STABILITY TESTS FOR TWO-DIMENSIONAL RECURSIVE FILTERS

by

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ABSTRACT

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The main objective of this work is to review the algebraic stability tests of two-dimensional recursive filters. Both frequency-domain methods and the data-domain method are presented. For the frequency-domain method, various existing algebraic methods are discussed. These include the Shanks, Huang, Maria-Fahmy, and Anderson-Jury methods. For the data-domain method, the extension of the Lyapunov theorem is presented including an approximate algebraic test. Both frequency-domain and data-domain proof are given for the approximate test. Some properties of a two-dimensional system which are different from a one-dimensional system are included. Several filters are evaluated by both methods and the agreement of the results is indicated.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
A	Capital letter denotes matrix
BIBO	Bounded-input bounded-output
E	Energy
FIR	Finite impulse response
IIR	Infinite impulse response
LSI	Linear shift-invariant
R^n	n-dimensional real vector space
$T[\cdot]$	The transformation operator
A^T	The transpose of the matrix A
$Z[\cdot]$	The z-transform operator
1-D	One-dimensional
2-D	Two-dimensional
$\in$	Belongs to
a^* or $\bar{a}$	The complex conjugate of a
A*B	A convolute with B
f'	The derivative of function f
$\|A\|$	The determinant of matrix A
$\equiv$	Equal by definition
$>$	Greater than
$\geqslant$	Greater than or equal to
$<$	Less than
$\leqslant$	Less than or equal to
$\sum$	The summation sign
$\oint_C$	The integration around the closed curve C

SYMBOL	DEFINITION
$\lambda[A]$	The eigenvalue of the matrix A
$\delta(m,n)$	2-D unit-sample sequence
$[a,b]$	The interval from a to b

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CHAPTER I

INTRODUCTION

Recently, two-dimensional signal processing has found a wide application in many fields. Many techniques have been employed in implementation. However, the recursive technique is one of the most important classes since it is shown to be the most efficient method. In designing the two-dimensional recursive filter, the designer is faced with two major problems, synthesis and stability. In this paper, only the latter problem will be discussed. The main goal of this work is to review the algebraic stability tests, including both frequency-domain and data-domain methods. The frequency-domain method is quite established [1-6]. However, the data-domain method is not successfully extended even in the scalar case. Recently, the extension of the Lyapunov theorem and an associate approximate test is given by Sendaula [7]. Since the stability of any type of recursive filter can be determined from the stability of the first-quadrant filter (or quarter-plane filter) by a suitable mapping of the original filter, only the stability test of the first-quadrant filter will be presented.

This paper is divided into five chapters. In Chapter II, the general theory of two-dimension signal and processing is presented. Most of the material is a straightforward extension of the one-dimensional case. Some special properties of the two-dimensional system are also indicated. A review of the existing algebraic stability test is presented in Chapter III. All the methods in this chapter employ the frequency domain technique. The data-domain technique is presented in Chapter IV. In this chapter, the state-space representation of the two-dimensional system is included. Next, the extension of the Lyapunov theorem of the two-dimensional system is introduced. Then the translation of the stability theorem to an approximate stability test is given. The proofs of the approximate test are given, both frequency-domain and data-domain proof. It is shown that this method yields the same result as the other methods in Chapter III. The conclusion is in the final chapter.

CHAPTER II

AN INTRODUCTION TO THE THEORY OF TWO-DIMENSIONAL SIGNAL PROCESSING

2-1 Introduction

There are many signals that are two-dimensional signals in nature, for example photographic data, for which two-dimensional signal processing techniques are required. Since the one-dimensional system is a special case of the two-dimensional system or multi-dimensional system, some properties of the two-dimensional system are just a straightforward extension of the one-dimensional system. Some are unique properties of the two-dimensional system which are not similar to the one-dimensional system. In this chapter, the fundamental theorem of the two-dimensional signal and system is summarized with the emphasis placed on the linear shift-invariant system (LSI). For more details see [8-10].

2-2 2-D Sequences

A two-dimensional (2-D) sequence is a function of two integer variables. As in the one-dimensional (1-D) case, it is useful to define the unit-sample and unit-step. The 2-D unit-sample sequence $\delta(m,n)$, usually referred to as discrete time impulse or simply impulse, is defined as

$$\delta(m,n) = \begin{cases} 1, & m=n=0 \\ 0, & \text{otherwise.} \end{cases} \qquad (2\text{-}1)$$

The 2-D unit-step sequence u(m,n) is defined as:

$$u(m,n) = \begin{cases} 1, & m \geqslant 0,\ n \geqslant 0 \\ 0, & \text{otherwise} \end{cases} \qquad (2\text{-}2)$$

A 2-D sequence is called a separable sequence if it can be expressed as a product of 1-D sequence in the form:

$$x(m,n) = x_1(m)x_2(n) \qquad (2\text{-}3)$$

It is sometimes useful to refer to the energy in a sequence. The energy E in a sequence x(m,n) is defined as:

$$E = \sum_m \sum_n |x(m,n)|^2 \qquad (2\text{-}4)$$

2-3 2-D Linear Shift-Invariant Systems (2-D LSI Systems)

A system is defined as a transformation or operator that maps an input sequence x(m,n) into output y(m,n). This is denoted as:

$$y(m,n) = T[x(m,n)] \qquad (2\text{-}5)$$

A system is said to be linear if $y_1(m,n)$ and $y_2(m,n)$ are the response of the system when the input, $x_1(m,n)$ and $x_2(m,n)$, respectively satisfy the relation

$$T[ax_1(m,n) + bx_2(m,n)] = aT[x_1(m,n)] + bT[x_2(m,n)]$$
$$= ay_1(m,n) + by_2(m,n) \qquad (2\text{-}6)$$

for any arbitrary constant a, b.

A system is said to be shift-invariant if and only if it satisfies

$$y(m-m_0, n-n_0) = T[x(m-m_0, n-n_0)] \qquad (2\text{-}7)$$

for all x and arbitrary integer m_0, n_0 where y(m,n) is the output of the system when the input is x(m,n).

A causal system is one for which the output for any $m=m_0$, $n=n_0$ depends on the input for $m \leq m_0$, $n \leq n_0$ only. From now on if a system is mentioned, it means a causal system if not stated otherwise.

As in the 1-D system, the 2-D LSI system can be completely specified by its impulse response h(m,n). The impulse response is the output of the system when the input is a 2-D unit-sample $\delta(m,n)$ as defined above. Moreover, the output y(m,n) of the 2-D LSI system is the convolution of the input sequence and the impulse response h(m,n), i.e.,

$$y(m,n) = \sum_{k}\sum_{l} x(k,l)h(m-k,n-l) \qquad (2\text{-}8)$$

or

$$y(m,n) = x(m,n)*h(m,n) \qquad (2\text{-}9)$$

where * represents a 2-D convolution.

A large class of the LSI system can be described by a linear difference equation:

$$\sum_{k=}^{M_1}\sum_{l=0}^{N_1} b_{k,l}y(m-k,n-l) = \sum_{k=}^{M_2}\sum_{l=0}^{N_2} a_{k,l}x(m-k,n-l) \qquad (2\text{-}10)$$

Generally, this class of systems need not be causal. Throughout this paper only this class of systems which are causal will be discussed. For this class of systems, the output y(m,n) can be computed recursively from the input x(m,n) and a set of initial conditions. This can be done by rewriting equation (2-10) as:

$$y(m,n) = (1/b_{i,j}) \sum_{k=l=0}^{M_2} \sum^{N_2} a_{k,l} x(m-k,n-l)$$

$$- (1/b_{i,j}) \sum_{\substack{k=l=0 \\ k,l \neq 0 \text{ simultaneously}}} \sum b_{k,l} y(m-k,n-l) \qquad (2\text{-}11)$$

The filter that is in this class is known as the recursive filter or infinite impulse response (IIR filter).

2-4 2-D Z-Transform

The 2-D z-transform $X(z_1,z_2)$ of a sequence $x(m,n)$ is defined as:

$$X(z_1,z_2) = \sum_m \sum_n x(m,n) z_1^{-m} z_2^{-n} \qquad (2\text{-}12)$$

where z_1 and z_2 are complex variables.

Note that many authors employ a slightly different definition of 2-D z-transform in their literature [1-3,5,6]. The 2-D z-transform $X(z_1,z_2)$ of 2-D sequence $x(m,n)$ is defined as:

$$X(z_1,z_2) = \sum_m \sum_n x(m,n) z_1^{m} z_2^{n} \qquad (2\text{-}13)$$

However, the first definition (2-12) will be employed here.

The inverse 2-D z-transform is given by the contour integral

$$x(m,n) = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} X(z_1,z_2) z_1^{m-1} z_2^{n-1} \, dz_1 dz_2 \qquad (2\text{-}14)$$

where the contours C_1 and C_2 are closed contours encircling the origin and are within the region of convergence.

A 2-D z-transform $X(z_1,z_2)$ is said to be separable if it can be expressed in the form:

$$X(z_1,z_2) = X_1(z_1)X_2(z_2) \qquad (2\text{-}15)$$

$X(z_1,z_2)$ will be separable if and only if the sequence $x(m,n)$ is a separable sequence. Generally, $x(m,n)$ and $X(z_1,z_2)$ are not separable.

Properties of 2-D Z-Transform

Let

$$Z[x(m,n)] = X(z_1,z_2) \qquad (2\text{-}16)$$

$$Z[y(m,n)] = Y(z_1,z_2) \qquad (2\text{-}17)$$

where $Z[\cdot]$ denotes the z-transform of the sequence inside. Some properties of the 2-D z-transform are summarized as follows:

1. Linearity

$$Z[ax(m,n) + by(m,n)] = aX(z_1,z_2) + bY(z_1,z_2) \qquad (2\text{-}18)$$

2. Shift of a sequence

$$Z[x(m+m_0,n+n_0)] = z_1^{m_0} z_2^{n_0} X(z_1,z_2) \qquad (2\text{-}19)$$

for any integer m_0, n_0

3. Multiplication by an exponential sequence

$$Z[a^m b^n x(m,n)] = X(a^{-1}z_1, b^{-1}z_2) \qquad (2\text{-}20)$$

4. Differentiation of $X(z_1,z_2)$

$$Z[mnx(m,n)] = \frac{d^2X(z_1,z_2)}{dz_1 dz_2} \qquad (2\text{-}21)$$

5. Conjugation of a complex sequence

$$Z[x^*(m,n)] = X^*(z_1^*, z_2^*) \quad (2\text{-}22)$$

where * denotes complex conjugate

6. $$Z[x(-m,-n)] = X(z_1^{-1}, z_2^{-1}) \quad (2\text{-}23)$$

7. Convolution of sequence

$$Z[x(m,n) * y(m,n)] = X(z_1,z_2)Y(z_1,z_2) \quad (2\text{-}24)$$

8. Parseval's relation

$$Z\left[\sum_m \sum_n x(m,n)y^*(m,n)\right] = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} X(v_1,v_2)$$

$$Y^*(v_1^{-1}, v_2^{-1}) v_1^{-1} v_2^{-1} \, dv_1 dv_2. \quad (2\text{-}25)$$

Since the 2-D z-transform of a convolution of two 2-D sequences is the product of their z-transforms, the input-output relation for a 2-D LSI system, expressed in terms of the z-transform, corresponds to a multiplication of the z-transforms of the input and the unit-sample response. The z-transform of the unit-sample is referred to as the system function or transfer function.

For a system that can be described by a linear constant-coefficient difference equation, the transfer function is a ratio of two variable polynomials, in particular, for the system that satisfies the difference equation

$$\sum_{k=l=0}^{M_1} \sum^{N_1} b_{k,l} y(m-k,n-l) = \sum_{k=l=0}^{M_2} \sum^{N_2} a_{k,l} x(m-k,n-l). \quad (2\text{-}26)$$

If the 2-D z-transform is applied to both sides of (2-26), it follows

$$Y(z_1,z_2)\left[\sum_{k=l=0}^{M_1}\sum^{N_1} b_{k,l} z_1^{-k} z_2^{-l}\right] = X(z_1,z_2)\left[\sum_{k=l=0}^{M_2}\sum^{N_2} a_{k,l} z_1^{-k} z_2^{-l}\right] \quad (2\text{-}27)$$

so that the transfer function $H(z_1,z_2) = Y(z_1,z_2)/X(z_1,z_2)$ is given by:

$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{\sum_{k=l=0}^{M_2}\sum^{N_2} a_{k,l} z_1^{-k} z_2^{-l}}{\sum_{k=l=0}^{M_1}\sum^{N_2} b_{k,l} z_1^{-k} z_2^{-l}} \quad (2\text{-}28)$$

In 1-D case, when the transfer function consists of a ratio of polynomials, it could be described in terms of poles and zeroes i.e., the root of denominator and numerator. In contrast, a general two variable polynomial can not be factored into first order polynomials, rather, a two variable polynomial can be factored into irreducible factors which are themselves two variable polynomials which can not be further factored. This problem sometimes is referred to in literature as root clustering in a complex plane which is opposite to the isolate singularity in the 1-D case. This problem makes the stability problem in the 2-D case much more difficult than the 1-D case and this is the major difference between the 1-D and 2-D systems.

CHAPTER III

STABILITY TESTS FOR 2-D RECURSIVE FILTERS: FREQUENCY-DOMAIN METHOD

3-1 Introduction

As noted in Chapter II, since the output of the recursive filter is the sum of the portion of the past inputs and outputs, it is possible for the output value become very large independent of the input. Therefore, the stability problem is one of the major problems in the 2-D filter. In this chapter, the existing algebraic methods will be reviewed. All the methods in this chapter employ the frequency-domain method or transform-method.

As in the 1-D case the concept of bounded-input bounded-output (BIBO) stability will be employed. It can be shown that the 2-D LSI system is stable if and only if

$$S \equiv \sum_k \sum_l |h(k,l)| < \infty \qquad (3\text{-}1)$$

i.e., the summability of the impulse response (see, for example [9]).

3-2 Shanks' Method

The first stability theorem for 2-D filter was introduced by Shanks [1]. The theorem can be restated as follows:

Theorem 3-1 (Shanks'):

A recursive (IIR) filter,

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)}, \qquad (3\text{-}2)$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are polynomial in z_1 and z_2, is BIBO stable if and only if there are no values of z_1 and z_2 such that $B(z_1,z_2) = 0$ for $|z_1| \geqslant 1$ and $|z_2| \geqslant 1$ simultaneously.

To apply Shanks' theorem is conceptually straightforward but computationally involved. One way to do this is to map $d_1 \equiv (z_1; |z_1| \geqslant 1)$ in z_1-plane into z_2-plane by the implicit mapping relation $B(z_1,z_2) = 0$. The filter is stable if and only if the image of d_1 in the z_2-plane completely lies inside the unit circle in the z_2-plane. Note that this method is not finite in its number of steps of calculation since the whole plane $d_1 \equiv (z_1; |z_1| \geqslant 1)$ is mapped into z_2-plane

Remarks

Generally, before applying Shanks' theorem to the system which can be described by (3-2), $A(z_1,z_2)$ and $B(z_1,z_2)$ are relatively prime i.e., there is no common factor between $A(z_1,z_2)$ and $B(z_1,z_2)$. However, there are two types of singularity for two variable rational function $H(z_1,z_2)$ [11]. The first type is called a pole or a nonessential singularity of the first kind which is a point (z_1,z_2) such that $B(z_1,z_2) = 0$ but $A(z_1,z_2) \neq 0$. This type of singularity is similar to the 1-D case. The second type is called a nonessential singularity of the second kind which is a point (z_1,z_2) such that $A(z_1,z_2) = B(z_1,z_2) = 0$. For this kind there is no 1-D analog. In this kind, there are no common factors that can be canceled out like in the 1-D case. For example, in

$$H(z_1,z_2) = \frac{(1 - z_1)(1 - z_2)}{2 - z_1 - z_2} \quad (3\text{-}3)$$

there is a nonessential singularity of the second kind at $z_1 = z_2 = 1$, where $H(z_1,z_2)$ is undefined.

It was shown by Goodman [11] that Shanks' theorem is essentially correct except the case may arise where $H(z_1,z_2)$ has a nonessential singularity of the second kind on T^2 where $T^2 = \{(z_1,z_2); |z_1| = 1, |z_2| = 1\}$. Therefore, Shanks' theorem can be modified as follows:

<u>Theorem 3-2</u>

A recursive filter, which is described by (3-2), is stable if there is no point (z_1,z_2) such that $B(z_1,z_2) = 0$ and $|z_1|$ and $|z_2|$ are greater than or equal to one simultaneously except possibly on $T^2 = \{(z_1,z_2);\ |z_1| = 1, |z_2| = 1\}$.

For some examples see Goodman [11].

<u>3-3 Huang's Method</u>

Huang [2] recognized that in mapping $d_1 = (z_1; |z_1| \geq 1)$ in z_1-plane into z_2 plane by the relation

$$B(z_1,z_2) = 0 \quad (3\text{-}4)$$

or

$$z_2 = f(z_1) \quad (3\text{-}5)$$

the extremum values of z_2 occur at $\partial d_1 = (z_1; |z_1| = 1)$.

Therefore, it is not necessary to map the whole d_1 to z_2-plane. Huang's theorem can be stated as follows:

Theorem 3-3 (Huang)

A causal recursive (IIR) filter with

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \qquad (3\text{-}6)$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are polynomials in z_1 and z_2 is stable if and only if

i. the map of $\partial d_1' = (z_1; |z_1| = 1)$ in the z_2-plane according to the implicit relation $B(z_1,z_2) = 0$, lies inside of $d_2 = (z_2; |z_2|<1)$, and

ii. no point in $d_1 = (z_1; |z_1| \geq 1)$ maps into the point $z_2 = \infty$ by the relation $z_2^{-n}B(z_1,z_2) = 0$, where n is the order of z_2 in $B(z_1,z_2)$.

For the proof see [12-14]. In applying Theorem 3-3, Huang suggested using bilinear transform by substituting

$$s_1 = \frac{z_1 - 1}{z_1 + 1} \qquad (3\text{-}7)$$

and

$$s_2 = \frac{z_2 - 1}{z_2 + 1} \qquad (3\text{-}8)$$

in (3-6) which becomes:

$$H(z_1,z_2) = \frac{C(s_1,s_2)}{D(s_1,s_2)} \qquad (3\text{-}9)$$

where C and D are polynomial in s_1 and s_2. Since the bilinear transformation transforms the inside of the unit circle in the z-plane to the left-half of the s-plane, the outside of the unit circle in the z-plane to the right-half of the s-plane, and the unit circle into the imaginary axis, then Theorem 3-3 can be restated as follows:

Theorem 3-4

A causal recursive filter $H(z_1,z_2)$ is stable if and only if

i. for all real finite ω_1 the complex polynomial in s_2, $D(j\omega_1,s_2)$ has no zero in the right-half of s_2-plane, and

ii. the real polynomial in s_1, $D(s_1,1)$ has no zero in the right-half of s_1-plane.

From Theorem 4-4, the second condition can be tested by many well-known criterions such as Hermite, Routh, Hurwitz, etc., since it is a one variable polynomial. For the first condition, $D(j\omega_1,s_2)$ can be written as a one variable polynomial with complex coefficients by regarding ω_1 as a parameter. Then, apply Hermite theorem [15], the first condition can be restated as follows:

Theorem 3-5

The first condition of Theorem 3-4 is equivalent to the following: Let $s_2 = j\omega_2$, express $D(j\omega_1,j\omega_2)$ in the forms:

$$D(j\omega_1,j\omega_2) = b_0(\omega_1)\omega_2^n + b_1(\omega_1)\omega_2^{n-1} + \dots + b_n(\omega_1) + j[a_0(\omega_1)\omega_2^n + a_1(\omega_1)\omega_2^{n-1} + \dots + a_n(\omega_1)]. \quad (3\text{-}10)$$

Where ω_1 and ω_2 are real, $a_i(\omega_1)$ and $b_i(\omega_1)$ are real polynomial in ω_1, and neither $a_0(\omega_1)$ nor $b_0(\omega_1)$ is zero. Let $H_{r,s}(\omega_1)$ be defined as:

$$H_{r,s} = a_r b_s - a_s b_r \qquad (3\text{-}11)$$

for $0 \leqslant r,s \leqslant n$. Let $D(\omega_1)$ denotes the n x n symmetrical polynomial matrix whose element $D_{i,j}(\omega_1)$, $(1 \leqslant i,j \leqslant n)$ is the sum of all those $H_{r,s}(\omega_1)$, $(0 \leqslant r,s \leqslant n)$ for which both

$$s + r = i + j - 1 \qquad (3\text{-}12)$$

and

$$s - r > i - j \qquad (3\text{-}13)$$

are satisfied. Then, the n successive principal minors of $D(\omega_1)$ must be positive for all real ω_1.

Note that each minor of $D(\omega_1)$ is polynomial in ω_1. Sturm's method can be employed to test whether each minor is positive for all real ω_1.

Sturm's Method

The polynomial p(x),

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \qquad (3\text{-}14)$$

is positive for all x in the interval [a,b] if and only if p(x) does not have zero of odd multiplicity in that interval. The number of zeroes in any interval can be determined as follows: Let

$$f_0 = p(x) \qquad (3\text{-}15)$$

$$f_1 = f_0' \qquad (3\text{-}16)$$

$$f_0 = q_1 f_1 + f_2$$

$$f_1 = q_2 f_2 + f_3$$

$$f_2 = q_3 f_3 + f_4$$

$$\dots\dots\dots\dots\dots \quad (3\text{-}17)$$

where the ' denotes derivative. The sequence f_1, f_2, f_3, ... , f_n is called the Sturm sequence. The number of zeroes in the interval a,b is equal to $v_a - v_b$ where v_a and v_b are the numbers of sign variation in the Sturm sequence when x is equaled to a and b respectively. In constructing Sturm sequence, if the process is terminated early at f_i $(i<n)$, f_i is the common factor of f_1 and f_0. If f_i is simple, the multiplicity of zero of f_i can be investigated. If f_i is too complicated, Sturm's method can be applied to it separately [16].

Example

Consider the filter

$$H(z_1,z_2) = \frac{1}{z_1 z_2 + a_0 + a_1 z_1 + a_2 z_2} \quad (3\text{-}18)$$

Applying Theorem 3-3

$$z_1 z_2 + a_0 + a_1 z_1 + a_2 z_2 = 0 \quad (3\text{-}19)$$

or

$$z_2 = \frac{a_0 + a_1 z_1}{z_1 + a_2} \quad (3\text{-}20)$$

equation (3-20) is the bilinear transformation which maps circle into circle. The image of the unit circle $\partial d_1 = (z_1 : |z_1| = 1)$ in the z_2-plane is then a circle. From (3-20), the center of this image circle is on real axis, and it intersects the real axis at

$$z_2 = -\frac{a_0 + a_1}{1 + a_2} \quad (3\text{-}21)$$

and

$$z_2 = -\frac{a_0 + a_1}{-1 + a_2} = -\frac{a_1 - a_0}{1 - a_2}$$

(3-22)

Then, the first condition of Theorem 3-3 is satisfied if and only if

$$\left|\frac{a_0 + a_1}{1 + a_2}\right| < 1 \quad (3\text{-}23)$$

$$\left|\frac{a_1 - a_0}{1 - a_2}\right| < 1$$

The second condition,

$$z_2^{-1}B(z_1,z_2) = z_1 + a_0 z_2^{-1} + a_1 z_1 z_2^{-1} + a_2$$

$$= 0 \quad (3\text{-}25)$$

$$\lim_{z_2 \to \infty} z_2^{-1}B(z_1,z_2) = z_1 + a_2 = 0 \quad (3\text{-}26)$$

$$z_1 = -a_2,$$

is satisfied if

$$|a_2| < 1 \quad (3\text{-}27)$$

Therefore, the filter (3-18) is stable if the inequalities (3-23), (3-24), and (3-27) are satisfied.

Huang's theorem (Theorem 3-3) can be generalized as follows [5]:

Theorem 3-6 (Strintzis)

The filter $H(z_1,z_2)$ where

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \qquad (3\text{-}28)$$

is BIBO stable if and only if

i. for some a, $|a| \geq 1$, $B(a,z_2) \neq 0$ when $|z_2| \geq 1$ (3-29)

ii. $B(z_1,z_2) \neq 0$, when $|z_1| \geq 1$ and $|z_2| = 1$ (3-30)

or under the following conditions:

i. for some a, $|a| \geq 1$, $B(a,z_2) \neq 0$, when $|z_2| \geq 1$ (3-31)

ii. for some b, $|b| = 1$, $B(z_1,b) \neq 0$, when $|z_1| \geq 1$ (3-32)

iii. $B(z_1,z_2) \neq 0$, when $|z_1| = |z_2| = 1$ (3-33)

3-4 Z-Plane Method

Anderson-Jury [3] and Maria-Fahmy [4] used Huang's theorem in testing stability. Instead of using bilinear transform, either Schur-Cohn matrix or Jury Table was employed. In [4], the procedure was based on the following theorem:

Theorem 3-7 (modified Jury Table)

Let $F(z)$ be the n^{th} degree polynomial given by

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \qquad (3\text{-}34)$$

where the coefficeints a_i, $i = 0,1,2,\ldots,n$ are complex numbers. The roots of $F(z)$ are inside the unit circle if and only if

$$b_0 < 0,\ c_0 > 0,\ d_0 > 0,\ \ldots,\ g_0 > 0,\ \ldots,\ t_0 > 0$$

where b_0, c_0, d_0, ..., t_0 are obtained as follows:

$$\begin{array}{llllll}
z^0 & z^1 & z^2 \ \ldots & z^{n-2} & z^{n-1} & z^n \\
a_0 & a_1 & a_2 & a_{n-2} & a_{n-1} & a_n \\
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 \\
b_0 & b_1 & b_2 & b_{n-2} & b_{n-1} & \\
\bar{b}_{n-1} & \bar{b}_{n-2} & \bar{b}_{n-3} & \bar{b}_1 & \bar{b}_0 & \\
c_0 & c_1 & c_2 & c_{n-2} & & \\
\bar{c}_{n-2} & \bar{c}_{n-3} & \bar{c}_{n-4} & \bar{c}_0 & & \\
d_0 & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
r_0 & r_1 & & & & \\
\bar{r}_1 & \bar{r}_0 & & & & \\
t_0 & & & & &
\end{array}$$

where

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ \bar{a}_n & \bar{a}_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ \bar{b}_{n-1} & \bar{b}_k \end{vmatrix}, \ \ldots,$$

and $\bar{a}_k$ is the complex conjugate of a_k.

To check the first condition in Huang's theorem (Theorem 3-3), $B(z_1,z_2)$ is viewed as a one variable polynomial of z_2 by regarding z_1 as a parameter. Then, construct a modified Jury Table. The first condition is reduced to checking the following:

$$b_0(x) < 0 \tag{3-35}$$

$$c_0(x) > 0,\ d_0(x) > 0, \ldots,\ t_0(x) > 0 \tag{3-36}$$

in the interval $-1 \leq x \leq 1$, where $z_1 = x + jy$ and $|z_1| = 1$. Note that these conditions can be checked by Sturm's method.

The second condition can be checked by finding

$$\lim_{z_2 \to \infty} z_2^{-n} B(z_1,z_2) = 0$$

which becomes a one variable polynomial, and applying the Jury Table to see whether z_1 has zeroes greater than one.

The method that was proposed in [3] used similar techniques but the Schur-Cohn matrix was employed instead of the modified Jury Table.

CHAPTER IV

STABILITY TEST FOR 2-D RECURSIVE FILTERS: DATA-DOMAIN METHOD

4-1 Introduction

As already shown in Chapter II, the 2-D LSI system can be completely specified by its impulse response. In this chapter, the 2-D LSI system will be described by means of state-space equation. This technique will give more information on the internal structure of the system. After the state-space description is given, the extension of Lyapunov lemma to the 2-D system is introduced. Then, the approximate stability test based on the theorem is presented.

4-2 State-Space Representation of 2-D Filters

Recently, the state-space descriptions have been given by many authors [17-19]. However, only the model given by Fornasini and Marchesini [19] will be employed here. They consider the following equations:

$$x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{i,j} \qquad (4\text{-}1)$$

$$y_{i,j} = Cx_{i,j} \qquad (4\text{-}2)$$

where i,j are positive integers denoting the vertical and horizontal coordinates, respectively. $\{x\} \in R^n$, is the state of the system. The inputs and outputs of the system are $\{u\} \in R^m$ and $\{y\} \in R^p$. The matrices A_0, A_1, A_2, B, and C are of the appropriate dimensions.

Transfer function matrix can be obtained by taking 2-D z-transform of (4-1) and (4-2), which become

$$z_1 z_2 X(z_1,z_2) = A_0 X(z_1,z_2) + A_1 z_1 X(z_1,z_2) + A_2 z_2 X(z_1,z_2) + BU(z_1,z_2) \quad (4\text{-}3)$$

and

$$Y(z_1,z_2) = CX(z_1,z_2). \quad (4\text{-}4)$$

Zero initial conditions have been assumed, since the transfer function matrix relate the input $U(z_1,z_2)$ and the output $Y(z_1,z_2)$ only. Straightforward manipulation yield:

$$X(z_1,z_2) = (z_1 z_2 I_n - A_0 - A_1 z_1 - A_2 z_2)^{-1} BU(z_1,z_2) \quad (4\text{-}5)$$

and

$$Y(z_1,z_2) = C(z_1 z_2 I_n - A_0 - A_1 z_1 - A_2 z_2)^{-1} BU(z_1,z_2). \quad (4\text{-}6)$$

From (4-6), it is clear that the transfer function is

$$H(z_1,z_2) = C(z_1 z_2 I_n - A_0 - A_1 z_1 - A_2 z_2)^{-1} B. \quad (4\text{-}7)$$

Comparing (4-7) and (2-28) of Chapter 2, which is repeated here for convenience,

$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{U(z_1,z_2)} = \frac{\sum_{k=l=0}^{M_2}\sum^{N_2} a_{k,l}\, z_1^{-k} z_2^{-l}}{\sum_{k=l=0}^{M_1}\sum^{N_1} b_{k,l}\, z_1^{-k} z_2^{-l}}$$

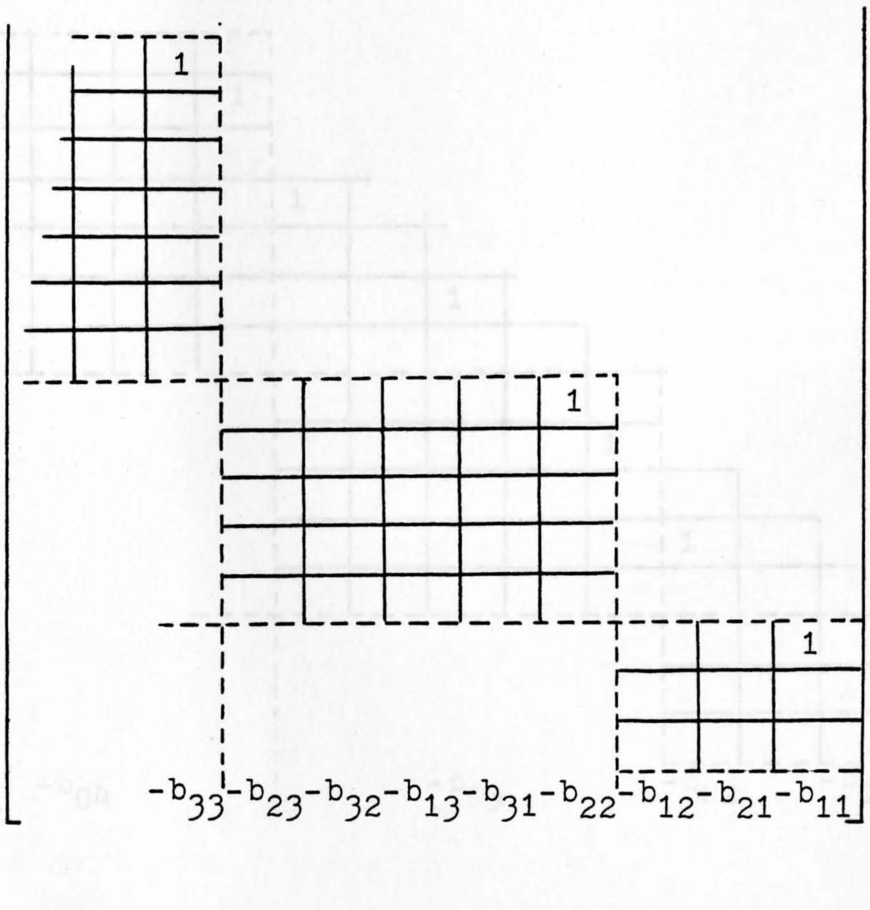
$$= \frac{A(z_1,z_2)}{B(z_1,z_2)} \qquad (4\text{-}8)$$

yield

$$B(z_1,z_2) = |(z_1 z_2 I_n - A_0 - A_1 z_1 - A_2 z_2)| , \qquad (4\text{-}9)$$

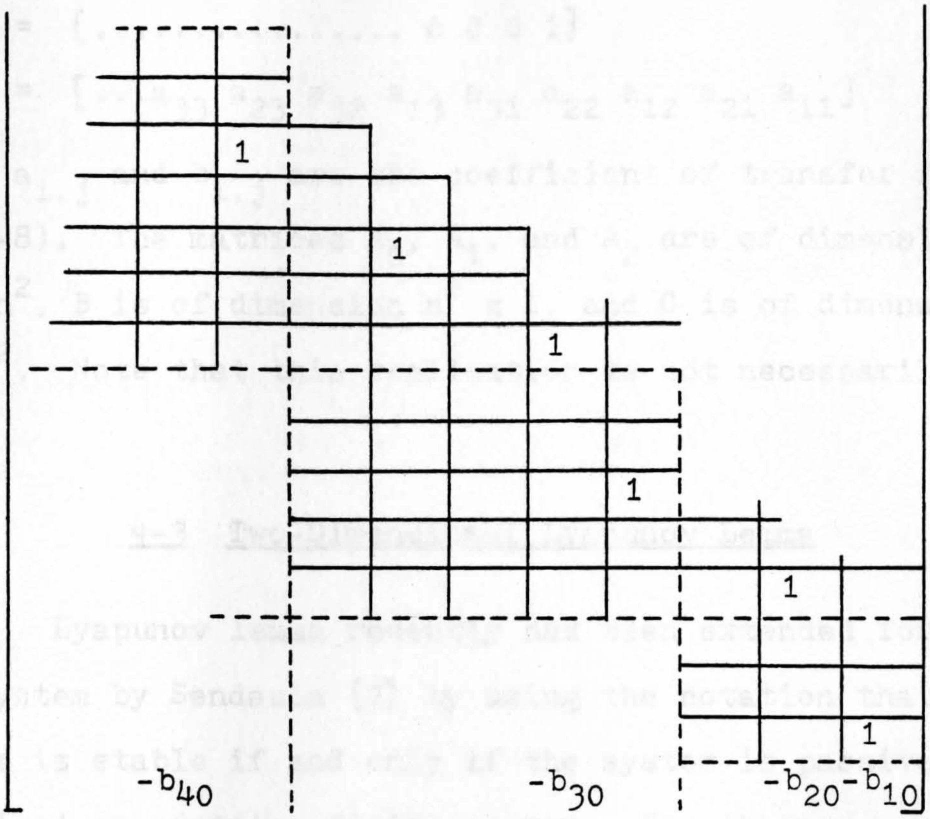
Therefore, if state-space description is known, the transfer function can be obtained by equation (4-7). In contrast, if the transfer function is known, the state-space equation (4-1) and (4-2) can be obtained by the following relation:

$A_0 =$



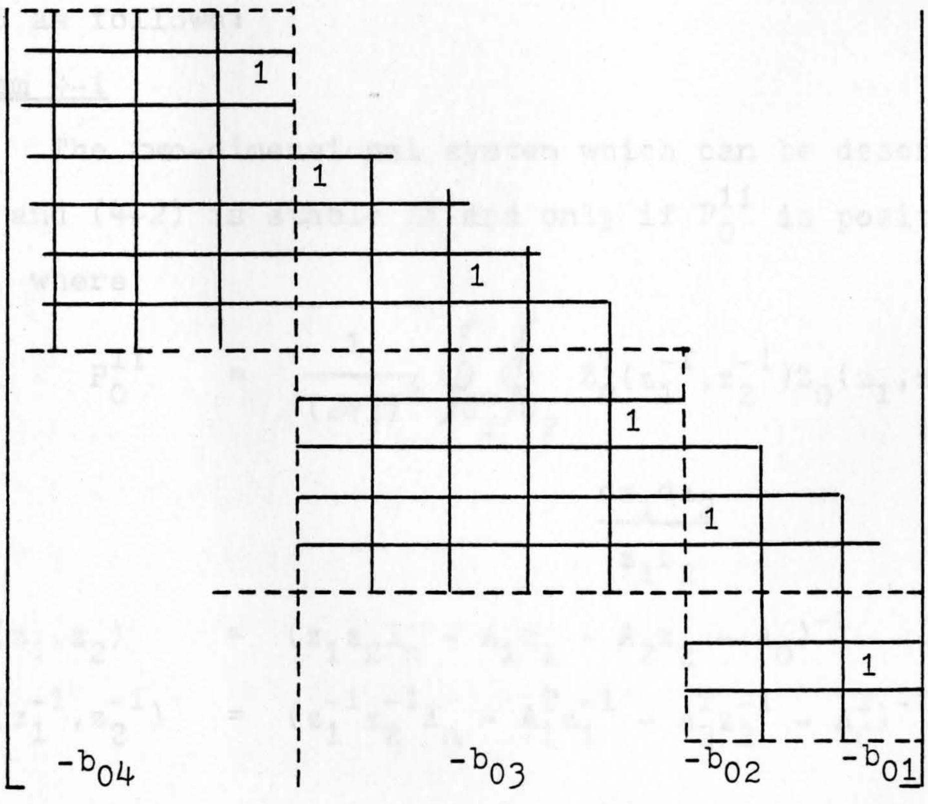
(4-9)

$A_1 =$



(4-10)

$A_2 =$



(4-11)

$$B^T = [\ldots\ldots\ldots\ldots\ldots\ldots \; 0 \; 0 \; 0 \; 1] \qquad (4\text{-}12)$$

$$C = [\ldots a_{33} \; a_{23} \; a_{32} \; a_{13} \; a_{31} \; a_{22} \; a_{12} \; a_{21} \; a_{11}] \qquad (4\text{-}13)$$

where $a_{i,j}$ and $b_{i,j}$ are the coefficient of transfer function in (4-8). The matrices A_0, A_1, and A_2 are of dimension $n^2 \times n^2$, B is of dimension $n^2 \times 1$, and C is of dimension $1 \times n^2$. Note that this realization is not necessarily minimal.

4-3 Two-Dimensional Lyapunov Lemma

Lyapunov lemma recently has been extended for the 2-D system by Sendaula [7] by using the notation that the system is stable if and only if the system is passive (dissipative) or contains finite energy. The theorem can be restated as follows:

Theorem 4-1

The two-dimensional system which can be described by (4-1) and (4-2) is stable if and only if P_0^{11} is positive definite, where

$$P_0^{11} = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \qquad (4\text{-}14)$$

$$Z_0(z_1, z_2) = (z_1 z_2 I_n - A_1 z_1 - A_2 z_2 - A_0)^{-1} \qquad (4\text{-}15)$$

$$Z_0^T(z_1^{-1}, z_2^{-1}) = (z_1^{-1} z_2^{-1} I_n - A_1^T z_1^{-1} - A_2^T z_2^{-1} - A_0^T)^{-1} \qquad (4\text{-}16)$$

and $\cdot^T$ denotes the transposition.

Remarks

It is clear that the above theorem gives the condition for the square summability of the impulse response. Recently, it was shown by Goodman [11] that the square summability does not imply BIBO stability and he stated the sufficient condition for square summability: that for the system described by $H(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$ is square summable if $H(z_1,z_2)$ is bounded in U^2 where $U^2 \equiv \{(z_1,z_2): |z_1|>1, |z_2|>1\}$. Unfortunately, the sufficient condition for the square summable which was given is not true. Consider the following examples:

Example 4-1

Let

$$H_1(z_1,z_2) = \frac{1}{2z_1z_2 - z_1 - z_2}. \qquad (4\text{-}17)$$

Consider

$$B(z_1,z_2) = 2z_1z_2 - z_1 - z_2 = 0$$

$$z_1 = \frac{z_2}{2z_2 - 1}. \qquad (4\text{-}18)$$

From (4-18), it is clear that $H(z_1,z_2)$ is analytic in U^2. It was shown in [11] that $H_1(z_1,z_2)$ is bounded and converge but not square summable. This contradicts the theorem. Note that the analyticity in U^2 does not imply the bound of impulse response but the bound of impulse response does imply the analyticity in U^2 [11].

Example 4-2

$$H_2(z_1,z_2) = \frac{1}{z_1 z_2 - 1} \tag{4-19}$$

$$B(z_1,z_2) = z_1 z_2 - 1 = 0 \tag{4-20}$$

$$z_1 = \frac{1}{z_2} \tag{4-21}$$

From (4-21), it is obvious that $H_2(z_1,z_2)$ is analytic in U^2. The impulse response of $H_2(z_1,z_2)$ is given by

$$h_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{4-22}$$

which is bounded in U^2. However, $H_2(z_1,z_2)$ is not square summable.

Example 4-3

Consider

$$H_3(z_1,z_2) = \frac{(z_1 - 1)(z_2 - 1)}{2z_1 z_2 - z_1 - z_2}. \tag{4-23}$$

It was shown [11] also that $H_3(z_1,z_2)$ is square summable but not summable. Notice that $H_1(z_1,z_2)$ and $H_3(z_1,z_2)$ have the same denominator, but are different in numerator. In $H_3(z_1,z_2)$, there is a nonessential singularity of the second kind at $z_1 = z_2 = 1$ (see the definition in the remarks of Chapter III). Therefore, it is likely that the square summability, but not summability, will occur only when there is nonessential singularity of the second kind on the biunit disk and $H(z_1,z_2)$ is bounded in U^2.

From Theorem 4-1 presented above, only the denominator of the transfer function is considered, which is the same as the other algebraic methods in Chapter III and the theorem by Shanks. Therefore, it will yield the same result as the other (which consider only denominator) do.

Theorem 4-1 can be applied by direct integration of (4-14), and the stability criterion will be obtained.

Example 4-4

Consider the filter

$$H(z_1,z_2) = \frac{1}{z_1z_2 - a_0 - a_1z_1 - a_2z_2} \tag{4-24}$$

which will be the same as the example in Chapter III if a_0, a_1, and a_2 is substituted by $-a_0$, $-a_1$, and $-a_2$. From (4-9), (4-10), and (4-11)

$$A_0 = a_0, \quad A_1 = a_1, \quad A_2 = a_2.$$

Then

$$Z_0(z_1,z_2) = \frac{1}{(z_1z_2 - a_0 - a_1z_1 - a_2z_2)} \tag{4-25}$$

$$Z_0^T(z_1^{-1},z_2^{-1}) = \frac{1}{(z_1^{-1}z_2^{-1} - a_0 - a_1z_1^{-1} - a_2z_2^{-1})} \tag{4-26}$$

Substitute $Z_0(z_1,z_2)$ and $Z_0^T(z_1^{-1},z_2^{-1})$ in (4-14), and it becomes:

$$P_0^{11} = \frac{1}{(2\pi j)^2}\oint_{C_2}\oint_{C_1}\frac{1}{(z_1z_2 - a_0 - a_1z_1 - a_2z_2)} \frac{1}{(z_1^{-1}z_2^{-1} - a_0 - a_1z_1^{-1} - a_2z_2^{-1})}\frac{dz_1dz_2}{z_1z_2}$$

$$P_0^{11} = \frac{1}{(2\pi j)^2} \oint_{C_2}\oint_{C_1} \frac{1}{(z_1 z_2 - a_0 - a_1 z_1 - a_2 z_2)} \frac{1}{(1 - a_0 z_1 z_2 - a_1 z_2 - a_2 z_1)} dz_1 dz_2$$

$$= \frac{1}{(2\pi j)^2} \oint_{C_2}\oint_{C_1} \frac{1}{(z_2 - a_1)\left\{z_1 - \dfrac{a_0 + a_2 z_2}{z_2 - a_1}\right\}} \frac{dz_1 dz_2}{(1 - a_1 z_2)\left\{1 - \dfrac{a_0 z_2 + a_2}{1 - a_1 z_2} z_1\right\}} \qquad (4\text{-}26)$$

There are two poles in (4-26), one at $(a_0 + a_2 z_2)/(z_2 - a_1)$ and the other at $(1 - a_1 z_2)/(a_0 z_2 + a_2)$

The residue due to $\dfrac{a_0 + a_2 z_2}{z_2 - a_1} = 1/[(z_2 - a_1)(1 - a_1 z_2) - (a_0 z_2 + a_2)(a_0 + a_2 z_2)]$. (4-27)

The residue due to $\dfrac{1 - a_1 z_2}{a_0 z_2 + a_2} = -1/[(z_2 - a_1)(1 - a_1 z_2) - (a_0 z_2 + a_2)(a_0 + a_2 z_2)]$. (4-28)

From (4-27) and (4-28), it is clear that only one pole can be inside the unit circle in order that $P_0^{11} \neq 0$. Assume $(a_0 + a_2 z_2)/(z_2 - a_1)$ is inside the unit circle. Therefore P_0^{11} becomes:

$$P_0^{11} = \frac{1}{2\pi j} \oint_{C_2} \frac{dz_2}{(z_2 - a_1)(1 - a_1 z_2) - (a_0 z_2 + a_2)(a_0 + a_2 z_2)}$$

$$P_0^{11} = \frac{1}{2\pi j}\oint_{C_2} \frac{dz_2}{(z_2 - a_1 z_2^2 - a_1 + a_1^2 z_2) - (a_0^2 z_2 + a_0 a_2 + a_0 a_2 z_2^2 + a_2^2 z_2)}$$

$$= \frac{1}{2\pi j}\oint_{C_2} \frac{dz_2}{-(a_1 + a_0 a_2) z_2^2 + (a_1^2 - a_0^2 - a_2^2 + 1) z_2 - (a_1 + a_0 a_2)}$$

$$= \frac{1}{2\pi j}\oint_{C_2} - \frac{dz_2}{(a_1 + a_0 a_2)(z_2 - (A + B))(z_2 - (A - B))} \qquad (4\text{-}29)$$

where

$$A = \frac{(a_1^2 - a_0^2 - a_2^2 + 1)}{2(a_1 + a_0 a_2)}$$

$$B = \sqrt{\frac{(a_1^2 - a_0^2 - a_2^2 + 1)}{(a_1 + a_0 a_2)^2} - 4}$$

$$= \frac{\sqrt{(a_1^2 - a_0^2 - a_2^2 + 1)^2 - 4(a_1 + a_0 a_2)^2}}{2(a_1 + a_0 a_2)} .$$

$$\text{The residue due to } A + B = -\frac{1}{(a_1 + a_0 a_2)2B} . \qquad (4\text{-}30)$$

$$\text{The residue due to } A - B = \frac{1}{(a_1 + a_0 a_2)2B} . \qquad (4\text{-}31)$$

From (4-30) and (4-31) it is obvious that either (A + B) or (A - B) can be inside the unit circle in order that P_0^{11} is not equal to zero. Assuming A - B is inside the unit circle. Then

$$P_0^{11} = \frac{1}{\sqrt{(a_1^2 - a_0^2 - a_2^2 + 1)^2 - 4(a_1 + a_0 a_2)^2}} \qquad (4\text{-}32)$$

From (4-32), P_0^{11} will be less than infinity, greater than one, and a real number when:

$$\text{i. } |(a_1^2 - a_0^2 - a_2^2 + 1)| > 2\,|(a_1 + a_0 a_2)| \qquad (4\text{-}33)$$

$$\text{ii. } |a_1|,\ |a_2|,\ |a_0| < 1\,. \qquad (4\text{-}34)$$

If a_0, a_1, and a_2 are replaced by $-a_0$, $-a_1$, and $-a_2$, it is easy to show that (4-33), and (4-34) are equivalent to the conditions in equations (3-23), (3-24), and (3-27) in the example of Chapter III.

4-4 The Approximate Stability Test

If the following identities:

$$\begin{aligned} Z_0(z_1,z_2) &= [z_1 z_2 I_n - A_0 - A_1 z_1 - A_2 z_2]^{-1} \\ &= (z_1 z_2)^{-1}[I_n + Z_0(z_1,z_2)(A_0 + A_1 z_1 \\ &\qquad + A_2 z_2)] \end{aligned} \qquad (4\text{-}35)$$

and

$$\begin{aligned} Z_0^T(z_1^{-1},z_2^{-1}) &= [z_1^{-1} z_2^{-1} I_n - A_0^T - A_1^T z_1^{-1} - A_2^T z_2^{-1}]^{-1} \\ &= (z_1 z_2)\,[I_n + (A_0^T + A_1^T z_1^{-1} + A_2^T z_2^{-1}) \\ &\qquad Z_0^T(z_1^{-1},z_2^{-1})] \end{aligned} \qquad (4\text{-}36)$$

are applied to Theorem 4-1, the following result is obtained:

Theorem 4-2

The approximate value of P_0^{11}, which is symmetric matrix, is the solution of

$$P_0^{11} = A_0^T P_0^{11} A_0 + A_1^T P_0^{11} A_1 + A_2^T P_0^{11} A_2 + I_n$$
$$+ A_0^T Q_{01} A_1 + A_1^T Q_{01} A_0 + A_0^T Q_{10} A_2$$
$$+ A_2^T Q_{10} A_0 + A_1^T Q_{11} A_2 + A_2^T Q_{11} A_1 \quad (4\text{-}37)$$

$$Q_{01} = Q_{10}^T A_0 + Q_{11} A_1 + P_0^{11} A_2 \quad (4\text{-}38)$$

$$Q_{10} = Q_{01}^T A_0 + P_0^{11} A_1 + Q_{11} A_2 \quad (4\text{-}39)$$

$$Q_{11} = P_0^{11} A_0 + Q_{01} A_1 + Q_{10} A_2 \quad (4\text{-}40)$$

where

$$Q_{01} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \quad (4\text{-}41)$$

$$Q_{10} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_2 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \quad (4\text{-}42)$$

$$Q_{11} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1^{-1} z_2 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}. \quad (4\text{-}43)$$

The proof of this theorem will be postponed to the next section.

For the scalar case, $A_0 = a_0$, $A_1 = a_1$, and $A_2 = a_2$, (4-37) - (4-39) become:

$$\begin{bmatrix} a_0^2 + a_1^2 + a_2^2 - 1 & 2a_0a_1 & 2a_0a_2 & 2a_1a_2 \\ a_2 & -1 & a_0 & a_1 \\ a_1 & a_0 & -1 & a_2 \\ a_0 & a_1 & a_2 & -1 \end{bmatrix} \begin{bmatrix} P_0^{11} \\ Q_{01} \\ Q_{10} \\ Q_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4\text{-}44)$$

which is a set of fourth order simultaneous equations. By straightforward manipulation (4-44) yields:

$$P_0^{11} = \frac{(a_1^2 - a_0^2 - a_2^2 + 1) - 2a_1(a_1 + a_0a_2)}{(a_1^2 - a_0^2 - a_2^2 + 1) - 4(a_1 + a_0a_2)}. \qquad (4\text{-}45)$$

From (4-45), P_0^{11} will be greater than one if

$$\text{i.}\quad |(a_1^2 - a_0^2 - a_2^2 + 1)| > 2|(a_1 + a_0a_2)| \qquad (4\text{-}46)$$

and

$$\text{ii.}\quad |a_1|, |a_2|, |a_0| < 1. \qquad (4\text{-}47)$$

It is clear that equations (4-46) and (4-47) are the same as (4-33) and (4-34) in Example 4-4. If (4-45) or (4-37) - (4-40) are used to calculate P_0^{11}, it is possible that $0 < P_0^{11} < 1$ but the filter is unstable. This is so because equations (4-37) - (4-40) are only approximations. Therefore, the filter will be stable if P_0^{11} is not only positive but also greater than 1.

In applying (4-37) - (4-40), P_0^{11}, Q_{01}, Q_{10}, and Q_{11} can be assumed to be symmetric. Then P_0^{11} can be solved explicitly. $\lambda[P_0^{11}] > 1$ can be checked by finding $[P_0^{11} - I_n]$, then using Sylvester's theorem which states that the symmetric matrix is positive definite if and only if the leading principle minors are all positive. In case the equation are linearly dependent or $[P_0^{11}] < 1$, the filter is unstable.

In calculation, the roundoff noise may cause error especially when $\lambda[P_0^{11}]$ is approximately 1. However, the stable filter will have $\lambda[P_0^{11}]$ nearly equal to one only if the filter is very stable i.e., the impulse response decreases very fast. Therefore, in a case that is difficult to decide, the impulse response of the filter should be found for a few values.

4-5 The Proofs of Theorem 2

4-5-1 The Interpretation of Q_{01}, Q_{10}, Q_{11}

Before proving the theorem, let us interpret the meaning of Q_{01}, Q_{10}, and Q_{11}. Consider

$$P_0^{11} = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}.$$

$Z_0(z_1, z_2)$ and $Z_0^T(z_1^{-1}, z_2^{-1})$ can be represented by the following series:

$$Z_0(z_1, z_2) = \sum_i \sum_j h_{i,j} z_1^{-i} z_2^{-j} \qquad (4\text{-}48)$$

$$Z_0^T(z_1^{-1}, z_2^{-1}) = \sum_i \sum_j h_{-i,-j} z_1^{i} z_2^{j} \qquad (4\text{-}49)$$

where

$$h_{-i,-j} = h_{i,j}, \quad \text{for every } i \text{ and } j. \qquad (4\text{-}50)$$

The plot of $Z_0(z_1, z_2)$ and $Z_0^T(z_1^{-1}, z_2^{-1})$ is shown in Fig. 4-1. Substitute (4-48) and (4-49) in P_0^{11}.

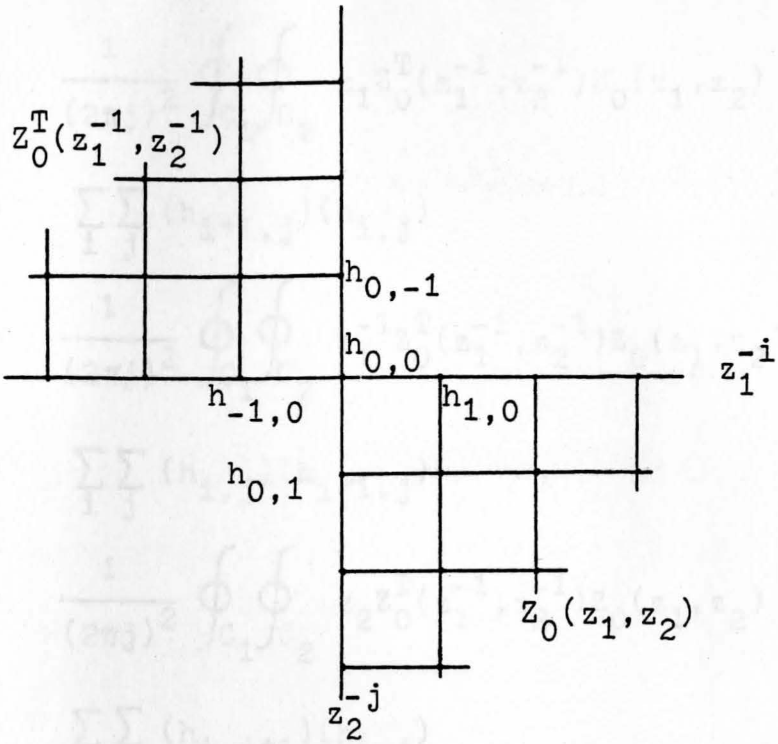


Figure 4-1 The plot of impulse response of $Z_0(z_1,z_2)$ and $Z_0^T(z_1^{-1},z_2^{-1})$

$$P_0^{11} = \frac{1}{(2\pi j)^2}\oint_{C_1}\oint_{C_2}\left[\sum_j\sum_i h_{-i,-j}\, z_1^i z_2^j\right]\left[\sum_j\sum_i h_{i,j}\, z_1^{-i}z_2^{-j}\right]\frac{dz_1dz_2}{z_1z_2} \qquad (4\text{-}51)$$

But from the complex variable theory

$$\frac{1}{(2\pi j)^2}\oint_{C_1}\oint_{C_2} a\, z_1^i z_2^j\,\frac{dz_1dz_2}{z_1z_2} = \begin{cases} a, & \text{if } i = j = 0 \\ 0, & \text{otherwise ;}\end{cases}$$

therefore,

$$P_0^{11} = \sum_i\sum_j (h_{i,j})^2. \qquad (4\text{-}52)$$

Similarly, it can be shown that

$$Q_{01} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i+1,j})(h_{i,j}) \tag{4-53}$$

$$Q_{01}^T = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1^{-1} Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i,j})(h_{i+1,j}) \tag{4-54}$$

$$Q_{10} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_2 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i,j+1})(h_{i,j}) \tag{4-55}$$

$$Q_{10}^T = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_2^{-1} Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i,j})(h_{i,j+1}) \tag{4-56}$$

$$Q_{11} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1^{-1} z_2 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i,j+1})(h_{i+1,j}) \tag{4-57}$$

$$\tilde{Q}_{11} = \frac{1}{(2\pi j)^2} \oint_{C_1}\oint_{C_2} z_1 z_2 Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \sum_i \sum_j (h_{i+1,j+1})(h_{i,j}) \; . \tag{4-58}$$

The representation of P_0^{11}, Q_{01}, Q_{01}^T, Q_{10}, Q_{10}^T, Q_{11}, and $\tilde{Q}_{11}$

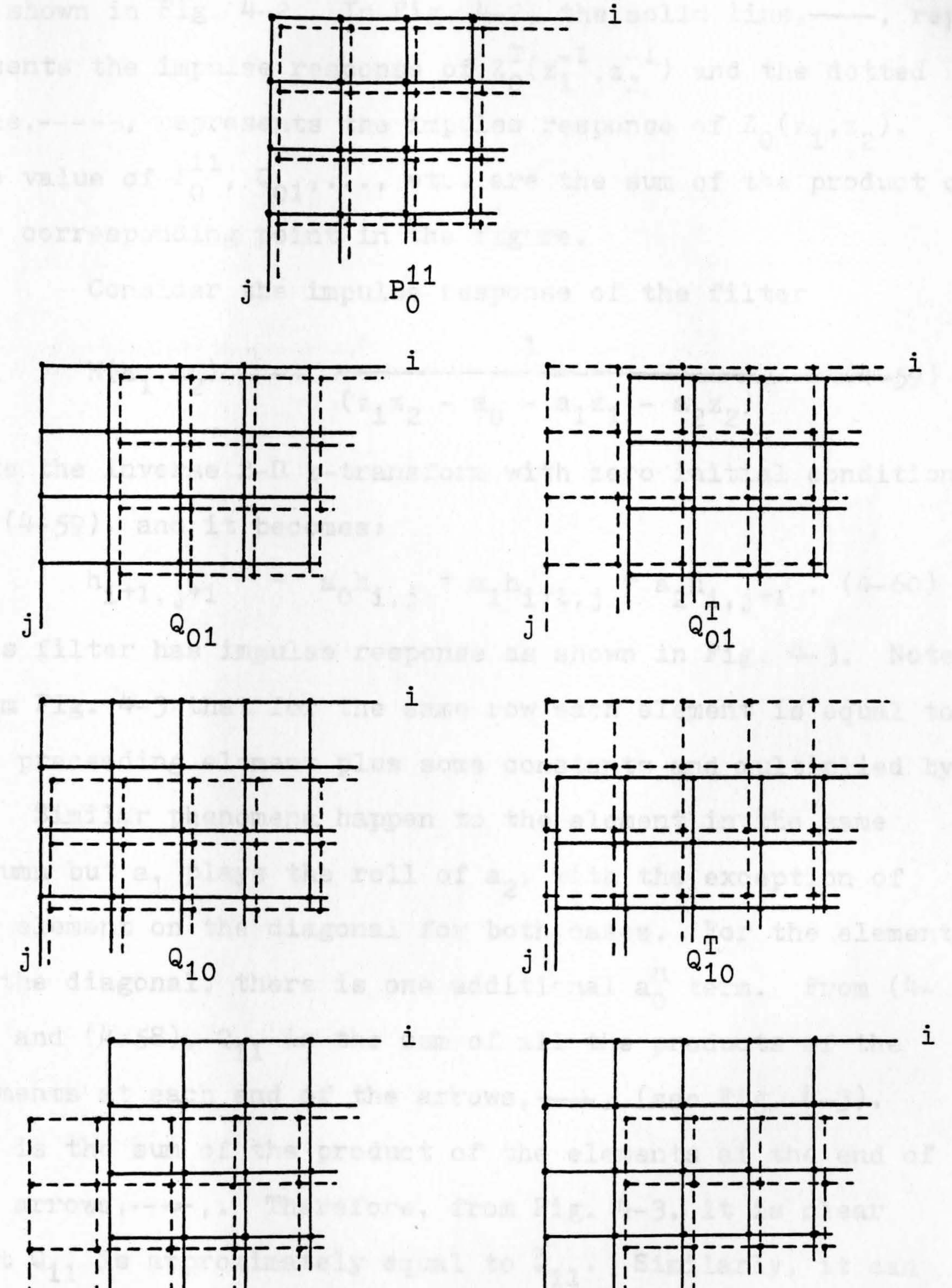


Figure 4-2 The representation of P_0^{11}, Q_{01}, Q_{01}^T, Q_{10}, Q_{10}^T, Q_{11}, and $\tilde{Q}_{11}$.

is shown in Fig. 4-2. In Fig. 4-2, the solid line,——, represents the impulse response of $Z_0^T(z_1^{-1},z_2^{-1})$ and the dotted line,-----, represents the impulse response of $Z_0(z_1,z_2)$. The value of P_0^{11}, Q_{01},..., etc. are the sum of the product of the corresponding point in the figure.

Consider the impulse response of the filter

$$H(z_1,z_2) = \frac{1}{(z_1z_2 - a_0 - a_1z_1 - a_2z_2)}. \qquad (4\text{-}59)$$

Take the inverse 2-D z-transform with zero initial condition of (4-59), and it becomes:

$$h_{i+1,j+1} = a_0h_{i,j} + a_1h_{i+1,j} + a_2h_{i,j+1} . \qquad (4\text{-}60)$$

This filter has impulse response as shown in Fig. 4-3. Note from Fig. 4-3 that for the same row each element is equal to the preceeding element plus some constants and multiplied by a_2. Similar phenomena happen to the element in the same column but a_1 plays the roll of a_2, with the exception of the element on the diagonal for both cases. For the element on the diagonal, there is one additional a_0^n term. From (4-57) and (4-58), Q_{11} is the sum of all the products of the elements at each end of the arrows,⟶, (see Fig. 4-3). $\tilde{Q}_{11}$ is the sum of the product of the elements at the end of the arrows,--->,. Therefore, from Fig. 4-3, it is clear that Q_{11} is approximately equal to $\tilde{Q}_{11}$. Similarly, it can be shown that for any filter Q_{11} and $\tilde{Q}_{11}$ are nearly equal. Note that the error between Q_{11} and $\tilde{Q}_{11}$ depends on P_0^{11}. If P_0^{11} is small, the error will be small and vice versa.

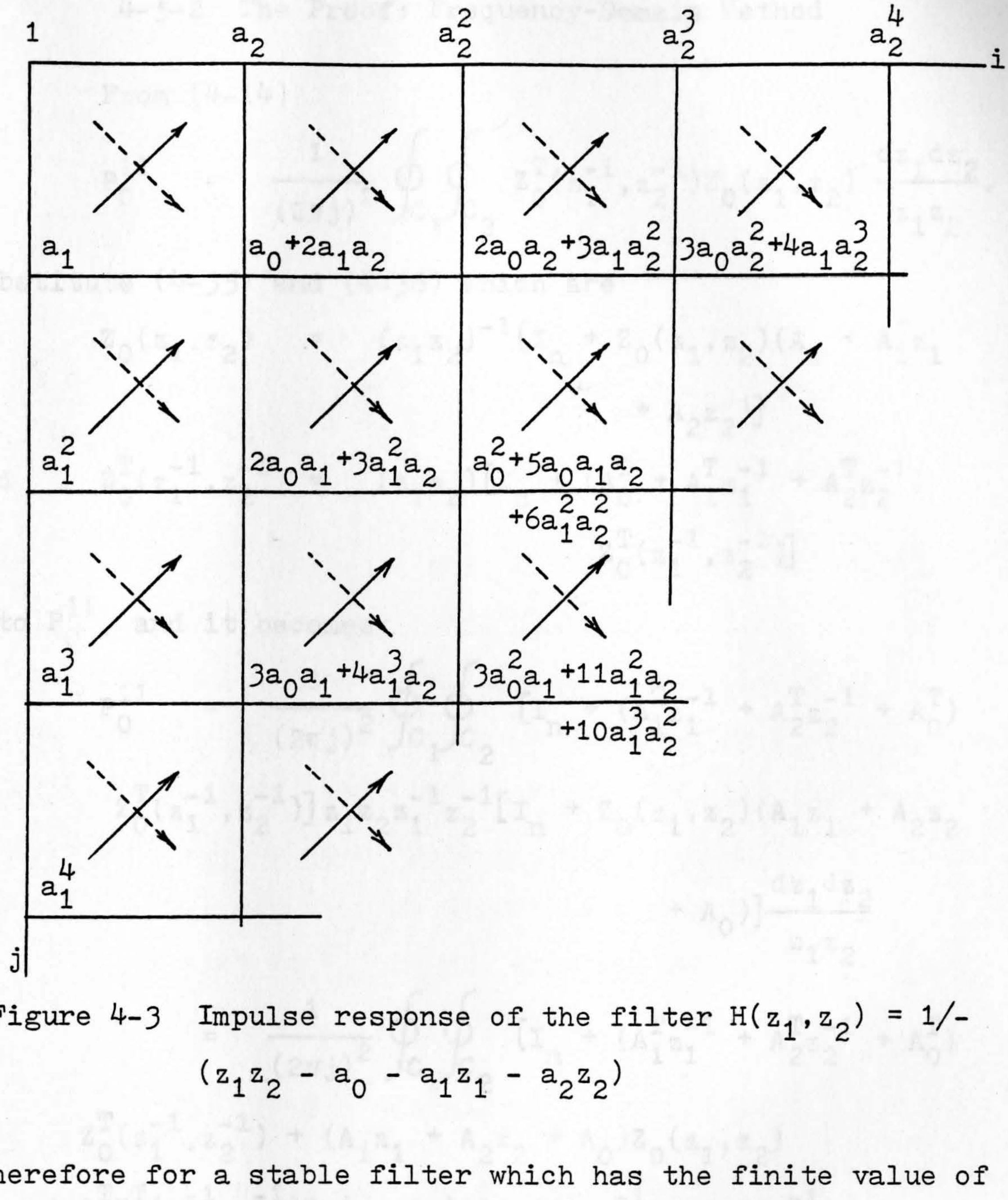


Figure 4-3 Impulse response of the filter $H(z_1,z_2) = 1/-(z_1z_2 - a_0 - a_1z_1 - a_2z_2)$

Therefore for a stable filter which has the finite value of P_0^{11}, Q_{11} and $\tilde{Q}_{11}$ will be very close, especially for the filter that has an impulse response which decreases very quickly.

4-5-2 The Proof: Frequency-Domain Method

From (4-14)

$$P_0^{11} = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}.$$

Substitute (4-35) and (4-36) which are

$$Z_0(z_1, z_2) = (z_1 z_2)^{-1} [I_n + Z_0(z_1, z_2)(A_0 + A_1 z_1 + A_2 z_2)]$$

and

$$Z_0^T(z_1^{-1}, z_2^{-1}) = (z_1 z_2)[I_n + (A_0^T + A_1^T z_1^{-1} + A_2^T z_2^{-1}) Z_0^T(z_1^{-1}, z_2^{-1})]$$

into P_0^{11}, and it becomes:

$$P_0^{11} = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} [I_n + (A_1^T z_1^{-1} + A_2^T z_2^{-1} + A_0^T) Z_0^T(z_1^{-1}, z_2^{-1})] z_1 z_2 z_1^{-1} z_2^{-1} [I_n + Z_0(z_1, z_2)(A_1 z_1 + A_2 z_2 + A_0)] \frac{dz_1 dz_2}{z_1 z_2}$$

$$= \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} [I_n + (A_1^T z_1^{-1} + A_2^T z_2^{-1} + A_0^T) Z_0^T(z_1^{-1}, z_2^{-1}) + (A_1 z_1 + A_2 z_2 + A_0) Z_0(z_1, z_2)$$
$$+ A_1^T Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2)(A_1 + A_2 z_1^{-1} z_2 + A_0 z_1^{-1})$$
$$+ A_2^T Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2)(A_1 z_1 z_2^{-1} + A_2 + A_0 z_2^{-1})$$
$$+ A_0^T Z_0^T(z_1^{-1}, z_2^{-1}) Z_0(z_1, z_2)(A_1 z_1 + A_2 z_2 + A_0)] \frac{dz_1 dz_2}{z_1 z_2} \quad (4\text{-}60)$$

It can be shown that

$$\frac{1}{(2\pi j)^2}\oint_{C_1}\oint_{C_2} I_n \frac{dz_1 dz_2}{z_1 z_2} = I_n \quad (4\text{-}61)$$

$$\frac{1}{(2\pi j)^2}\oint_{C_1}\oint_{C_2} (A_1 z_1 + A_2 z_2 + A_0) Z_0(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} = 0 \quad (4\text{-}62)$$

$$\frac{1}{(2\pi j)^2}\oint_{C_1}\oint_{C_2} (A_1^T z_1^{-1} + A_2^T z_2^{-1} + A_0^T) Z_0^T(z_1^{-1}, z_2^{-1}) \frac{dz_1 dz_2}{z_1 z_2} = 0. \quad (4\text{-}63)$$

Substitute (4-61) - (4-63) and, using the notation defined in (4-41) - (4-43), equation (4-60) becomes (4-37).

Similarly, if only $Z_0(z_1, z_2)$ defined in (4-35) (not substituted for $Z_0^T(z_1^{-1}, z_2^{-1})$) is substituted in Q_{01}, Q_{10}, and $\tilde{Q}_{11}$, (4-37) - (4-40) will be obtained. For the last equation (4-40), $\tilde{Q}_{11}$ is equated to Q_{11} since, from the previous subsection, Q_{11} and $\tilde{Q}_{11}$ are approximately equal.

Note that if $\tilde{Q}_{11}$ is not substituted for Q_{11}, an infinite set of equations will be obtained.

4-5-3 Data-Domain Proof

As defined in section 4-5-1

$$P_0^{11} = \sum_i \sum_j (h_{i,j})(h_{i,j}) \quad (4\text{-}64)$$

$$Q_{01} = \sum_i \sum_j (h_{i+1,j})(h_{i,j}) \quad (4\text{-}65)$$

$$Q_{10} = \sum_i \sum_j (h_{i,j+1})(h_{i,j}) \quad (4\text{-}66)$$

$$Q_{11} = \sum_i \sum_j (h_{i+1,j})(h_{i,j+1}) \tag{4-67}$$

$$\tilde{Q}_{11} = \sum_i \sum_j (h_{i+1,j+1})(h_{i,j}). \tag{4-68}$$

Rewrite P_0^{11} as follows:

$$P_0^{11} = I_n + \sum_i \sum_j (h_{i,j})^T(h_{i,j})$$
$i, j \neq 0$ simultaneously

$$= I_n + \sum_i \sum_j (h_{i+1,j+1})^T(h_{i+1,j+1})$$
$$+ A_2^TA_2(I_n - A_2^TA_2)^{-1} + A_1^TA_1(I_n - A_1^TA_1)^{-1}. \tag{4-69}$$

From the relation

$$h_{i+1,j+1} = A_0h_{i,j} + A_1h_{i+1,j} + A_2h_{i,j+1}. \tag{4-70}$$

Substitute (4-70) into (4-69)

$$P_0^{11} = I_n + \sum_i \sum_j (h_{i,j}^TA_0^T + h_{i+1,j}^TA_1^T + h_{i,j+1}^TA_2^T)$$
$$(A_0h_{i,j} + A_1h_{i+1,j} + A_2h_{i,j+1})$$
$$+ A_2^TA_2(I_n - A_2^TA_2)^{-1}$$
$$+ A_1^TA_1(I_n - A_1^TA_1)^{-1}. \tag{4-71}$$

By noting that

$$P_0^{11} = h_{i+1,j}^TA_1^TA_1h_{i+1,j} + A_1^TA_1(I_n - A_1^TA_1)^{-1}$$
$$= h_{i,j+1}^TA_2^TA_2h_{i,j+1} + A_2^TA_2(I_n - A_2^TA_2)^{-1}$$

and using the notation from (4-64) - (4-67), (4-37) will be obtained.

By using techniques similar to P_0^{11} and from the frequency-domain proof, (4-37) -(4-40) can be obtained.

CHAPTER V

CONCLUSION

The algebraic stability tests were given for both frequency- and data-domain methods. Various methods were given for the frequency-domain method. For the data-domain method, based on energy argument, the Lyapunov lemma was extended to the 2-D case. Although the given test is only an approximation, it was shown that for the scalar case it yielded the same result as the other methods. This test can be performed by a finite number of algebraic calculations, which can be programmed to the computer easily. Moreover, the calculations need only to solve the simultaneous equation and evaluate the values of the determinant, therefore no advance knowledge is required.

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