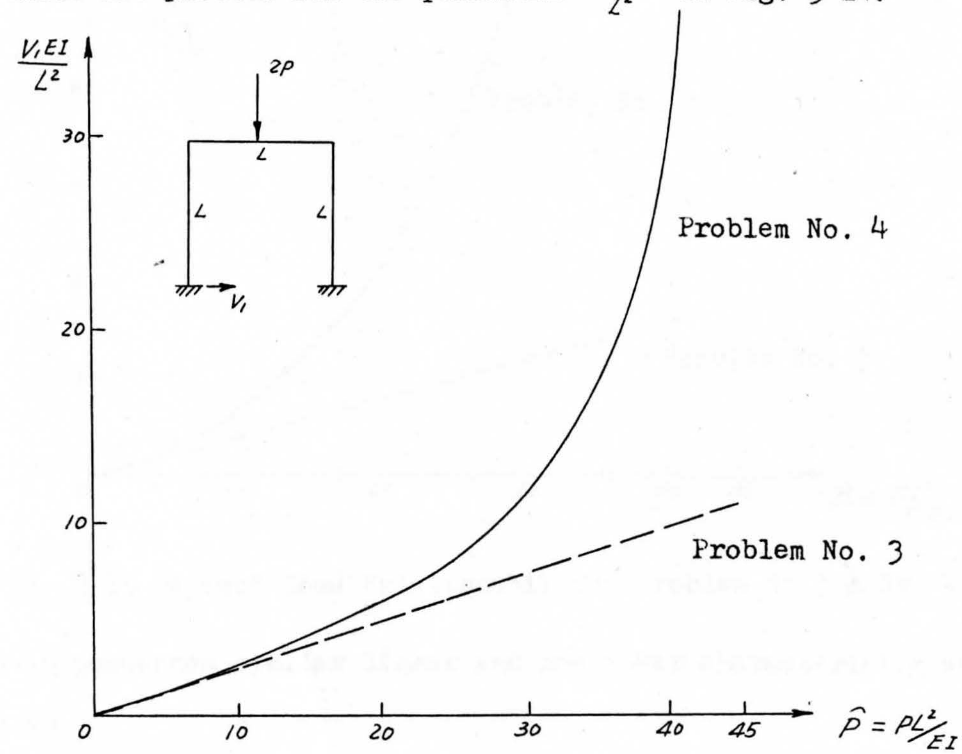


Fig. 3-14 Shear Force-Load Relationship for Problems No. 3 & No.4

Fig. 3-14 possesses similar linear and nonlinear characteristics as Fig. 3-13.

The moment M_1 on the Problem No.3 obtains:

$$M_1 = -PL/12 \qquad \text{or} \qquad M_1 L/EI = -\hat{P}/12$$

where
$$\hat{P} = PL^2/EI$$
and in Problem No. 4

$$M_1 = \frac{PL(\hat{P}+60)}{16(\hat{P}-45)} \qquad \text{or} \qquad \frac{M_1 L}{EI} = \frac{-\hat{P}}{12}\left(\frac{1+\hat{P}/60}{1-\hat{P}/45}\right)$$

Fig. 3-15 shows the $\frac{-M_1 L}{EI} - \hat{P}$ relationship:

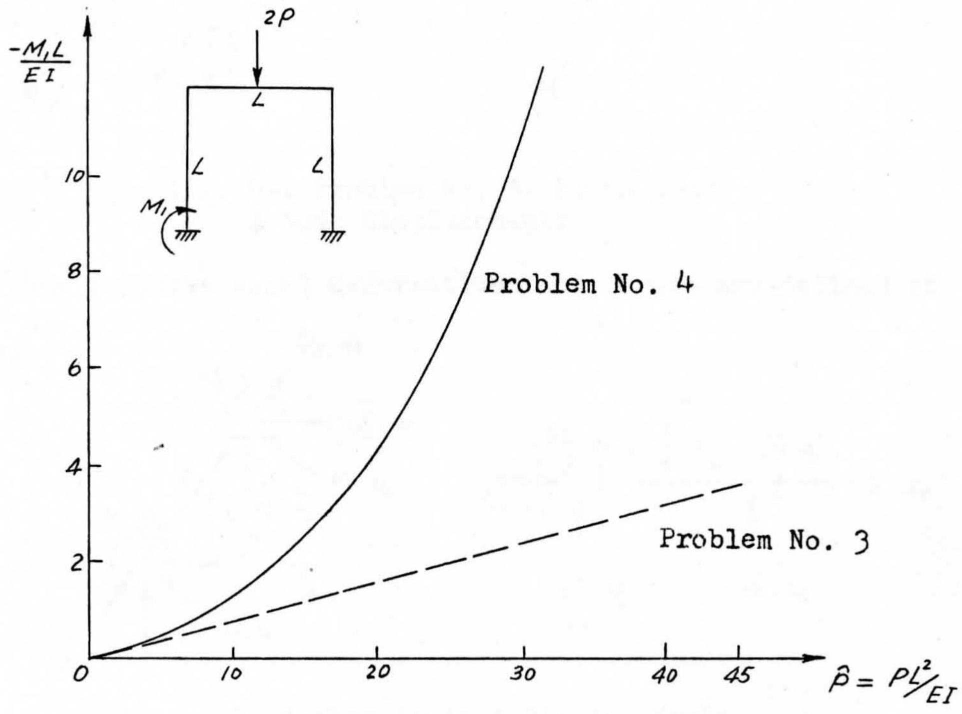


Fig. 3-15 Moment-Load Relationship for Problem No.3 & No. 4

Fig. 3-15 possesses similar linear and nonlinear characteristic as Fig. 3-13.

CHAPTER IV

ANALYSIS OF NONORTHOGONAL FRAMES WITH HORIZONTAL MEMBERS

4.1 Problem No. 5: Nonorthogonal Member-Loaded Frame, Axial Deformation Included

The rigid frame is shown in Fig. 4-1 has the same EI and L on both members.

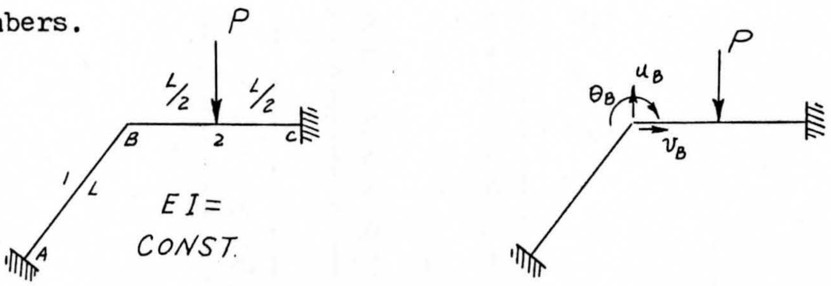


Fig. 4-1 Problem No. 5· Frame Load & Node Displacements

The positive nodal deformations and forces are defined as follows:

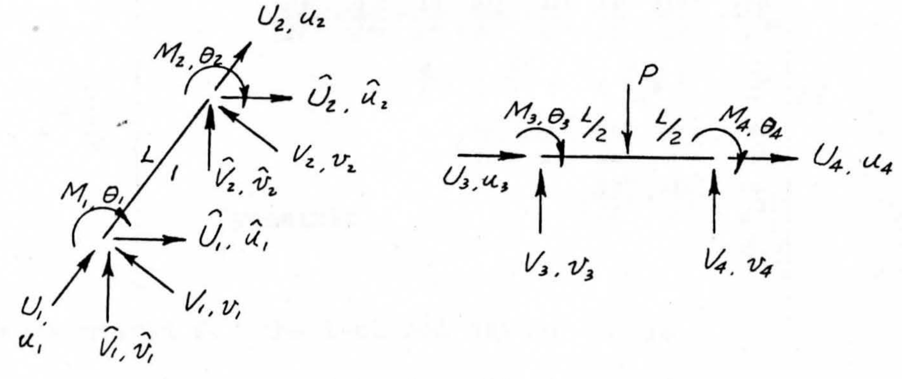


Fig. 4-2 Member Loads & Displacements

Considering the axial effect on both members, the displacement matrix $\{\Delta\}$ is written as follows:

$$\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

The boundary conditions are specified as

$$u_1 = v_1 = \theta_1 = 0 \qquad u_4 = v_4 = \theta_4 = 0$$

$$u_2 = u_3 = u_B \qquad v_2 = v_3 = v_B$$

$$\theta_2 = \theta_B \qquad \theta_3 = \theta_B$$

which take the matrix form

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \theta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

For the horizontal member, considering the axial effect, the stiffness matrix for the member is:

$$[\hat{K}]_E = \begin{bmatrix} \left(\frac{Ac^2}{IL} + \frac{12s^2}{L^3}\right) & \left(\frac{Acs}{IL} - \frac{12cs}{L^3}\right) & \left(\frac{6s}{L^2}\right) & \left(\frac{-Ac^2}{IL} - \frac{12s^2}{L^3}\right) & \left(\frac{-Acs}{IL} + \frac{12cs}{L^3}\right) & \left(\frac{6s}{L^2}\right) \\ & \left(\frac{As^2}{IL} - \frac{12c^2}{L^3}\right) & \left(\frac{-6c}{L^2}\right) & \left(\frac{-Acs}{IL} + \frac{12cs}{L^3}\right) & \left(\frac{-As^2}{IL} - \frac{12c^2}{L^3}\right) & \left(\frac{-6c}{L^2}\right) \\ & & \left(\frac{4}{L}\right) & \left(\frac{-6s}{L^2}\right) & \left(\frac{6c}{L^2}\right) & \left(\frac{2}{L}\right) \\ & & & \left(\frac{Ac^2}{IL} + \frac{12s^2}{L^3}\right) & \left(\frac{Acs}{IL} - \frac{12cs}{L^3}\right) & \left(\frac{-6s}{L^2}\right) \\ & \text{Symmetric} & & & \left(\frac{As^2}{IL} + \frac{12c^2}{L^3}\right) & \left(\frac{6c}{L^2}\right) \\ & & & & & \left(\frac{4}{L}\right) \end{bmatrix} EI$$

The stiffness matrix for the inclined member 2 is

$$[\hat{K}]_E = \begin{bmatrix} \frac{A}{IL} & 0 & 0 & \frac{-A}{IL} & 0 & 0 \\ & \frac{12}{L^3} & \frac{-6}{L^2} & 0 & \frac{-12}{L^3} & \frac{-6}{L^2} \\ & & \frac{4}{L} & 0 & \frac{6}{L^2} & \frac{2}{L} \\ & & & \frac{A}{IL} & 0 & 0 \\ & \text{Symmetric} & & & \frac{12}{L^3} & \frac{6}{L^2} \\ & & & & & \frac{4}{L} \end{bmatrix} EI$$

Combining the latter two matrices together, the global stiffness matrix of the frame becomes

$$|\hat{K}| = \begin{bmatrix} [\hat{K}]_1 , & [0] \\ [0] , & [\hat{K}]_2 \end{bmatrix}$$

The structural stiffness matrix in turn is obtained as

$$[r]^T[\hat{k}][r] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{K} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$[r]^T[\hat{k}][r] = \frac{EI}{L^3} \begin{bmatrix} \phi(c^2+1)+12s^2 , & cs(\phi-12) & -6sL \\ cs(\phi-12) & (\phi s^2+12c^2+12) & 6L(c-1) \\ -6sL & 6L(c-1) & 8L^2 \end{bmatrix}$$

The inverse of above equation is:

$$\left[[r]^T[K][r]\right]^{-1} = \frac{(EI)^2/L^4}{D} \begin{bmatrix} [8\phi s^2+12(5c^2+6c+5)] , & [-8cs\phi+36s+60cs] , & [6s(1-c)\phi+72s(c+1)]/L \\ & [8(c^2+1)\phi+60s^2] , & \left[\frac{6(c-1)^2\phi+72s^2}{L}\right] \\ \text{Symmetric} & & \left[\frac{24\phi(c^2+1)+144s^2+\phi^2 s^2}{L^2}\right] \end{bmatrix}$$

where

$$D = \frac{(EI)^3}{L^7}\left[\phi\, 24(5c^2+6c+5) + 8\phi^2 s^2 + 288s^2\right]$$

The fixed-end moments and shears on the member 2 are:

$$\begin{Bmatrix} M_3 \\ M_4 \end{Bmatrix} = \begin{Bmatrix} \frac{PL}{8} \\ \frac{-PL}{8} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} \frac{-P}{2} \\ \frac{-P}{2} \end{Bmatrix}$$

The nodal forces matrix becomes:

$$\{Q\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P/2 \\ PL/8 \\ 0 \\ -P/2 \\ PL/8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

The nodal displacement matrix $\{\Delta\}$ is computed as follows:

$$\{\Delta\} = \Big[[r]^T [\hat{k}] [r] \Big]^{-1} \{Q\}$$

or

$$\{\Delta\} = \frac{(EI)^2/L^4}{D} \begin{Bmatrix} \frac{5\phi(13C+3)P}{4} - \frac{3S(7C+3)P}{L^4} \\ -\frac{P\phi(13C^2+6C+13)}{4} - 21S^2P \\ \frac{6\phi CP}{L} - \frac{18PL^2}{L} + \frac{P\phi^2 S^2}{L} \end{Bmatrix}$$

The total member-end force is:

$$\{P\} = \left\{ \begin{array}{c} \dfrac{[35\phi(7c+3)P - \phi^2 s(13c+3)P \cdot \frac{1}{4}]}{B} \\ \dfrac{[-3\phi(7c^2+6c+7)P - \frac{13}{4}\phi^2 s^2 P]}{B} \\ \dfrac{[1.5\phi PL(3c^2+18c+3) - 18Ps^2 L - 0.25P\phi^2 s^2 L]}{B} \\ \dfrac{[-35\phi(7c+3)P + \phi^2 s(13c+3)P \cdot \frac{1}{4}]}{B} \\ \dfrac{[3\phi(7c^2+6c+7)P + \frac{13}{4}\phi^2 s^2 P]}{B} \\ \dfrac{[1.5\phi PL(3c^2+10c+3) + 18Ps^2 L - 0.5P\phi^2 L \cdot s^2]}{B} \\ \dfrac{[35(7c+3)\phi P - 0.25\phi^2 s(13c+3)P]}{B} \\ -\dfrac{P}{2} + [3P\phi(13c^2+18c+13) + 0.75Ps^2\phi^2 + 144s^2 P]\dfrac{1}{B} \\ \dfrac{PL}{8} + \dfrac{[3\phi PL(13c^2+22c+13)\frac{1}{2} - 54Ps^2 L - P\phi^2 s^2 L \cdot \frac{1}{2}]}{B} \\ \dfrac{[-3\phi sP(7c+3) + \phi^2 Ps(13c+3)\frac{1}{4}]}{B} \\ -\dfrac{P}{2} + \dfrac{[-3\phi P(13c^2+18c+13) - 144s^2 P - 0.75\phi^2 s^2 P]}{B} \\ -\dfrac{PL}{8} + \dfrac{[-1.5\phi P(13c^2+14c+13) - 90PsL^2 - \frac{1}{4}P\phi^2 s^2 L]}{B} \end{array} \right\}$$

where $$B = 8s^2\phi^2 + 24(5c^2+6c+5)\phi + 288s^2$$

If the effects of axial displacement are neglected, this implies that the axial stiffness coeffieient $\frac{AE}{L}$ approaches infinity. This is accomplished mathematically by requiring $\phi \rightarrow \infty$. The resulting problem solution is abtained from the results of Problem No. 5 by factoring the parameter ϕ from the numerator and denominator of each term, and taking the limit as $\phi \rightarrow \infty$ in the remaining quotient.

Preforming these operations, the nodal displacement matrix becomes

$$\{\Delta\} = \frac{\frac{(EI)^2}{L^4}}{D} \left\{ \begin{array}{l} \phi\left(\dfrac{sP(13c+3)}{4} - \dfrac{3Ps(7c+3)}{\phi}\right) \\ \phi\left(\dfrac{-P(13c^2+6c+13)}{4} - \dfrac{21s^2 P}{\phi}\right) \\ \phi^2\left(\dfrac{Ps^2}{8} - \dfrac{6Pc}{\phi} - \dfrac{18Ps^2}{\phi^2}\right) \end{array} \right\}$$

The total member-end force matrix reduces to the form

$$
\{P\} = \begin{Bmatrix}
\left[-\frac{PS(13C+3)}{4} + \frac{3SP(7C+3)}{\phi}\right]\frac{\phi^2}{B} \\
\left[\frac{-13S^2P}{4} - \frac{3\phi(7+7C^2+6C)P}{\phi^2}\right]\frac{\phi^2}{B} \\
\left[-\frac{PS^2L}{4} + \frac{3PL(18C+3+3C^2)}{2\phi} - \frac{18PS^2L}{\phi^2}\right]\frac{\phi^2}{B} \\
\left[+\frac{SP(13C+3)}{4} - \frac{3SP(7C+3)}{\phi}\right]\frac{\phi^2}{B} \\
\left[\frac{13S^2P}{4} + \frac{3(7+7C^2+6C)P}{\phi}\right]\frac{\phi^2}{B} \\
\left[-\frac{PLS^2}{2} + \frac{3PL(10C+3C^2+3)}{2\phi} + \frac{18PS^2L}{\phi^2}\right]\frac{\phi^2}{B} \\
\left[-\frac{PS(13C+3)}{4} + \frac{3S(7C+3)P}{\phi}\right]\frac{\phi^2}{B} \\
-\frac{P}{2} + \left[\frac{3PS^2}{4} + \frac{3P(13C^2+18C+13)}{\phi} + \frac{144S^2P}{\phi^2}\right]\frac{\phi^2}{B} \\
\frac{PL}{8} + \left[-\frac{PS^2L}{2} - \frac{3PL(13C^2+22C+13)}{2\phi} - \frac{54PS^2L}{\phi^2}\right]\frac{\phi^2}{B} \\
\left[\frac{PS(13C+3)}{4} - \frac{3SP(7C+3)}{\phi}\right]\frac{\phi^2}{B} \\
-\frac{P}{2} + \left[-\frac{3S^2P}{4} - \frac{3P(13C^2+18C+13)}{\phi} - \frac{144S^2P}{\phi^2}\right]\frac{\phi^2}{B} \\
\frac{-PL}{8} + \left[-\frac{PS^2L}{4} - \frac{3P(13C^2+14C+13)L}{2\phi} - \frac{90PS^2L}{\phi^2}\right]\frac{\phi^2}{B}
\end{Bmatrix}
$$

4.2 Problem No. 6: Nonorthogonal Member-Loaded Frame, No Axial Deformations

The rigid frame, the same as Problem No. 5, is shown in Fig. 4-1, with the same EI and L on both member.

The displacement matrix $\{\Delta\}$ is written as follows:

$$
\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}
$$

By the same procedure as Problem No. 5, we have the nodal force matrix as

$$\{Q\} = \left\{ \begin{matrix} 0 \\ \frac{-P}{2} \\ \frac{PL}{8} \end{matrix} \right\}$$

The nodal displacement matrix is

$$\{\Delta\} = \frac{\frac{(EI)^2}{L^4}}{D} \left\{ \begin{matrix} \phi\left(\frac{SP(13C+3)}{4} - \frac{3PS(7C+3)}{\phi}\right. \\ \phi\left(\frac{-P(13C^2+6C+13)}{4} - \frac{21S^2P}{\phi}\right) \\ \phi^2\left(\frac{PS^2}{8} - \frac{6PC}{\phi} - \frac{12PS^2}{\phi^2}\right) \end{matrix} \right\} \tag{4-1a}$$

And, the total member-end force matrix is

$$\{P\} = \left\{ \begin{matrix} \left[\frac{-PS(13C+3)}{4} + \frac{3SP(7C+3)}{\phi}\right] \frac{\phi^2}{B} \\ \left[\frac{-13S^2P}{4} - \frac{3\phi(7+7C^2+6C)P}{\phi^2}\right] \frac{\phi^2}{B} \\ \left[\frac{-PS^2L}{4} + \frac{3PL(18C+3+3C^2)}{2\phi} - \frac{18PS^2L}{\phi^2}\right] \frac{\phi^2}{B} \\ \left[\frac{SP(13+3)}{4} - \frac{3SPC7C+3)}{\phi}\right] \frac{\phi^2}{B} \\ \left[\frac{13S^2P}{4} + \frac{3(7+7C^2+6C)P}{\phi}\right] \frac{\phi^2}{B} \\ \left[-\frac{PLS^2}{2} + \frac{3PL(10C+3C^2+3)}{2\phi} + \frac{18PS^2L}{\phi^2}\right] \frac{\phi^2}{B} \\ \left[\frac{-PS(13C+3)}{4} + \frac{3S(7C+3)P}{\phi}\right] \frac{\phi^2}{B} \\ \frac{-P}{2} + \left[\frac{3PS^2}{4} + \frac{3P(13C^2+18C+13)}{\phi} + \frac{144S^2P}{\phi^2}\right] \frac{\phi^2}{B} \\ \frac{PL}{8} + \left[-\frac{PS^2L}{2} - \frac{3PL(13C^2+22C+13)}{2\phi} - \frac{54PS^2L}{\phi^2}\right] \frac{\phi^2}{B} \\ \left[\frac{PS(13C+3)}{4} - \frac{3SP(7C+3)}{\phi}\right] \frac{\phi^2}{B} \\ -\frac{P}{2} + \left[-\frac{3S^2P}{4} - \frac{3P(13C^2+18C+13)}{\phi} - \frac{144S^2P}{\phi^2}\right] \frac{\phi^2}{B} \\ \frac{-PL}{8} + \left[-\frac{PS^2L}{4} - \frac{3PL(13C^2+14C+13)}{2\phi} - \frac{90PLS^2}{\phi^2}\right] \frac{\phi^2}{B} \end{matrix} \right\} \tag{4-1b}$$

where $B = 8S^2\phi^2 + 24(5C^2+6C+5) + 288S^2$

Taking the limit as ϕ approaches infinity yields

$$\{\Delta\} = \begin{Bmatrix} 0 \\ 0 \\ PL^2/64 \end{Bmatrix} \frac{1}{EI} \qquad (4\text{-}2)$$

and

$$\{P\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P/2 \\ PL/8 \\ 0 \\ -P/2 \\ -PL/8 \end{Bmatrix} - \begin{Bmatrix} -(13C+3)P/32S \\ -13P/32 \\ -PL/32 \\ (13C+3)P/32S \\ 13P/32 \\ -PL/16 \\ -(13C+3)P/32S \\ 3P/32 \\ -PL/16 \\ (13C+3)P/32S \\ -3P/32 \\ -PL/32 \end{Bmatrix} = \begin{Bmatrix} -(13C+3)P/32S \\ -13P/32 \\ -PL/32 \\ (13C+3)P/32S \\ 13P/32 \\ -PL/16 \\ -(13C+3)P/32S \\ -13P/32 \\ PL/16 \\ (13C+3)P/32S \\ -19P/32 \\ -5PL/32 \end{Bmatrix} \qquad (4\text{-}3)$$

A plot of the relationship between rotation (θ_B), moment (M_1), and applied force (P) as given by Equations (4-1a), (4-1b), (4-2) and (4-3) for the angle $\theta = 45^\circ$, with $\phi = 10^4$, and $\phi = \infty$ is ahown in Fig. 4-3.

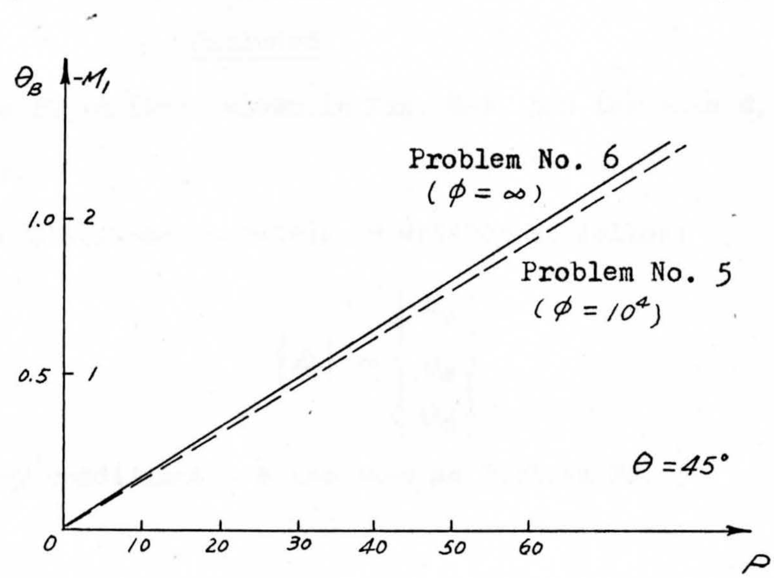


Fig. 4-3 $\begin{Bmatrix} \theta_B - P \\ -M_1 - P \end{Bmatrix}$ Relationship for Problems No.5 & No.6

The final free body diagram is shown in Fig. 4-4.

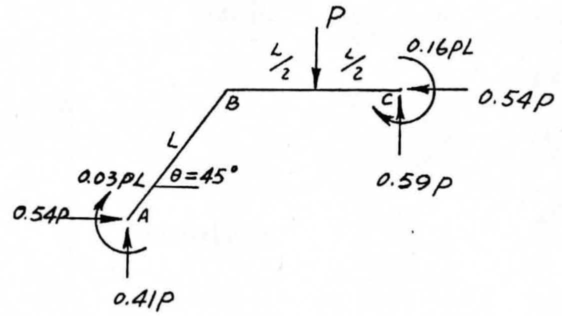


Fig. 4-4 Free Body Diagram for Problem No. 6

The equilibrium conditions of the frame are checked as follows:

(1) $\Sigma F_x = 0$

$$0.54P - 0.54P = 0$$

(2) $\Sigma F_y = 0$

$$0.41P + 0.59P - P = 0$$

(3) $\Sigma M_c = 0$

$$0.16PL + 0.03PL - P \cdot L/2 - 0.54P \cdot (0.71L) + 0.41P \cdot (1.71L) = 0$$

O.K.

4.3 Problem No. 7 Nonorthogonal Member-Loaded Frame, Axial Force Excluded

The rigid frame shown in Fig. 4-1, has the same E, I, and L on each member.

The displacement matrix is written as follow:

$$\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

The boundary conditions are the same as Problem No. 5.

In bending theory, we neglect the axial force effect, $\left[\tilde{K}\right]_G = \left[0\right]$

so $\left[\tilde{K}\right] = \left[\tilde{K}\right]_E$ the stiffness matrix on the member 1 becomes:

$$\left[\hat{K}\right]_1 = EI \begin{bmatrix} 12S^2/L^3 & -12CS/L^3 & 6S/L^2 & -12S^2/L^3 & 12CS/L^3 & 6S/L^2 \\ & 12C^2/L^3 & -6C/L^2 & 12CS/L^3 & -12C^2/L^3 & -6C/L^2 \\ & & 4/L & -6S/L^2 & 6C/L^2 & 2/L \\ & & & 12S^2/L^3 & -12CS/L^3 & -6S/L^2 \\ & & & & 12C^2/L^3 & 6C/L^2 \\ & \text{Symmetric} & & & & 4/L \end{bmatrix}$$

The stiffness matrix on the member 2 is obtained as:

$$\left[\hat{K}\right]_2 = EI \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 12/L^3 & -6/L^2 & 0 & -12/L^3 & -6/L^2 \\ & & 4/L & 0 & 6/L^2 & 2/L \\ & & & 0 & 0 & 0 \\ & & & & 12/L^3 & 6/L^2 \\ & \text{Symmetric} & & & & 4/L \end{bmatrix}$$

The global stiffness matrix of the frame yields:

$$\left[\hat{K}\right] = \begin{bmatrix} \left[\hat{K}\right]_1 & \left[0\right] \\ \left[0\right] & \left[\hat{K}\right]_2 \end{bmatrix}$$

This same result is obtained from Problem 5 by setting the value of the parameter $\phi = 0$.

The structural stiffness matrix becomes:

$$[r]^T[\hat{k}][r] = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1&0&0&0 \end{bmatrix} \left[\hat{k}\right] \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$$

or

$$[r]^T[\hat{k}][r] = EI \begin{bmatrix} 12S^2/L^3 & -12CS/L^3 & -6S/L^2 \\ -12CS/L^3 & 12(C^2+1)/L^3 & 6(C-1)/L^2 \\ -6S/L^2 & 6(C-1)/L^2 & 8/L \end{bmatrix}$$

The inverse of the above equation is:

$$\left[[r]^T[\hat{k}][r]\right]^{-1} = \frac{L^7}{288S^2EI} \begin{bmatrix} \frac{1}{L^4}(60C^2+72C+60), & \left(\frac{60CS+36S}{L^4}\right) & \left(\frac{72CS+72S}{L^5}\right) \\ & \frac{60S^2}{L^4} & \frac{72S^2}{L^5} \\ \text{Symmetric} & & \frac{144S^2}{L^6} \end{bmatrix}$$

The nodal forces matrix is:

$$\{Q\} = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1&0&0&0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P/2 \\ PL/8 \\ 0 \\ -P/2 \\ PL/8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

The nodal displacement matrix becomes:

$$\{\Delta\} = \begin{Bmatrix} \dfrac{P(7C+3)L^3}{96SEI} \\ \dfrac{7PL^3}{96EI} \\ \dfrac{PL^2}{16EI} \end{Bmatrix} \tag{4-4}$$

The total member end forces yields:

$$\{P\} = \{P_0\} + [K][r][\Delta] = \begin{Bmatrix} 0 \\ 0 \\ -PL/16 \\ 0 \\ 0 \\ PL/16 \\ 0 \\ 0 \\ -PL/16 \\ 0 \\ -P \\ -7PL/16 \end{Bmatrix} \tag{4-5}$$

Fig. 4-5 shows $\hat{P}$-θ_B relationship:

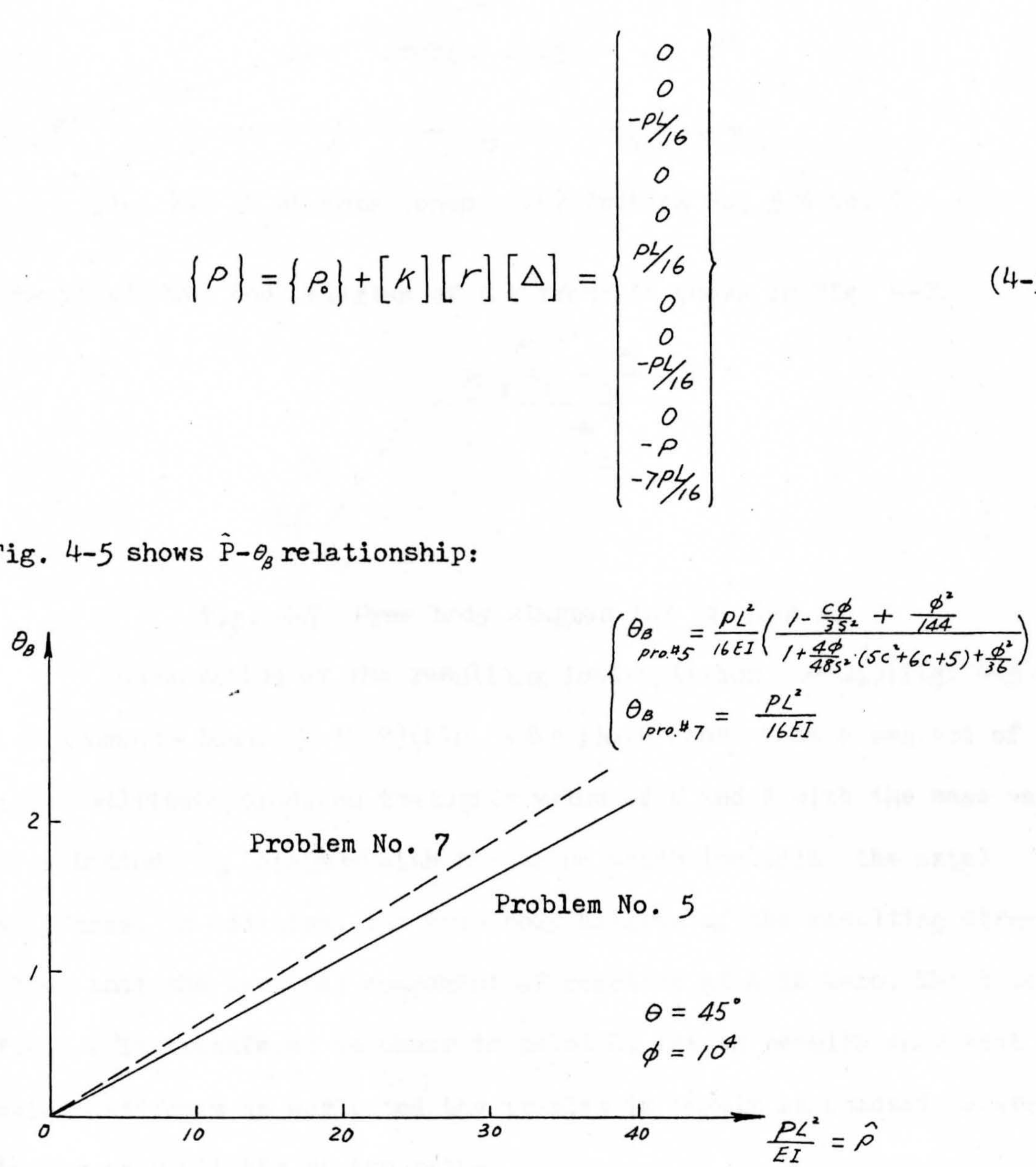
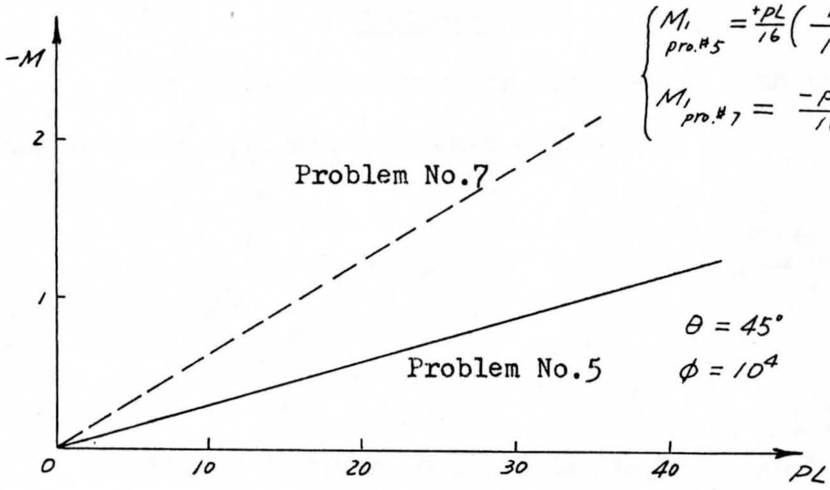


Fig. 4-5 Load-Rotation Relationship for Problems No. 5 & No. 7

Fig. 4-6 shows M_1-P relationship:



$$\begin{cases} M_{1_{pro.\#5}} = \frac{+PL}{16}\left(\dfrac{1 - \frac{\phi^2}{72} + \frac{\phi c(18c+3c^2+3)}{12S^2}}{1 + \frac{\phi^2}{36} + \frac{\phi}{12}(5c^2+6c+5)} \right) \\ M_{1_{pro.\#7}} = \frac{-PL}{16} \end{cases}$$

Fig. 4-6 M_1-P Relationship for Problem No. 5 & No. 7

The final free body diagram of the frame is shown in Fig. 4-7.

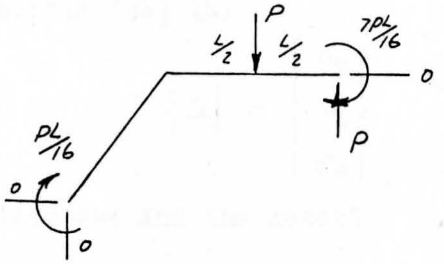


Fig. 4-7 Free body Diagram for Problem No. 7

Observation of the resulting load-rotation (P-θ_B)(Fig. 4-5) and moment- load (M_1-P)(Fig. 4-6) plots show that a neglect of axial stiffness produces the higher value of P and M with the same value of rotation θ_B compared with the value which includes the axial stiffness. In addition, the free body diagram of the resulting structures shows that the vertical component of reaction at A is zero. The total force P is transfered as shear to point C. Theses results show that if axial stiffness is neglected the problem is purely an academic exercise having no realistic design value.

4.4 Problem No. 8 Nonorthogonal Node-Loaded Frame, Axial Forces Included

The rigid frame shown in Fig. 4-8(a) with the load at the node has the same E,I, and L on each member.

Fig. 4-8 Frame Load & Node Displacements for Problem No. 8

The frame is statically indeterminate and kinematically indeterminate to the third degree. The node displacements are shown in Fig. 4-5(b). The displacement matrix $\{\Delta\}$ is:

$$\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

The positive displacement and the rotation on each member are shown in Fig. 4-1.

Applying the boundary conditions:

$$u_1 = u_4 = 0 \qquad v_1 = v_4 = 0 \qquad \theta_1 = \theta_4 = 0$$

$$u_2 = u_3 = 0 \qquad v_2 = v_3 = 0 \qquad \theta_2 = \theta_3 = 0$$

gives

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \theta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

Taking into account the axial force effect, the stiffness matrix $[\hat{K}]$ for member 1 is the combined elastic stiffness matrix and the geometric stiffness matrix as given by Equation 1-36, where the axial force is defined as the parameter.

The stiffness matrix $[\hat{K}]$ for member 2 is equal to $[\hat{K}]_E - R[\bar{K}]_G$. The axial force R in the member 2 is different than the axial force P in the member 1, hence

$$[\hat{K}]_2 = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ & \left(\frac{12EI}{L^3} - \frac{6R}{5L}\right) & \left(\frac{-6EI}{L^2} + \frac{R}{10}\right) & 0 & \left(\frac{-12EI}{L^3} + \frac{6R}{5L}\right) & \left(\frac{-6EI}{L^2} + \frac{R}{10}\right) \\ & & \left(\frac{4EI}{L} - \frac{2RL}{15}\right) & 0 & \left(\frac{6EI}{L^2} - \frac{R}{10}\right) & \left(\frac{2EI}{L} + \frac{LR}{30}\right) \\ & & & \frac{AE}{L} & 0 & 0 \\ & \text{Symmetric} & & & \left(\frac{12EI}{L^3} - \frac{6R}{5L}\right) & \left(\frac{6EI}{L^2} - \frac{R}{10}\right) \\ & & & & & \left(\frac{4EI}{L} - \frac{2LR}{15}\right) \end{bmatrix}$$

The global stiffness matrix of the frame is obtained as:

$$[\hat{K}] = \begin{bmatrix} [\hat{K}]_1 & [0] \\ [0] & [\bar{K}]_2 \end{bmatrix}$$

The structural stiffness matrix is:

$$
[r]^T[\hat{k}][r] = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1&0&0&0 \end{bmatrix} [\hat{k}] \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}
$$

or upon multiplication gives

$$
[r]\,[\hat{k}]\,[r] = \begin{bmatrix} \left[\left(\frac{A(c^2+1)}{IL} + \frac{12S^2}{L^3}\right)EI - \frac{6S^2P}{5L}\right], & \left[\left(\frac{Acs}{IL} - \frac{12cs}{L^3}\right)EI + \frac{+6csP}{5L}\right], & \left[\frac{-6SEI}{L^2} + \frac{5P}{10}\right] \\ & \left[EI\left(\frac{AS^2}{IL} + \frac{12c(c^2+1)}{L^3}\right) - \frac{6Pc^2+6R}{5L}\right], & \left[\frac{6c(c-1)EI}{L^2} + \frac{-PC+R}{10}\right] \\ \text{Symmetric} & & \left[\frac{8EI}{L} - \frac{2L(P+R)}{15}\right] \end{bmatrix} \quad (4\text{-}6)
$$

From the free body diagram of node B shown as follows:

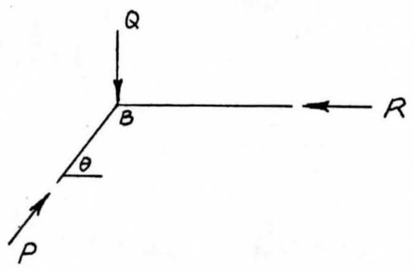


Fig. 4-9 Free Body Diagram of Node B, Problem No. 8

Satisfaction of force equilibrium at node B requires that

$$
Q = P\sin\theta \quad , \quad R = P\cdot\cos\theta
$$

Thus,

$$P = \frac{Q}{\sin\theta} \tag{4-7a}$$

$$R = \frac{\cos\theta \cdot Q}{\sin\theta} \tag{4-7b}$$

Substituting Equation 4-7a & 4-7b into Equation 4-6 yields

$$[r]^T[\hat{k}][r] = \begin{bmatrix} \left[\phi(c^2+1) + s^2\left(12 - \frac{6}{5}\frac{Q}{s}\right)\right], & \left[cs\phi - cs\left(12 - \frac{6Q}{5s}\right)\right], & \left[s\left(-6 + \frac{Q}{10s}\right)\right] \\ & \left[\phi s^2 + 12(1+c^2) - \frac{6}{5}\left(c^2\frac{Q}{s} + \frac{Qc}{s}\right)\right], & 6[c-1], \\ \text{Symmetric} & & \left[8 - \frac{2}{15}\left(\frac{Q}{s} + \frac{Qc}{s}\right)\right] \end{bmatrix} \tag{4-8}$$

For convenience, the following algebraic equations are defined:

$$a_1 = c^2 + 1 \qquad c_1 = s^2$$

$$a_2 = \left(12 - \frac{6Q}{5s}\right) \qquad c_2 = 12 + (1 + c^2) - \frac{6}{5}\left(c^2\frac{Q}{s} + \frac{Qc}{s}\right)$$

$$b_1 = cs \qquad c_3 = 6(c - 1)$$

$$b_2 = -cs\left(12 - \frac{6Q}{5s}\right) \qquad d_3 = 8 - \frac{2}{15}\left(\frac{Q}{s} + \frac{Qc}{s}\right)$$

$$b_3 = s\left(-6 + \frac{Q}{10 \cdot s}\right)$$

Equation 4-8 becomes:

$$[r][\hat{k}][r] = \begin{bmatrix} a_1\phi + a_2, & b_1\phi + b_2 & b_3 \\ b_1\phi + b_2, & c_1\phi + c_2 & c_3 \\ b_3 & c_3 & d_3 \end{bmatrix}$$

The inverse of the latter matrix is formulated using the following cofactor definitions:

$$C_{11} = \phi\left(c_1 d_3 + \frac{c_2 d_3}{\phi} - \frac{c_3^2}{\phi} \right)$$

$$C_{12} = \phi\left(-b_1 d_3 - \frac{b_2 d_3}{\phi} + \frac{b_3 c_3}{\phi} \right)$$

$$C_{13} = \phi\left(b_1 c_3 - b_3 c_1 + \frac{b_2 c_3}{\phi} - \frac{b_3 c_2}{\phi} \right)$$

$$C_{22} = \phi\left(a_1 d_3 + \frac{a_2 d_3}{\phi} - \frac{b_3^2}{\phi} \right)$$

$$C_{23} = \phi\left(-a_1 c_3 + b_1 b_3 - \frac{a_2 c_3}{\phi} + \frac{b_2 b_3}{\phi} \right)$$

$$C_{33} = \phi^2 (a_1 c_1 - b_1^2) + \frac{a_1 c_2 + a_2 c_1 - 2 b_1 b_2}{\phi} + \frac{a_2 c_2 - b_2^2}{\phi^2}$$

The determinant of the matrix is:

$$D = \phi^2 \Big[(a_1 c_1 d_3 - b_1^2 b_3) + \frac{1}{\phi} (a_1 c_2 d_3 + a_2 c_1 d_3 + 2 b_1 b_3 c_3 - c_1 b_3^2 - a_1 c_3^2 - 2 b_1 b_2 d_3) + \frac{1}{\phi^2} (2 b_2 b_3 c_3 + a_2 c_2 d_3 - c_2 b_3^2 - a_2 c_3^2 - b_2^2 d_3) \Big]$$

hence,

$$\Big[[r] [\hat{k}] [r] \Big]^{-1} = \frac{1}{D} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

The nodal forces matrix becomes

$$\{Q\} = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1&0&0&0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -Q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -Q \\ 0 \end{Bmatrix}$$

The nodal displacement $\{\Delta\}$ is computed as follow:

$$\{\Delta\} = -\Big[[r]^T[\hat{k}][r]\Big]^{-1}\{Q\}$$

or

$$\{\Delta\} = \frac{-1}{D}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{Bmatrix} 0 \\ -Q \\ 0 \end{Bmatrix}$$

which simplifies to

$$\{\Delta\} = \frac{+1}{D}\begin{Bmatrix} C_{12}Q \\ C_{22}Q \\ C_{23}Q \end{Bmatrix}$$

The total member-end forces $\{P\}$ becomes

$$\{P\} = \{P_0\} + [\hat{k}]\,[r]\{\Delta\}$$

and in component form

$$\{P\} = \left\{\begin{array}{l}
+[(\phi c^2 - 12s^2) + \frac{6}{5}sQ]\frac{C_{11}}{D}\cdot Q + [-\phi cs + 12cs - \frac{6cQ}{5}]\frac{C_{22}Q}{D} + (6sL - QL/_{10})\frac{C_{23}Q}{D} \\
[-\phi cs + 12cs - \frac{6cQ}{5}]\frac{C_{11}Q}{D} + [-s^2\phi - 12c^2 + \frac{6c^2Q}{5s}]\frac{C_{22}Q}{D} + (-6cL + \frac{cQ}{5})\frac{C_{23}Q}{D} \\
[-6sL - \frac{QL}{10}]\frac{C_{11}Q}{D} + [-6cL + \frac{cQ}{5}]\frac{C_{22}}{D}Q + [2L^2 + QL^4/_{30\cdot s}]\frac{C_{23}Q}{D} \\
+[a_1\phi + a_2]\frac{C_{11}Q}{D} + [b_1\phi + b_2]\frac{C_{22}Q}{D} + [b_3]\frac{C_{23}Q}{D} \\
-Q + [b_1\phi + b_2]\frac{C_{11}Q}{D} + [c_1\phi + c_2]\frac{C_{22}Q}{D} + [c_3]\frac{C_{23}Q}{D} \\
b_3\frac{C_{11}Q}{D} + c_3\frac{C_{22}Q}{D} + d_3\frac{C_{23}Q}{D} \\
\phi\frac{C_{11}Q}{D} \\
(12L - \frac{6cQ}{5s})\frac{C_{22}Q}{D} + (-6L + \frac{cQ}{10s})\frac{C_{23}Q}{D} \\
(-6L + \frac{cQ}{10\cdot s})\frac{C_{22}Q}{D} + (4L^2 - \frac{2Q}{15})\frac{C_{23}Q}{D} \\
-\phi\frac{C_{11}Q}{D} \\
-(12L - \frac{6cQ}{5s})\frac{C_{22}Q}{D} - (-6L + \frac{cQ}{10s})\frac{C_{23}Q}{D} \\
(-6L + \frac{cQ}{10s})\frac{C_{22}Q}{D} - (2L^2 + \frac{L^4Q}{s})\frac{C_{23}Q}{D}
\end{array}\right\}$$

From Equation 4-10, the shear force V_1 becomes

$$\hat{V}_1 = \left\{(-\phi cs + 12cs - \frac{6cQ}{s})(\phi c_1 d_3 + c_2 d_3 - c_3) + \left[(-\phi s^2 - 12c^2) + \frac{6c^2 Q}{5s}\right](\phi a_1 d_3 + a_2 d_3 - b_3^2) + (-6cL + \frac{cQ}{s})(-\phi a_1 c_3 + b_1 b_3 - a_2 c_3 + b_2 b_3)\right\} \frac{Q}{D}$$

or

$$\hat{V}_1 = \frac{-\phi^2(c_1 d_3 cs + a_1 d_3 s^2) + \phi(12cs\,c_1 d_3 - c_2 d_3 cs + \cdots\cdots) + (12cs\,c_2 d_3 + \cdots\cdots)}{\phi^2(a_1 c_1 d_3 - b_1^2 b_3) + \phi(a_1 c_2 d_3 + \cdots\cdots) + (2b_2 b_3 c_3 + \cdots\cdots)}$$

Taking $\lim\limits_{\phi \to \infty} \hat{V}_1$, one obtains:

$$\hat{V}_1 = \frac{[c_1 d_3 cs + a_1 d_3 s^2]Q}{a_1 c_1 d_3 - b_1^2 b_3}$$

or

$$\hat{V}_1 = \frac{cs^2 + c^2 + 1}{c^2 + 1}\left(\frac{1 - \dfrac{(c^2 + cs^2 + 1 - c^2 s^2 + c^3 + c)\cdot Q}{60(cs^2 + c^2 + 1)}}{1 - \dfrac{(1 + c + c^2 + c^3)\cdot Q}{60s(cs^2 + c^2 + 1)}}\right) Q \qquad (4\text{-}11)$$

A plot of the $\hat{V}_1$-Q relationship is shown in Fig. 4-11.

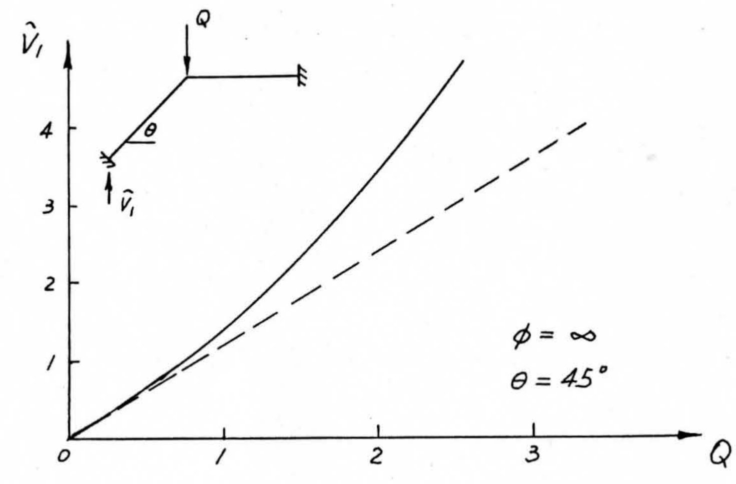


Fig. 4-10 $\hat{V}_1$-Q Relationship for Problem No. 8

It is clear from Fig. 4-11 that the relationship beeween the shear force $\hat{V}_1$ and applied force Q is non-linear. The tangent line to the initial point on the curve varies with different value of angle θ .

The relationship between the nodal vertical displacement v_B and the nodal force Q is shown in Equation 4-9 as

$$\left\{ v_B \right\} = \left\{ \frac{C_{22} Q}{D} \right\}$$

which upon substitution becomes:

$$\left\{ v_B \right\} = \frac{\phi \left(a_1 d_3 + \frac{a_2 d_3}{\phi} - \frac{b_3 c_2}{\phi^2} \right) Q}{\phi^2 \left[\left(a_1 c_1 d_3 - b_1^2 d_3 \right) + \frac{1}{\phi} \left(a_1 a_3 d_3 + a_2 c_1 + d_3 + \cdots \right) + \frac{1}{\phi^2} \left(2 b_1 b_2 c_3 + \cdots \right) \right]}$$

From the above equation it can be seen that the limiting condition $\phi \rightarrow 0$ and $\phi \rightarrow \infty$ are not defined mathematically for a practical situation (i.e. $\phi \rightarrow \infty$, $V_B = 0$, and $\phi \rightarrow 0$, v_B is undefined). This condition requires a substitution of a reasonable numerical value for ϕ in the calculations and hence extremely complicates the algebraic calculation of the solution. Thus, a practical solution must be obtained numerically by use of a computer.

CHAPTER V

ANALYSIS OF THE SYMMETRIC NONORTHOGONAL FRAMES WITH BOTH MEMBERS INCLINED

5.1 Problem No. 9: Node Loaded Frame, Axial Deformations and Axial Forces Included

The rigid frame with a load at the node shown in Fig. 5-1 is analysed for both members having common parameter L,E, and I constant.

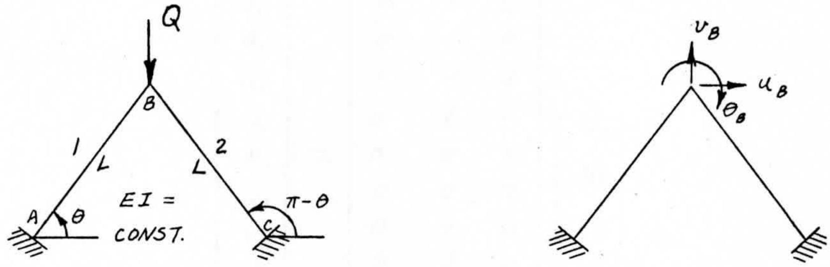


Fig. 5-1 Problem No.9 Frame Load & Node Displacements

The frame is statically indeterminate to the third degree, and kinematically indeterminate to the third degree. The redundant displacements are shown in Fig. 5-1 (b).

The positive nodal deformations and forces are defined in Fig. 5-2 as follow:

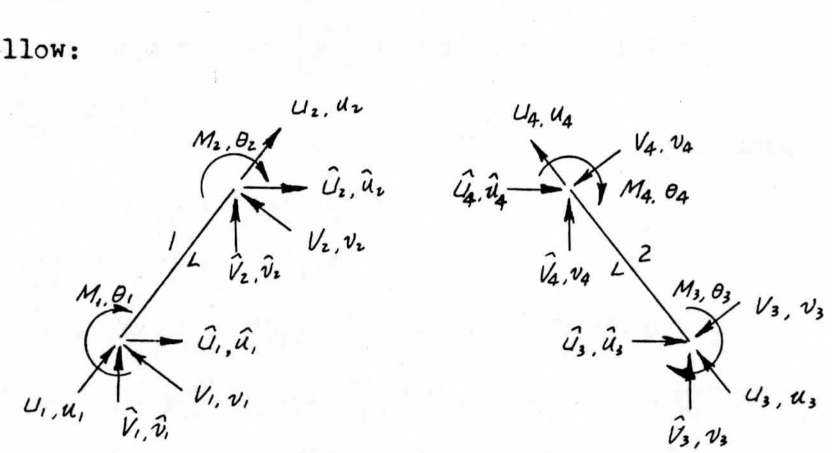


Fig. 5-2 Member Loads & Displacements for Problem No. 9

The displacement matrix $\{\Delta\}$ is written as follows:

$$\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

Applying the boundary condition:

$$\theta_1 = \theta_3 = 0 \qquad \hat{u}_1 = \hat{u}_3 = 0 \qquad \hat{v}_1 = \hat{v}_3 = 0$$

$$\theta_2 = \theta_4 = \theta_B \qquad \hat{u}_2 = \hat{u}_4 = u_B \qquad \hat{v}_2 = v_4 = v_B$$

one obtains:

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \theta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \theta_2 \\ \hat{u}_3 \\ \hat{v}_3 \\ \theta_3 \\ \hat{u}_4 \\ \hat{v}_4 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix} \tag{5-1}$$

The stiffness matrix $[\hat{K}]$ includes elastic bending stiffness matrix $[\hat{K}]_E$ and the geometric stiffness matrix $[\hat{K}]_G$, which is

$$[\hat{K}] = [\hat{K}]_E - P[K]_G$$

For the member 1 $[\hat{K}]$ is shown as follows:

$$[\hat{K}]_1 = \begin{bmatrix}
\left[\left(\frac{AC}{IL}+\frac{12S^2}{L^3}\right)EI-\frac{6S^2P}{5L}\right] & & & & & \text{Symmetric} \\
\left[\left(\frac{ACS}{IL}-\frac{12CS}{L^3}\right)EI+\frac{6CSP}{5L}\right], & \left[\left(\frac{AS^2}{IL}+\frac{12C^2}{L^3}\right)EI-\frac{6C^2P}{5L}\right] & & & & \\
\left[\frac{6SEI}{L^2}-\frac{PS}{10}\right], & \left[\frac{-6CEI}{L^2}+\frac{CP}{10}\right], & \left[\frac{4EI}{L}-\frac{2PL}{15}\right] & & & \\
\left[\left(\frac{-AC^2}{IL}-\frac{12S^2}{L^3}\right)EI+\frac{6S^2P}{5L}\right], & \left[\left(\frac{-ASC}{IL}+\frac{12CS}{L^3}\right)EI-\frac{6CSP}{5L}\right], & \left[\frac{-6SEI}{L^2}+\frac{SP}{10}\right], & \left[\left(\frac{AC^2}{IL}+\frac{12S^2}{L^3}\right)EI-\frac{6S^2P}{5L}\right] & & \\
\left[\left(\frac{-ACS}{IL}+\frac{12CS}{L^3}\right)EI-\frac{6CSP}{5L}\right], & \left[\left(\frac{-AS^2}{IL}-\frac{12C^2}{L^3}\right)EI+\frac{6C^2P}{5L}\right], & \left[\frac{-6CEI}{L^2}-\frac{CP}{10}\right], & \left[\left(\frac{ACS}{IL}-\frac{12CS}{L^3}\right)EI+\frac{6CSP}{5L}\right], & \left[\left(\frac{AS^2}{IL}-\frac{12C^2}{L^3}\right)EI-\frac{6C^2P}{5L}\right] & \\
\left[\frac{6SEI}{L^2}-\frac{SP}{10}\right], & \left[\frac{-6CEI}{L^2}+\frac{CP}{10}\right], & \left[\frac{2EI}{L}+\frac{PL}{30}\right], & \left[\frac{-6SEI}{L^2}+\frac{SP}{10}\right], & \left[\frac{6CEI}{L^2}-\frac{CP}{10}\right], & \left[\frac{4EI}{L}-\frac{2PL}{15}\right]
\end{bmatrix}$$

where $s = \sin\theta \qquad c = \cos\theta$

For the member 2 , $[\hat{K}]$ is defined as:

$$
[\hat{K}]_2 = \left[\begin{array}{llllll}
\left[\left(\frac{Ac}{IL}+\frac{12s^2}{L}\right)EI-\frac{6s^2P}{5L}\right], & & & & & \text{Symmetric} \\
\left[\left(\frac{Acs}{IL}-\frac{12cs}{L^3}\right)EI+\frac{6csP}{5L}\right], & \left[\left(\frac{As^2}{IL}+\frac{12c^2}{L^3}\right)EI-\frac{6c^2P}{5L}\right], & & & & \\
\left[\frac{6sEI}{L^2}-\frac{Ps}{10}\right], & \left[\frac{-6cEI}{L^2}+\frac{cP}{10}\right], & \left[\frac{4EI}{L}-\frac{2PL}{15}\right], & & & \\
\left[\left(\frac{-Ac^2}{IL}-\frac{12s^2}{L^3}\right)EI+\frac{6s^2P}{5L}\right], & \left[\left(\frac{-Acs}{IL}+\frac{12cs}{L^3}\right)EI-\frac{6csP}{5L}\right], & \left[\frac{-6sEI}{L^2}+\frac{sP}{10}\right], & \left[\left(\frac{Ac^2}{IL}+\frac{12s^2}{L^3}\right)EI-\frac{6s^2P}{5L}\right] & & \\
\left[\left(\frac{-Acs}{IL}+\frac{12cs}{L^3}\right)EI-\frac{6csP}{5L}\right], & \left[\left(\frac{-As^2}{IL}-\frac{12c^2}{L^3}\right)EI+\frac{6c^2P}{5L}\right], & \left[\frac{-6cEI}{L^2}-\frac{cP}{10}\right], & \left[\left(\frac{Acs}{IL}-\frac{12cs}{L^3}\right)EI+\frac{6csP}{5L}\right], & \left[\left(\frac{As^2}{IL}-\frac{12c^2}{L^3}\right)EI-\frac{6c^2P}{5L}\right] & \\
\left[\frac{6sEI}{L^2}-\frac{sP}{10}\right], & \left[\frac{-6cEI}{L^2}+\frac{cP}{10}\right], & \left[\frac{2EI}{L}+\frac{PL}{30}\right], & \left[\frac{-6sEI}{L^2}+\frac{sP}{10}\right], & \left[\frac{6cEI}{L^2}-\frac{cP}{10}\right], & \left[\frac{4EI}{L}-\frac{2PL}{15}\right]
\end{array}\right]
$$

where $s = \sin(\pi-\theta) \qquad c = \cos(\pi-\theta)$

The global stiffness matrix of the frame is:

$$
[\hat{K}] = \begin{bmatrix} [K]_1 & [0] \\ [0] & [K]_2 \end{bmatrix}
$$

The structural stiffness matrix becomes:

$$
[r]^T[\hat{K}][r] = \begin{bmatrix}
0&0&0&1&0&0&0&0&0&1&0&0 \\
0&0&0&0&1&0&0&0&0&0&1&0 \\
0&0&0&0&0&1&0&0&0&0&0&1
\end{bmatrix} [\hat{K}] \begin{bmatrix}
0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1
\end{bmatrix}
$$

$$[r]^T[\hat{k}][r] = \frac{2EI}{L^3}\begin{bmatrix} (\phi c^2+12s^2-\frac{6}{5}\hat{P}s^2), & 0 & (-6s+\frac{\hat{P}s}{10})\cdot L \\ 0 & ,(\phi s^2+12c^2-\frac{6}{5}\hat{P}c^2), & 0 \\ (-6s+\frac{\hat{P}s}{10})\cdot L & 0 & (4-\frac{2\hat{P}}{15})L^2 \end{bmatrix}$$

where $\phi = \frac{AL^2}{I}$, $\hat{P} = \frac{PL^2}{EI}$

For the critical buckling load condition, the node displacements are arbitrary, hence the following determinate equations holds:

$$\left|[r]^T[\hat{k}][r]\right| = 0$$

or

$$-\frac{372}{125}\hat{P}^3 + \left[\frac{32\phi(32c^4+31s^4)+14112c^2s^2}{100}\right]\hat{P}^2 - \left[\frac{32\phi^2+48\phi(29s^4+32c^4)+25344}{15}\right]\hat{P}$$

$$+ \left[64\phi^2c^2s^2 + 96\phi(8c^4+5s^4) + 5760 = 0\right. \qquad (5\text{-}2a)$$

where

$$\phi = \frac{AL^2}{I} , \qquad \hat{P} = \frac{PL^2}{EI}$$

Equation 5-2a is factored into the following product form:

$$(\phi s^2+12c^2-\frac{6c^2}{5}\hat{P})\left[\frac{3}{20}\hat{P}^2 - \frac{78s^2+2\phi c^2}{15}\hat{P} + (12s^2+4\phi c^2)\right] = 0 \qquad (5\text{-}2b)$$

The three roots of Equation 5-2b become:

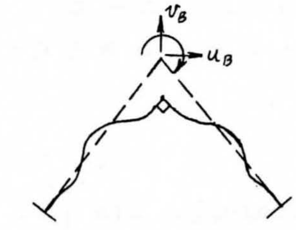
$$\hat{P}_1 = \frac{5(\phi s^2+12c^2)}{6c^2}$$

$$\hat{P}_2 = \frac{4\left[(39s^2+c^2\phi)+\sqrt{1116s^4-57c^2s^2\phi+\phi^2c^4}\right]}{9s^2}$$

$$\hat{P}_3 = \frac{4\left[(39s^2+c^2\phi)-\sqrt{1116s^4-57c^2s^2\phi+\phi^2c^4}\right]}{9s^2}$$

Numerically it may be shown that $\hat{P}_3$ corresponds to the lowest critical buckling mode.

The critical buckling load corresponds to a mode pattern, shown in Fig. 5-3.



$$\begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix} \propto \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$u_B = \theta_B = 0 \qquad v_B = 1$

Fig. 5-3 Problem No. 9 Buckling Mode Shape

The inverse of the structural stiffness matrix is:

$$\left[[r]^T[K][r]\right]^{-1} = \frac{1}{D}\begin{bmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{bmatrix}$$

where

$$m_{11} = \frac{4E^2I^2}{L^4}\left[\frac{4c}{25}\hat{\rho}^2 - \frac{2(s^2\phi + 48c^2)}{15}\hat{\rho} + 4(s^2\phi + 12c^2)\right]$$

$$m_{13} = \frac{4E^2I^2}{L^5}\left[\frac{-3c^2s}{25}\hat{\rho}^2 + \frac{84c^2s + \phi s^3}{10}\hat{\rho} - 6(s^2\phi + 9c^2s)\right]$$

$$m_{22} = \frac{4E^2I^2}{L^4}\left[\frac{3s^2}{20}\hat{\rho}^2 - \frac{2(39s^2 + \phi c^2)}{15}\hat{\rho} + 4(3s^2 + \phi c^2)\right]$$

$$m_{33} = \frac{4E^2I^2}{L^6}\left[\frac{36c^2s^2}{25}\hat{\rho}^2 - \frac{6}{5}[\phi(c^4+s^4) + 24c^2s^2]\hat{\rho} + \phi^2c^2s^2 + 12\phi(c^4+s^4) + 144c^2s^2\right]$$

$$D = \frac{E^3I^3}{L^7}\left[\left(\phi s^2 + 12c^2 - \frac{6c^2}{5}\hat{\rho}\right)\left(\frac{3s^2}{20}\hat{\rho}^2 - \frac{78s^2 + 2\phi c^2}{15}\hat{\rho} + 12s^2 + 4\phi c^2\right)\right]$$

The nodal forces matrix is obtained as:

$$\{Q\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -Q/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -Q/2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -Q \\ 0 \end{Bmatrix} \tag{5-3}$$

The nodal displacement $\{\Delta\}$ are computed:

$$\{\Delta\} = \begin{Bmatrix} 0 \\ \dfrac{\hat{Q}}{2(\phi S^2 + 12C^2 - \frac{6C^2}{5}\hat{P})} \\ 0 \end{Bmatrix} \tag{5-4}$$

The value of the member forces $\{P\}$ are shown as follows:

$$\{P\} = \frac{\hat{Q}}{2[(\phi S^2 + 12C^2 - \frac{6C^2}{5}\hat{P})]} \begin{Bmatrix} -cS\phi + 12cS - \frac{6cS}{5}\hat{P} \\ -\phi S^2 - 12C^2 + \frac{6C^2\hat{P}}{5} \\ 6cL - \frac{c\hat{P}L}{10} \\ \phi cS - 12cS + \frac{6cS\hat{P}}{5} \\ \phi S^2 + 12C^2 - \frac{6C^2\hat{P}}{5} \\ 6cL - \frac{c\hat{P}L}{10} \\ \phi cS - 12cS + \frac{6cS\hat{P}}{5} \\ -\phi S^2 - 12C^2 + \frac{6C^2\hat{P}}{5} \\ -6cL + \frac{c\hat{P}L}{10} \\ -\phi cS + 12cS - \frac{6cS}{5}\hat{P} \\ \phi S^2 + 12C^2 - \frac{6C^2\hat{P}}{5} \\ -6cL + c\hat{P}L \end{Bmatrix}$$

or

$$
\{P\} = \begin{Bmatrix}
(-cs\phi + 12cs - 1.2cs\hat{p})/B \\
Q/2 \\
(6cL - c\hat{p}L/10)/B \\
(\phi cs - 12cs + 1.2cs\hat{p})/B \\
-Q/2 \\
(+6cL - \frac{c\hat{p}L}{10})/B \\
(\phi cs - 12cs + 1.2cs\hat{p})/B \\
Q/2 \\
(-6cL - c\hat{p}L/10)/B \\
(-\phi cs + 12cs - 1.2cs\hat{p})/B \\
-Q/2 \\
(-6cL - c\hat{p}/10)/B
\end{Bmatrix} \tag{5-5}
$$

where

$$
B = \frac{\hat{Q}}{2\,(\phi s^2 + 12c^2 - 1.2c^2\hat{p})}
$$

The forces on each members are shown below:

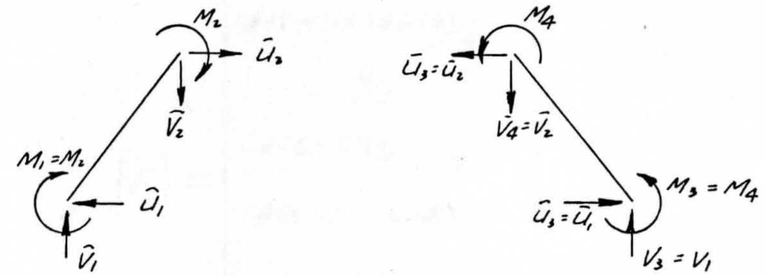


Fig. 5-4 Free Body Diagram for Each Member in Problem No. 9

The relationship between the member axial force P and applied node force Q shown in Fig. 5-5.

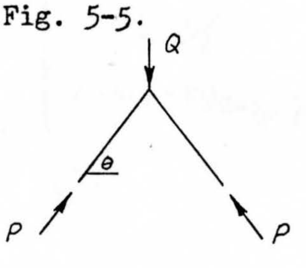


Fig. 5-5 Free Body Diagram at the Node B in Problem No. 9

Force equilibrium requires the condition

$$2P\sin\theta = Q$$

or

$$P = \frac{Q}{2\cdot\sin\theta} \tag{5-6}$$

Substituting Equation 5-6 into Equations 5-4 and 5-5, one obtains:

$$\{\Delta\} = \begin{Bmatrix} 0 \\ \dfrac{\hat{Q}}{2(\phi S^2 + 12C^2 - \frac{3C^2 Q}{5S})} \\ 0 \end{Bmatrix} \tag{5-7}$$

and

$$\{P\} = \begin{Bmatrix} (-\phi CS + 12CS - 0.6CQ)/B \\ Q/2 \\ (6CL - CQ/20\cdot S)/B \\ (\phi CS - 12CS + 0.6CQ)/B \\ -Q/2 \\ (6CL - CQ/20\cdot S)/B \\ (\phi CS - 12CS + 0.6CQ)/B \\ Q/2 \\ (-6CL - CQ/20\cdot S)/B \\ (-\phi CS + 12CS - 0.6CQ)/B \\ -Q/2 \\ (-6CL - CQ/20\cdot S)/B \end{Bmatrix} \tag{5-8}$$

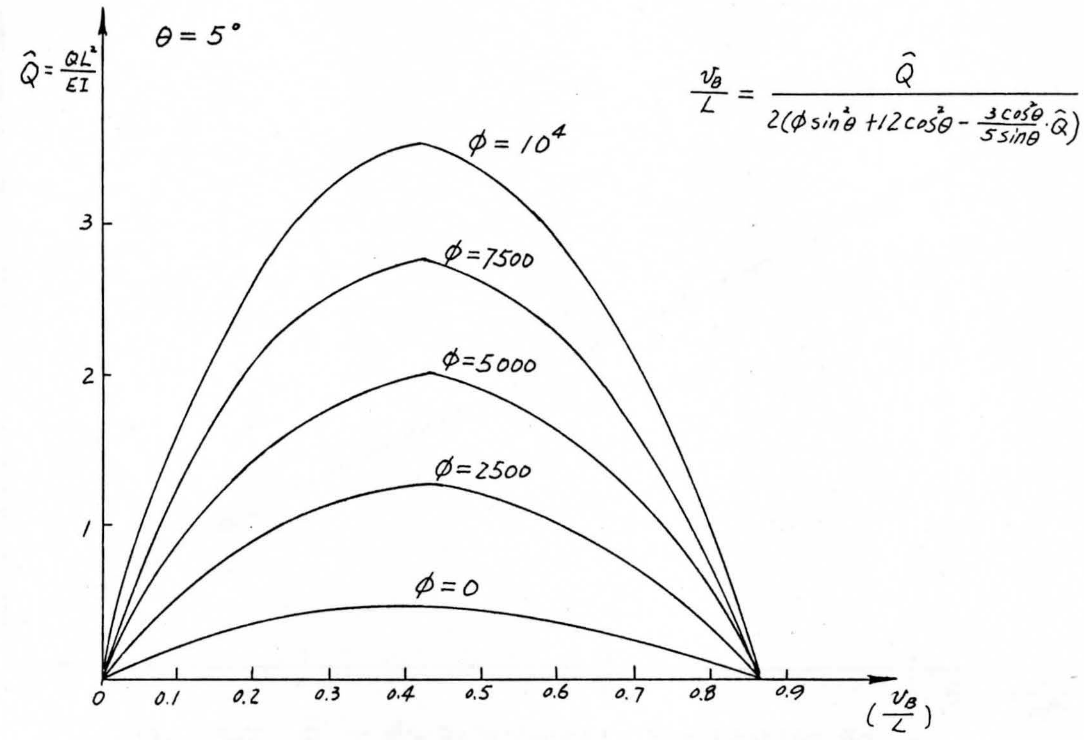


Fig. 5-6 $\frac{v_B}{L} - \hat{Q}$ Relationship with Different ϕ

Fig. 5-6 Shows the vertical node displacement plotted versus the applied node load $\hat{Q}$ for $\theta = 5°$ with the parameter ϕ a variable, The larger the value of ϕ is, the more stiff the member is, hence, the higher maximum load can be taken, A plot of the maximum value of $\hat{Q}$ from Fig. 5-6 versus the parameter ϕ is shown in Fig. 5-7.

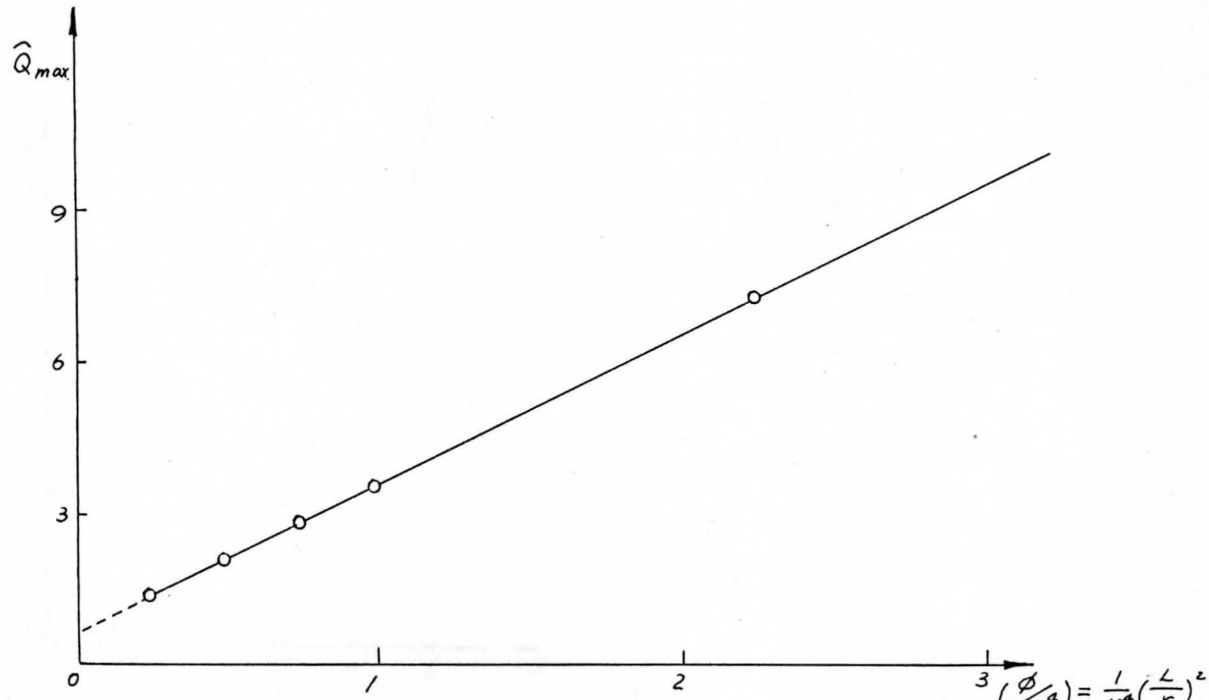


Fig. 5-7 $\hat{Q}_{max}$ — $\phi/10^4$ Relationship for Problem No. 9

A plot of the relationship between critical buckling load $\hat{Q}$ and vertical displacement $\hat{v}_B$ as given by Equation 5-7 for the different angles θ, with $\phi = 10^4$, is shown in Fig. 5-8.

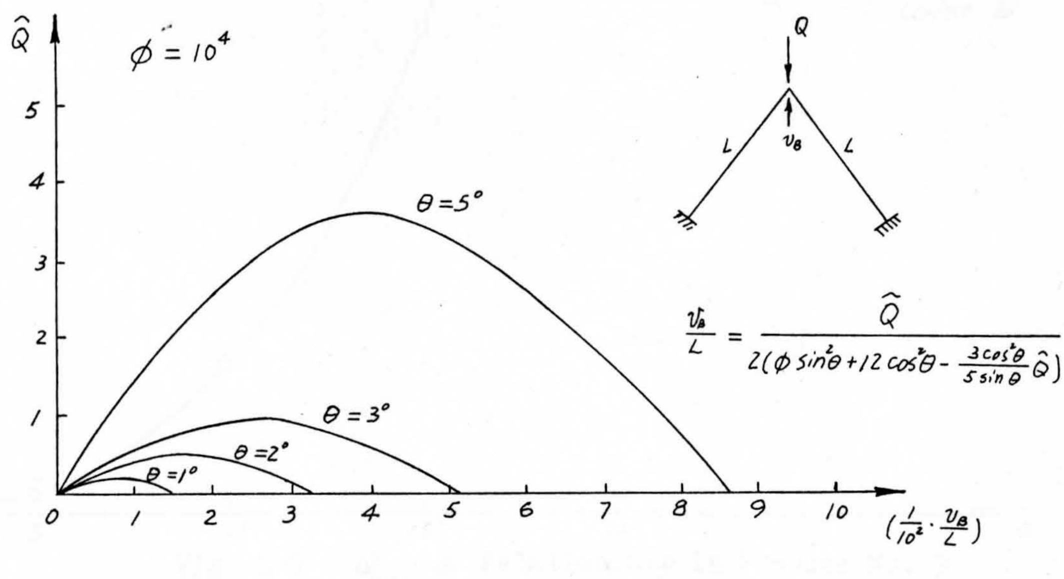


Fig. 5-8 Load-Displacement Relationship for Problem No. 9

A plot of the maximum condition of Fig. 5-8 is shown in Fig. 5-9.

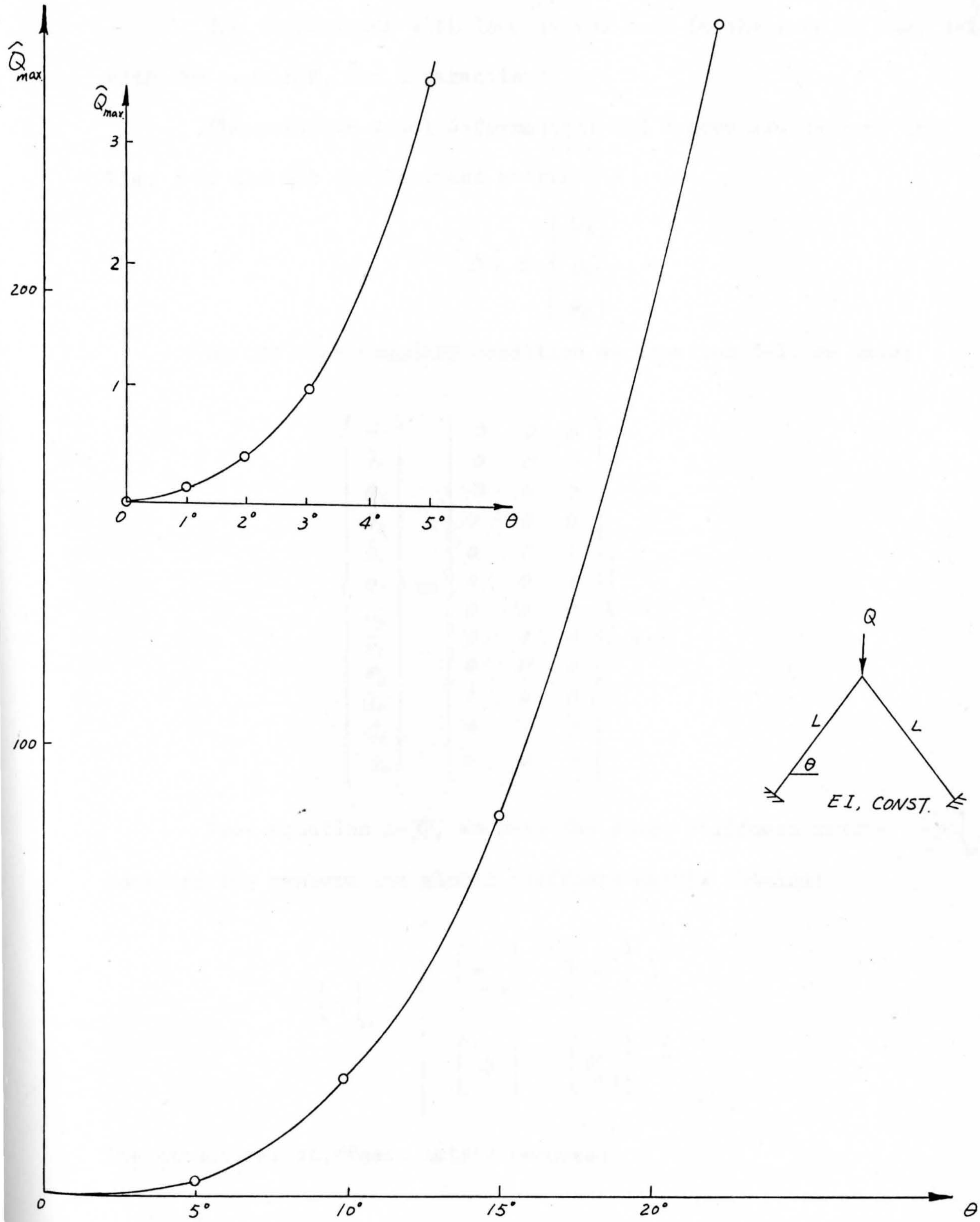


Fig. 5-9 $Q_{max} - \theta$ Relationship in Problem No. 9

5.2 Problem No. 10 Axial-Loaded Frame, Exact Solution

The rigid frame with load on the node is the same as Fig. 5-1, with the same L,E, and I parameters.

The positive nodal deformations and forces are defined in Fig. 5-2, and the displacement matrix $\{\Delta\}$ is

$$\{\Delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

By the same boundary condition as Equation 5-1, we have:

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \theta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \theta_2 \\ \hat{u}_3 \\ \hat{v}_3 \\ \theta_3 \\ \hat{u}_4 \\ \hat{v}_4 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

From Equation 1-37, we have the exact stiffness matrix $[K]_{ex.}$, combined two members the global stiffness matrix obtains:

$$[K]_{ex.} = \begin{bmatrix} [K_{\theta=\theta}]_1 & [0] \\ [0] & [K_{\theta=\pi-\theta}]_2 \end{bmatrix}$$

The structural stiffness matrix becomes:

$$[r]^T[K][r] = \begin{bmatrix} \left[\phi \tilde{c}^2 + \dfrac{\hat{k}^3 \cdot s \cdot \tilde{s}}{2(\cos\hat{k}-1)+\hat{k}\sin\hat{k}}\right], & 0\,, & \dfrac{-\hat{k}^2 \tilde{s}(c-1)\cdot L}{2\cdot(\cos\hat{k}-1)+\hat{k}\sin\hat{k}} \\ 0\,, & \left[-\dfrac{\bar{k}^3 \cdot s \cdot \tilde{c}^2}{2(\cos\hat{k}-1)+\hat{k}\sin\hat{k}} + \phi\tilde{s}^2\right], & 0 \\ \left[\dfrac{-\hat{k}^2 \cdot \tilde{s}(c-1)\cdot L}{2(\cos\hat{k}-1)+\hat{k}\sin\hat{k}}\right], & 0 & \left[\dfrac{-\hat{k}(s-\hat{k}\cdot c)\cdot L^2}{2(\cos\hat{k}-1)+\hat{k}\sin\hat{k}}\right] \end{bmatrix} \frac{2EI}{L^3}$$

where $\hat{k} = kL \qquad k = \sqrt{P/EI}$

For the arbitrary solution of the node displacements

$$\left|[r]^T[K][r]\right| = 0$$

hence,

$$\left[\phi\tilde{s}^2 - \frac{\hat{k}^3 \cdot s \cdot \tilde{c}^2}{T}\right]\left[-\phi\tilde{c}^2\hat{k}(s-\hat{k}c)L^2 - \frac{L^2\hat{k}^4\cdot\tilde{s}\left[s(s-\hat{k}c)+\tilde{s}(c-1)^2\right]}{T}\right] = 0$$

where $T = 2(\cos\hat{k}-1)+\hat{k}\sin\hat{k}$

The inverse of the structural stiffness matrix is:

$$\left[[r]^T[K][r]\right]^{-1} = \frac{1}{D}\begin{bmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{bmatrix}$$

where

$$m_{11} = \frac{-\phi\tilde{s}^2\cdot\hat{k}(s-\hat{k}c)\cdot L^2}{T} + \frac{\hat{k}^4 s\cdot\tilde{c}^2\cdot(s-\hat{k}c)\cdot L^2}{T^2}$$

$$m_{13} = \frac{-\phi\tilde{s}^3\hat{k}^2(c-1)\cdot L}{T} + \frac{\hat{k}^5\cdot s\cdot\tilde{s}\cdot\tilde{c}^2(c-1)\cdot L}{T^2}$$

$$m_{22} = \frac{-\phi\tilde{c}^2\hat{k}(s-\hat{k}c)\cdot L^2}{T} - \frac{L^2\tilde{s}\hat{k}^4\left[s(s-\hat{k}c)+\tilde{s}(c-1)^2\right]}{T^2}$$

$$m_{33} = \phi^2\tilde{c}^2\tilde{s}^2 - \frac{\phi\hat{k}^3 s(\tilde{c}^4+\tilde{s}^3)}{T} - \frac{\hat{k}^6 s^2\cdot\tilde{s}\cdot\tilde{c}}{T^2}$$

$$D = \frac{8E^3I^3}{TL^3}\left[\phi\tilde{s}^2 - \frac{\hat{k}^3 s\cdot\tilde{c}^2}{T}\right]\left[-\phi\tilde{c}^2\hat{k}(s-\hat{k}c)L^2 - \frac{L^2\hat{k}^4\cdot\tilde{s}\left[s(s-\hat{k}c)+\tilde{s}(c-1)^2\right]}{T}\right]$$

The nodal forces matrix is the same as Equation 5-3

$$\{Q\} = \begin{Bmatrix} 0 \\ -Q \\ 0 \end{Bmatrix}$$

The nodal displacement matrix $\{\Delta\}$ is computed as:

$$\{\Delta\} = \begin{Bmatrix} 0 \\ \dfrac{Q}{2\frac{EI}{L^3}\left[\phi\bar{s}^2 - \frac{k^3 s\cdot\bar{c}^2}{T}\right]} \\ 0 \end{Bmatrix} \tag{5-9}$$

The values of the member forces obtain:

$$\{P\} = \frac{Q}{2\left[\phi\bar{s}^2 - \frac{k^3 s\bar{c}^2}{T}\right]} \begin{Bmatrix} -\phi\bar{c}\bar{s} - \frac{k^3 s\cdot\bar{s}\cdot\bar{c}}{T} \\ -\phi\bar{s}^2 + \frac{k^3\bar{c}^2\cdot s}{T} \\ \frac{k^2 L\cdot\bar{c}(c-1)}{T} \\ \phi\bar{c}\cdot\bar{s} + \frac{k^3 s\cdot\bar{s}\bar{c}}{T} \\ \phi\bar{s}^2 - \frac{k^3 s\bar{c}^2}{T} \\ \frac{k^2 L\cdot\bar{c}(c-1)}{T} \\ \phi\bar{c}\bar{s} + \frac{k^3 s\cdot\bar{s}\bar{c}}{T} \\ -\phi s^2 + \frac{k^3 s\cdot\bar{c}^2}{T} \\ -\frac{k^2 L\cdot\bar{c}(c-1)}{T} \\ -\phi\bar{s}\bar{c} - \frac{k^3 s\cdot\bar{c}\bar{s}}{T} \\ \phi\bar{s}^2 - \frac{k^3 s\bar{c}^2}{T} \\ \frac{-k^2 L\bar{c}(c-1)}{T} \end{Bmatrix}$$

The scalar relation between $\hat{Q}$ and $\frac{v_B}{L}$ is given by Equation 5-9, as

$$\left\{\frac{v_B}{L}\right\} = \frac{\hat{Q}}{2\left[\phi\bar{s}^2 - k^3 s\bar{c}^2/T\right]} \tag{5-10}$$

Fig. 5-10 shows the $Q - (\frac{v_B}{L})$ relationship of Equation 5-10 for different values of the slope angle θ with ϕ equal to 10^4.

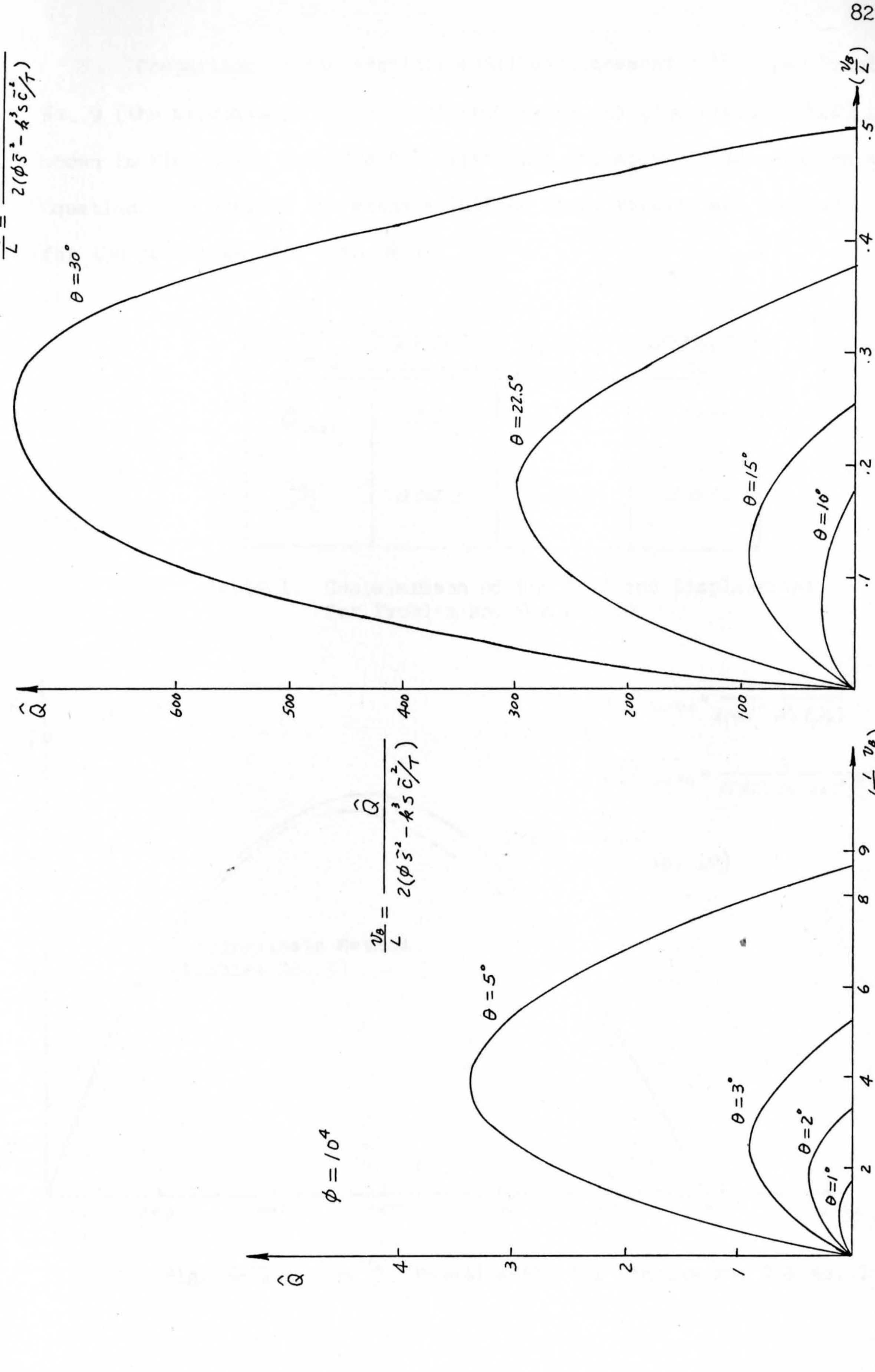


Fig. 5-10 $\hat{Q}$ - (v_B/L) Relationship for problem No. 10

Comparison of the vertical nodal displacement (v_B/L) for Problem No. 9 (the approximate method) and Problem No. 10 (the exact method) is shown in Fig. 5-11. Equation 5-7 represents the approximate solution and Equation 5-10 defines the exact solution. These results are plotted for the geometric case with $\theta = 5°$.

$\theta = 5°$	EXACT. (Pro. #10)	APPROX. (Pro. # 9)	DIFFERENCE %
$\hat{Q}_{max.}$	3.53	3.49	1.1 %
v_B/L	0.0415	0.040	3.6 %

Table 1. Compararison of the Load and Displacement for Problem No. 9 & No. 10

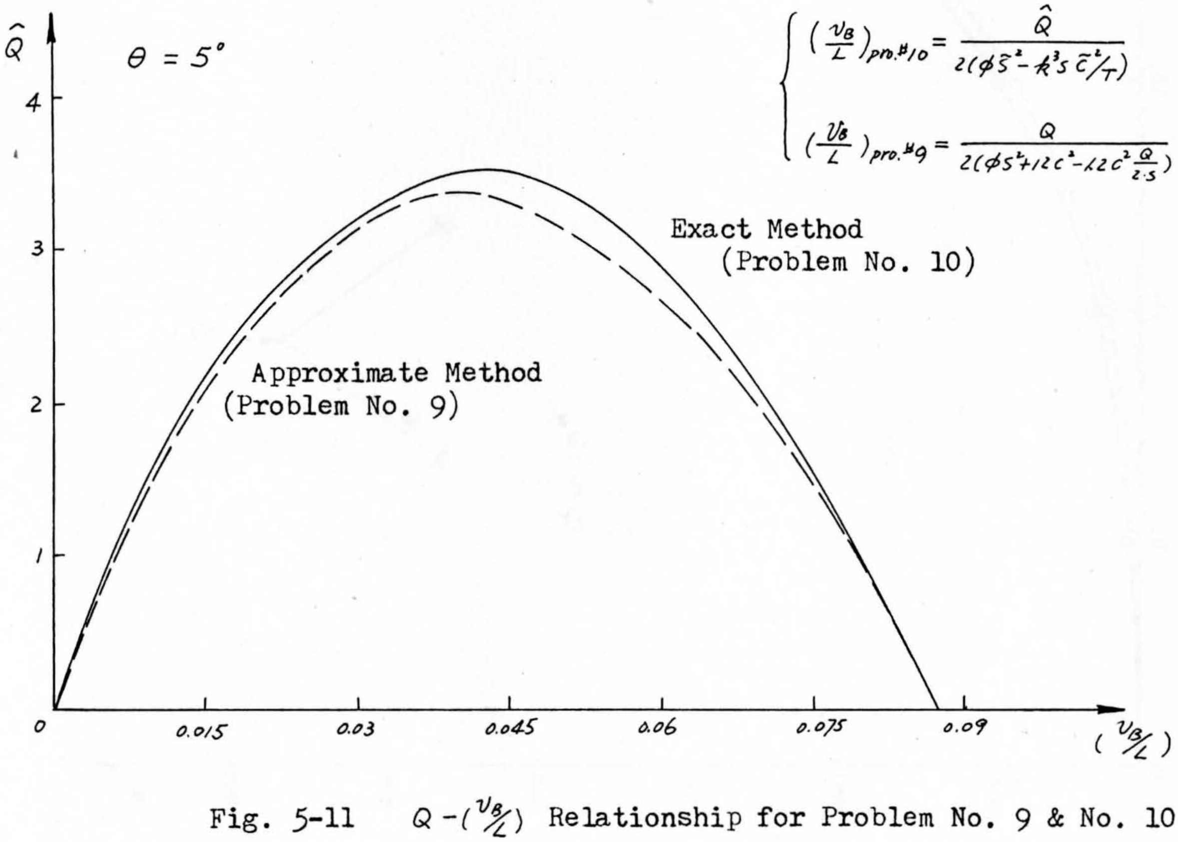


Fig. 5-11 $Q-(v_B/L)$ Relationship for Problem No. 9 & No. 10

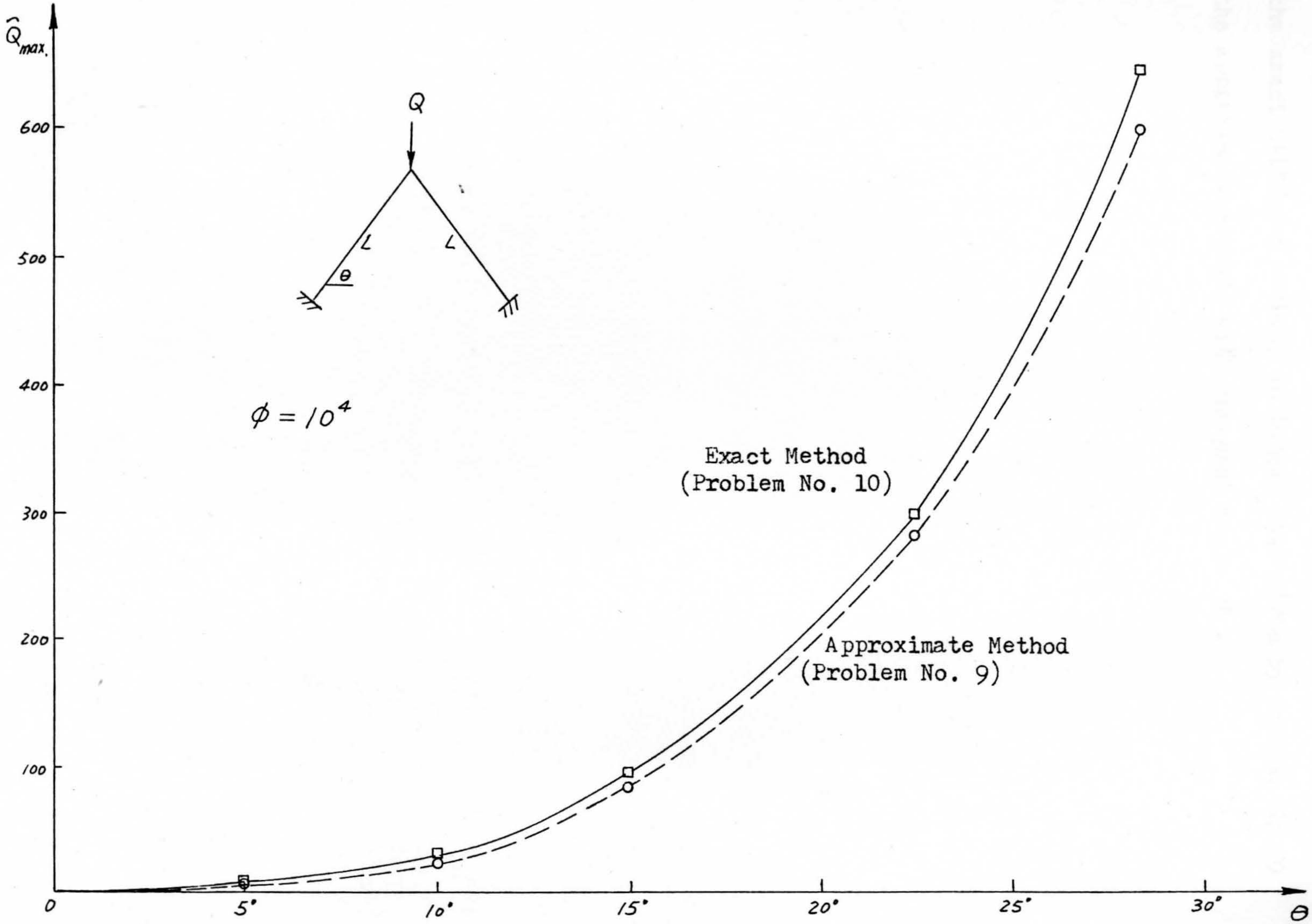


Fig. 5-12 $\bar{Q}_{max} - \theta$ Relationship for Problem No. 9 & No. 10

A comparison plot of $\hat{Q}_{max.}$ from Fig. 5-9, and Fig. 5-10 is shown in Fig. 5-12. It may be seen that $\hat{Q}_{max.}$ given by Problem No. 10 (the exact stiffness method) is larger than given by Problem No. 9 (the approximate method) with the same angle θ .

CHAPTER VI

DISCUSSION AND CONCLUSION

6.1 Discussion

In analysis of the orthogonal planar frame, if the stiffness matrix $\left[\hat{K}\right]$ is taken as the elastic stiffness matrix $\left[\hat{K}\right]_E$ only, this leads to the linear solutions relating forces and displacements. If both the elastic stiffness matrix $\left[\hat{K}\right]_E$ and the geometrical stiffness $\left[\bar{K}\right]_G$ are utilized, that is, the effect of axial force on bending is included, this leads to a non-linear solutions relating forces and displacements. If the effect of axial stiffness $\left(\frac{AE}{L}\right)$ is excluded from the analysis, the solution of the bending problem is not effected. For the static buckling problem, the number of critical buckling loads and associated mode shapes is equal to the order of the condensed stiffness matrix. In the biaxial critical load condition, the value of the critical buckling load is determined by geometrical interaction between both axial forces which produces a condition of the lowest critical buckling load being associated with a higher order mode shape. From Fig. 3-6, the left section of the solid line shown represents the lowest critical buckling load defined for the first mode shape. The right section represents the lowest critical buckling load in the second mode shape; the intersection of two curves is approximately at $\left\{\begin{matrix} Q = 31.5 \\ P = 6.0 \end{matrix}\right\}$ which is obtained numerically in Appendix I. It can be seen that the critical buckling load is lower when biaxial forces are applied.

In the symmetric orthogonal frame, the geometrical stiffness matrix is included in the column members only when the load is applied vertically on the node. In horizontal members the geometrical stiffness matrix may be excluded. For the case where the geometrical stiffness is included, the effect is that nonlinear force-displacement conditions result as shown in Fig. 3-14 & Fig. 3-15. All values of shear forces, moments and displacements versus the same applied load show nonlinear characteristics with values higher than those given by the linear analysis. This shows these solutions are more accurate than linear solutions which are only first approximations to the actual solutions.

For the nonorthogonal frame, neglecting the axial stiffness term (i.e. $\frac{AE}{L} \rightarrow 0$), the solution mathematically satisfies the equilibrium but physically leads to an unacceptable solution. The axial stiffness in nonorthogonal frames causes coupling which effects the bending situation, so, it must be included in the stiffness matrix $[\tilde{K}]_E$. For orthogonal frames, this coupling does not occur and hence the $\frac{AE}{L}$ term may be neglected in the analysis reducing the stiffness matrix from a (6x6) matrix to a (4x4) matrix. The displacement-load relationship for the nonorthogonal frames exists only when $\frac{AE}{L}$ has a reasonable numerical value. Axial displacements become zero as ϕ approaches infinity and v is indefinite as ϕ approaches zero.

For the symmetrical nonorthogonal node-loaded frames, the larger the inclination angle θ with the horizontal, the greater the applied load that is supported for a given displacement. Fig. 5-7 shows that frames with greater axial stiffness support higher loads. For the same problem, the answer obtained by use of the exact stiffness matrix always yields higher values of load carrying capacity than that obtained by the approximate stiffness method.

6.2 Conclusion

The total stiffness matrix $\left[K\right]$ contains not only elastic bending stiffness $\left[\hat{K}\right]_E$ and geometrical stiffness $\left[\hat{K}\right]_G$ but also must include the axial stiffness components $\frac{AE}{L}$. The exact theory yields a stiffness matrix with trigonometric elements which may be expanded in infinite series in the form:

$$\left[K\right] \approx \left[\hat{K}\right]_E - P\left[\hat{K}\right]_G - \cdots\cdots$$

For convenience, this two term form is used as the approximate stiffness matrix which shows close correlation to that given by the exact form (see Fig. 5-11).

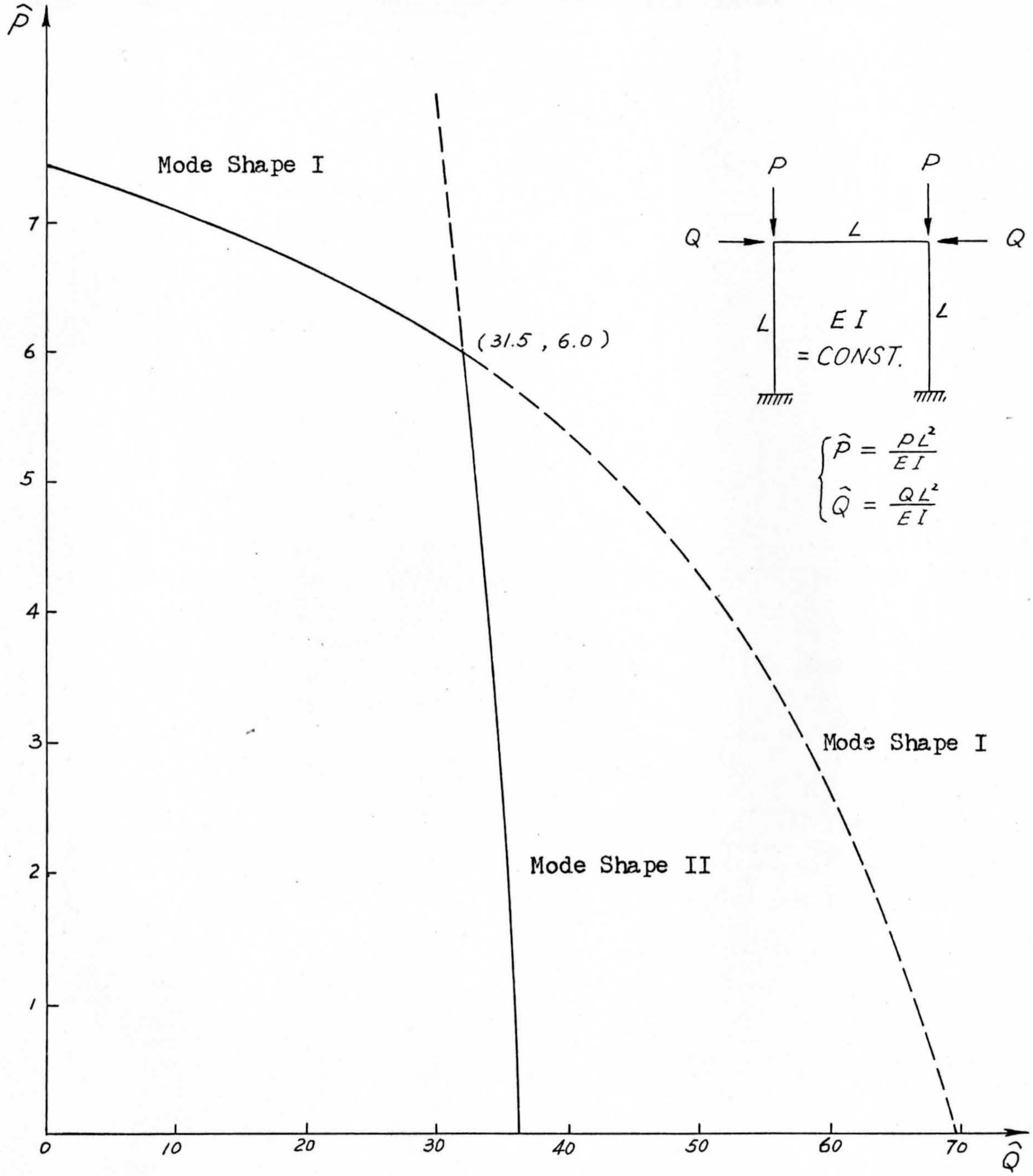
For symmetrically loaded orthogonal frames the $\frac{AE}{L}$ terms may be excluded from the analysis with no resulting complications in the solutions. For symmetrically loaded nonorthogonal frames, the axial stiffness component $\frac{AE}{L}$ must be included in order to produce reasonable numerical results. Mathematical analysis of nonorthogonal frames is much more complicated and a reliance on matrix techniques is an absolute necessity.

APPENDIX I

Equation 3-6 shows:

$$-\frac{1}{1500}(60\hat{P}^3+123\hat{P}^2\hat{Q}+60\hat{P}\hat{Q}^2)+\frac{1}{75}(383\hat{P}^2+442\hat{P}\hat{Q}+30\hat{Q}^2)$$
$$-\frac{1}{5}(856\hat{P}+212\hat{Q})+1008=0$$

The relationship between P and Q is plotted as follows:



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