STATIC AND DYNAMIC STABILITY OF SLOPED COLUMNS -

A MATRIX APPROACH

by

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ABSTRACT

STATIC AND DYNAMIC STABILITY OF SLOPED COLUMNS - A MATRIX APPROACH

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The purpose of this thesis is to investigate the static stability characteristics and the free vibration phenomena of a beam-column inclined from the vertical alignment. A modern compact matrix approach is utilized to formulate the mathematical solutions. A variety of end support conditions are considered, including fixed ends, simple supports, and a combination of those.

For the static stability problem, critical buckling loads are determined for various angles of inclination of the beam-column. For each end support condition, nonlinear load-deflection curves are plotted and critical buckling loads are determined from the maximum point of those curves.

For the free vibration problem, the natural frequencies of free vibration of the sloped beam-column are determined for various values of the inclination angle. For each end support condition, axial force-natural frequency curves are constructed using an approximate linearized condition.

In general, it is found that the critical buckling

load decreases as the angle of inclination (measured from the horizontal) decreases. For the free vibration problem it is found that, as the induced axial force increases, the resonant frequency decreases. When the induced axial force reaches the critical buckling load, the resonant frequency is zero.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
a	Inclination angle
A	Area of cross section
e	Strain
E	Modulus of elasticity
$\hat{F}$	Nondimensional applied force
$\hat{F}_n$	Normalized applied force
I	Moment of inertia about z axis
I_R	Reissner's functional
$\hat{k}^2$	Nondimensional induced axial force
M	Bending moment
N	Axial force
R	Slenderness ratio
T	Kinetic energy
u,v	Axial and lateral displacements
U,V	Horizontal and vertical end forces
x,y	Axial and lateral coordinates
β_o	Normalization constant
ϑ	Node rotation angle
Θ	End rotation angle
λ	Lagrangian multiplier or constraint reaction
τ	Stress
ψ	Axial strain function
ω	Resonant frequency

LIST OF FIGURES

CHAPTER I

INTRODUCTION

The problem of the natural frequency of a beam-column has been extensively investigated.[1]* It has been found that, in general, as the axial force increases the resonant frequency decreases. At the value of axial force equal to the critical buckling load, the resonance frequency is zero. The purpose of this thesis is to investigate this phenomena for the case where the beam-column is inclined from the vertical alignment.

In this thesis work, the statical problem of the combined effects of axial strain and bending is reformulated using a more concise, modern matrix approach. This mathematical procedure is utilized to form the basic approach to the solution of the dynamics problem which is by nature a far more complex mathematical entity.

The simple truss shown in Figure 1.1 is called a "Mises Truss", the static stability of which was first investigated for axial strain.[2] Later, the static stability of the same system was investigated, using exact nonlinear theory, for bending deformation only.[3] In a recent study, the static stability problem of combined axial strain and bending deformation using an approximate large deflection

* Number in parenthesis refers to literature cited in the bibliography.

theory and a classical differential equation approach, has been investigated.[4]

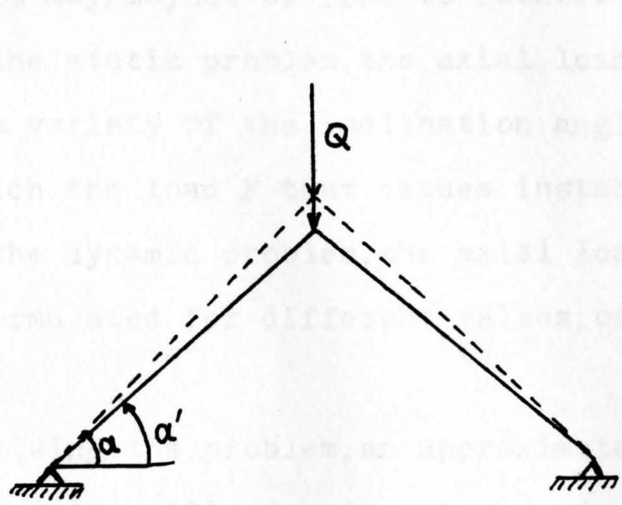


Fig. 1.1 Mises Truss.

Because of symmetry, one may investigate only one element of the system setting proper deflection restrictions. This element is shown in Figure 1.2. Since both axial strain and bending are considered, this member is a beam-column, and it is investigated for various end conditions.

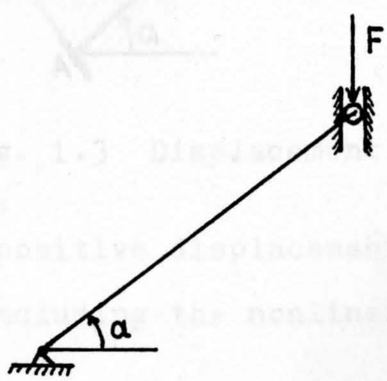


Fig. 1.2 Sloped Beam-Column.

The deflection restrictions are as follows:

A. End A has no deflections although it may/maynot be free to rotate.

B. End B only moves in the vertical direction and it may/maynot be free to rotate.

For the static problem, the axial load-axial deflection curves, for a variety of the inclination angles a, are obtainned, from which the load F that causes instability is determined. For the dynamic problem, the axial load-frequency curves are formulated for different values of the inclination angle a.

In solving the problem, an approximate large deflection theory is utilized. In other words, in the axial strain the bending effect is included.

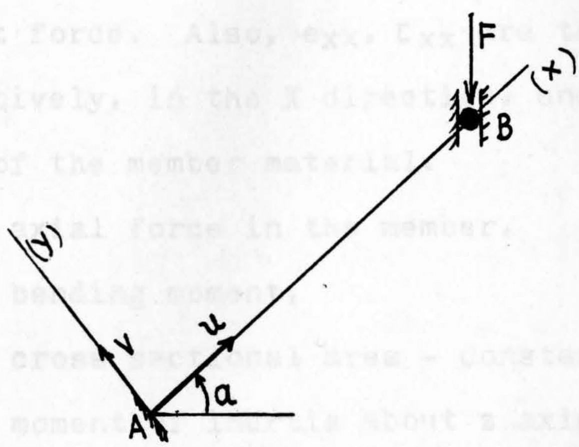


Fig. 1.3 Displacement Functions, u-v.

In Figure 1.3 the positive displacement functions are shown. The axial strain including the nonlinear term is

$$\psi = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

The total strain of the beam-column is

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + y\left(\frac{\partial^2 v}{\partial x^2}\right)$$

or $$e_{xx} = u_{,x} + \frac{1}{2} v_{,x}^2 + y\, v_{,xx}$$

where the subscript denotes the appropriate derivative.

The differential equation for the static problem is developed using Reissner's Functional (I_R)[5]

$$I_R = \iiint_V \left[\tau_{xx}\, e_{xx} - \frac{1}{2E} \tau_{xx}^2 \right] dV + $$
$$+ F\,(u_{(L)} \sin\alpha + v_{(L)} \cos\alpha) - \lambda\, (u_{(L)} \cos\alpha - v_{(L)} \sin\alpha).$$

where the term containing F represents the work done by the applied nodal force, λ defines a Lagrange multiplier physically representing the horizontal end constraint force at B, and the entire term containing λ expresses the work done by this constraint force. Also, e_{xx}, τ_{xx} are the strain and stress, respectively, in the X direction, and E the modulus of elasticity of the member material.

Defining N-the axial force in the member,

M-the bending moment,

A-the cross sectional area - constant,

I-the moment of inertia about z axis,

the stress τ_{xx} consists of the axial stress $\frac{N}{A}$ with corresponding strain $e_{xx} = u_{,x} + \frac{1}{2} v_{,x}^2$ and the bending stress $\frac{My}{I}$ with corresponding strain $y\, v_{,xx}$.

Substituting into the Reissner's Functional, it follows that,

$$I_R = \iiint_V \left\{ \frac{N}{A} \left(u_{,x} + \frac{1}{2} v_{,x}^2 \right) + \frac{My}{I} (y\, v_{,xx}) - \frac{1}{2E} \left(\frac{N}{A} + \frac{My}{I} \right)^2 \right\} dV +$$
$$+ F\,(u_{(L)} \sin\alpha + v_{(L)} \cos\alpha) - \lambda\, (u_{(L)} \cos\alpha - v_{(L)} \sin\alpha) \quad (1.1a)$$

Integrating the latter equation over the area, one obtains

$$I_R = \int_L \left\{ N\left(u_{,x} + \frac{1}{2} v_{,x}^2\right) + M\, v_{,xx} - \frac{N^2}{2EA} - \frac{M^2}{2EI} \right\} dx +$$
$$+ F\,(u_{(L)} \sin\alpha + v_{(L)} \cos\alpha) - \lambda\,(u_{(L)} \cos\alpha - v_{(L)} \sin\alpha)$$

It follows that

$$I_R = \int_L \left\{ N\left(u_{,x} + \frac{1}{2} v_{,x}^2\right) + M\, v_{,xx} - \frac{N^2}{2EA} - \frac{M^2}{2EI} \right\} dx +$$
$$+ \left[F\, u \sin\alpha + F\, v \cos\alpha - \lambda\, u \cos\alpha + \lambda\, v \sin\alpha \right]_{x=L} \qquad (1.1b)$$

Performing the variational operations (i.e. δu, δv, δN, δM) on the parameters of Equation (1.1b), one obtains

$$\delta I_R = \int_L \left\{ \left(u_{,x} + \frac{1}{2} v_{,x}^2\right) \delta N + N\, \delta\left(u_{,x} + \frac{1}{2} v_{,x}^2\right) + M\, \delta(v_{,xx}) + \right.$$
$$\left. + v_{,xx}\, \delta M - \frac{N}{EA}\, \delta N - \frac{M}{EI}\, \delta M \right\} dx +$$
$$+ \left[F \sin\alpha\, \delta u + F \cos\alpha\, \delta v - \lambda \cos\alpha\, \delta u + \lambda \sin\alpha\, \delta v \right]_{x=L}$$

but $\quad \delta\left(u_{,x} + \frac{1}{2} v_{,x}^2\right) = \delta(u_{,x}) + \frac{1}{2}\, \delta(v_{,x}^2)$

Combining the latter two equations, yields

$$\delta I_R = \int_L \left\{ \left(u_{,x} + \frac{1}{2} v_{,x}^2\right) \delta N + N\, \delta(u_{,x}) + \frac{1}{2} N\, \delta(v_{,x}^2) + M\, \delta(v_{,xx}) + \right.$$
$$\left. + v_{,xx}\, \delta M - \frac{N}{EA}\, \delta N - \frac{M}{EI}\, \delta M \right\} dx +$$
$$+ \left[F \sin\alpha\, \delta u + F \cos\alpha\, \delta v - \lambda \cos\alpha\, \delta u + \lambda \sin\alpha\, \delta v \right]_{x=L} \qquad (1.2)$$

Separately performing the integration of the terms in Equation (1.2), yields

$$\int_L N\, \delta(u_{,x}) \cdot dx = \int_L N\, d(\delta u) = N\, \delta u \Big|_0^L - \int_L N_{,x}\, \delta u\, dx \qquad (1.3)$$

$$\int_L \frac{1}{2} N\, \delta(v_{,x}^2)\, dx = \int_L N\, v_{,x}\, \delta(v_{,x})\, dx =$$
$$= \int_L N\, v_{,x} \cdot d(\delta v)$$

or $\quad \int_L \frac{1}{2} N\, \delta(v_{,x}^2)\, dx = N\, v_{,x}\, \delta v \Big|_0^L - \int_L (N\, v_{,x})_{,x}\, \delta v\, dx \qquad (1.4)$

$$\int_L M\,\delta(v_{,xx})\,dx = \int_L M\,d[\delta(v_{,x})] = M\,\delta(v_{,x})\Big|_0^L - \int_L \delta(v_{,x})\,dM =$$

$$= M\,\delta(v_{,x})\Big|_0^L - \int_L M_{,x}\,\delta(v_{,x})\,dx =$$

$$= M\,\delta(v_{,x})\Big|_0^L - \int_L M_{,x}\,d(\delta v) =$$

$$= M\,\delta(v_{,x})\Big|_0^L - M_{,x}\,\delta v\Big|_0^L + \int_L M_{,xx}\,\delta v\,dx$$

or

$$\int_L M\,\delta(v_{,xx})\,dx = M\,\delta(v_{,x}) - M_{,x}\,\delta v\Big]_0^L + \int_L M_{,xx}\,\delta v\,dx \qquad (1.5)$$

Substituting Equations (1.3), (1.4) and (1.5) into Equation (1.2), yields

$$\delta I_R = \int_L \Big\{ (u_{,x} + \tfrac{1}{2} v_{,x}^2)\,\delta N - (N v_{,x})_{,x}\,\delta v + M_{,xx}\,\delta v + v_{,xx}\,\delta M - \frac{N}{EA}\,\delta N - \frac{M}{EI}\,\delta M - N_{,x}\,\delta u \Big\}\,dx +$$

$$+ \Big[N\,\delta u + N\,v_{,x}\,\delta v + M\,\delta(v_{,x}) - M_{,x}\,\delta v \Big]_0^L +$$

$$+ \Big[F \sin\alpha\,\delta u + F\cos\alpha\,\delta v - \lambda\cos\alpha\,\delta u + \lambda\sin\alpha\,\delta v \Big]_{x=L}$$

Properly rearranging the terms of the latter equation, one obtains

$$\delta I_R = \int_L \Big\{ u_{,x} + \tfrac{1}{2} v_{,x}^2 - \frac{N}{EA} \Big\}\,\delta N\,dx + \int_L \Big\{ M_{,xx} - (N v_{,x})_{,x} \Big\}\,\delta v\,dx +$$

$$+ \int_L \Big\{ v_{,xx} - \frac{M}{EI} \Big\}\,\delta M\,dx - \int_L N_{,x}\,\delta u\,dx +$$

$$+ \Big[(N + F\sin\alpha - \lambda\cos\alpha)\,\delta u + M\,\delta(v_{,x}) + (N v_{,x} - M_{,x} + F\cos\alpha + \lambda\sin\alpha)\,\delta v \Big]_{x=L} +$$

$$+ \Big[(N)\,\delta u + (N v_{,x} - M_{,x})\,\delta v + (M)\,\delta(v_{,x}) \Big]_{x=0}$$

The variation of the Functional must vanish (i.e. $\delta I_R = 0$).

Because the variation variables inside the integral are arbitrary, for the integral to be zero, the coefficients of the variational elements are equated to zero, thus producing the stress-strain relations

$$u_{,x} + \frac{1}{2} v_{,x}^2 = \frac{N}{EA} \tag{1.6a}$$

$$N = \text{constant} \tag{1.6b}$$

$$M = EI\, v_{,xx} \tag{1.7}$$

and the differential equation of equilibrium

$$M_{,xx} - (N\, v_{,x})_{,x} = 0 \tag{1.8}$$

Combining Equations (1.7), (1.8) and (1.6b) one obtains

$$(EI\, v_{,xx})_{,xx} - N\, v_{,xx} = 0$$

Considering the parameters E and I as constants gives

$$\left.\begin{aligned} & EI\, v_{,xxxx} - N\, v_{,xx} = 0 && (1.9a) \\ \text{or} \quad & EI \frac{\partial^4 v}{\partial x^4} - N \frac{\partial^2 v}{\partial x^2} = 0 && (1.9b) \end{aligned}\right\} \quad 0 \leq x \leq L$$

From the variation process and utilizing Equation (1.7), the following boundary conditions result:

	either	or	
at x = L	$u = 0$	$N + F\sin\alpha - \lambda\cos\alpha = 0$	(1.10a)
	$v = 0$	$EI\, v_{,xxx} - N\, v_{,x} = F\cos\alpha + \lambda\sin\alpha$	(1.10b)
	$v_{,x} = 0$	$v_{,xx} = 0$	(1.10c)
at x = 0	$u = 0$	$N = 0$	(1.10d)
	$v = 0$	$EI\, v_{,xxx} - N\, v_{,x} = 0$	(1.10e)
	$v_{,x} = 0$	$v_{,xx} = 0$	(1.10f)

Equation (1.6), in combination with Equation (1.10a), indicates that the axial force N and the axial strain ψ are constants throughout the beam-column.

Now we proceed to determine the differential equations of motion and the boundary conditions governing the dynamics problem. The same displacement functions and sign convention as used in the static problem are used for the dynamic problem. For the solution of the problem Hamiltons's Principle using Reissner's Functional is utilized. The kinetic energy of the beam-column, denoted by T, is given by the expression

$$T = \int_L \frac{1}{2} \rho A (\dot{v}^2 + \dot{u}^2)\, dx \tag{1.11}$$

where ρ - the mass density of the beam-column

A - the cross section area

u(x,t), v(x,t)- the axial and transverse displacement functions, respectively,

and ($\dot{}$) denotes the appropriate time derivative. Reissner's Functional I_R is defined by Equation (1.1a). Hamilton's Principle(5), using Reissner's Functional, is defined as

$$\delta \left[\int_{t_1}^{t_2} (T - I_R)\, dt \right] = 0 \tag{1.12}$$

Performing the variational operation on T only, since $\delta(I_R)$ has been determined from the static problem, one obtains

$$\delta T = \delta \left\{ \int_{t_1}^{t_2} \left[\int_L \frac{1}{2} \rho A (\dot{v}^2 + \dot{u}^2)\, dx \right] dt \right\}$$

Interchanging the order of integration, the latter equation becomes

$$\delta T = \delta \left\{ \int_L \left[\int_{t_1}^{t_2} \frac{1}{2} \rho A (\dot{v}^2 + \dot{u}^2)\, dt \right] dx \right\}$$

Performing the variational operation, yields

$$\delta T = \int_L \left[\int_{t_1}^{t_2} (\rho A \dot{v}\, \delta\dot{v} + \rho A \dot{u}\, \delta\dot{u})\, dt \right] dx$$

or

$$\delta T = \int_L \left[\int_{t_1}^{t_2} \rho A \dot{v}\, \delta \dot{v}\, dt + \int_{t_1}^{t_2} \rho A \dot{u}\, \delta \dot{u}\, dt \right] dx$$

Integrating with respect to time, yields

$$\delta T = \int_L \left[\rho A \dot{v}\, \delta v \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\rho A \dot{v})\, \delta v\, dt + \rho A \dot{u}\, \delta u \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\rho \dot{A} \dot{u})\, \delta u\, dt \right] dx$$

or

$$\delta T = \int_L \left[\rho A \dot{v}\, \delta v \Big|_{t_1}^{t_2} + \rho A \dot{u}\, \delta u \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ (\rho \dot{A} \dot{v})\, \delta v + (\rho \dot{A} \dot{u})\, \delta u \right\} dt \right] dx$$

or

$$\delta T = \int_L \left[\rho A \dot{v}\, \delta v + \rho A \dot{u}\, \delta u \right]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \left\{ (\rho \dot{A} \dot{v})\, \delta v + (\rho \dot{A} \dot{u})\, \delta u \right\} dx\, dt \qquad (1.13)$$

The variation on Reissner's Functional yields

$$\delta \left[\int_{t_1}^{t_2} I_R\, dt \right] = \int_{t_1}^{t_2} \left[\delta I_R \right] dt$$

Substituting the latter equation and Equation (1.13) into Equation (1.12), gives

$$\int_0^L \left[\rho A \dot{v}\, \delta v + \rho A \dot{u}\, \delta u \right]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \left[\int_0^L \left\{ (\rho \dot{A} \dot{v})\, \delta v + (\rho \dot{A} \dot{u})\, \delta u \right\} dx + \delta I_R \right] dt = 0$$

But at t_1 and t_2 $v = u = 0$, and the latter equation yields

$$\int_{t_1}^{t_2} \left[\int_0^L \left\{ (\rho \dot{A} \dot{v})\, \delta v + (\rho \dot{A} \dot{u})\, \delta u \right\} dx + \delta I_R \right] dt = 0 \qquad (1.14)$$

For the integral of the latter equation to be always equal to zero the terms to be integrated must be zero. Combining the inertial terms with the terms of the Reissner's Functional that have the same variational elements, one obtains the differential equations of motion as follows:

$$(EI\, v_{,xx})_{,xx} - (N\, v_{,x})_{,x} + (\rho \dot{A} \dot{v}) = 0 \qquad (1.15a)$$

$$N_{,x} - (\rho \dot{A} \dot{u}) = 0 \tag{1.15b}$$

$$u_{,x} + \frac{1}{2} v_{,x}^2 = \frac{N}{EA} \tag{1.15c}$$

Considering the parameters E, I, ρ and A as constants and using Equation (1.15c), the two previous equations become

$$\left.\begin{aligned} &EI\, v_{,xxxx} - (N v_{,x})_{,x} + \rho A \ddot{v} = 0 \quad (1.16) \\ &EA\,(u_{,x} + \tfrac{1}{2} v_{,x}^2)_{,x} - \rho A \ddot{u} = 0 \quad (1.17) \end{aligned}\right\} 0 \leq x \leq L$$

which are the two differential equations of motion. The boundary conditions are found to be:

	either	or is prescribed
at x = 0	$N = 0$	u
	$EI\, v_{,xxx} - N v_{,x} = 0$	v
	$EI\, v_{,xx} = 0$	$v_{,x}$
at x = L	$N + F \sin\alpha + \lambda \cos\alpha = 0$	u
	$EI\, v_{,xxx} - N v_{,x} = F\cos\alpha - \lambda \sin\alpha$	v
	$EI\, v_{,xx} = 0$	$v_{,x}$

Recall that ($\cdot$) denotes appropriate derivative with respect to time and the subscript following comma denotes appropriate derivative with respect to the subscript variable. Also, the term $u_{,x} + \frac{1}{2} v_{,x}^2$ is the axial strain function in the beam-column defined as the ψ function.

CHAPTER II

DERIVATION OF THE EXACT STIFFNESS MATRIX FOR AN INCLINED BEAM-COLUMN (STATIC)

From Chapter I the differential equation of equilibrium for a statical beam-column is written as

$$EI \frac{\partial^4 v}{\partial x^4} - N \frac{\partial^2 v}{\partial x^2} = 0$$

or

$$EI\, v^{IV} - N\, v^{II} = 0$$

where the superscript implies the order of differentiation. Dividing by EI and setting $K^2 = -\frac{N}{EI}$, (i.e. compression force), it follows that,

$$v^{IV} + K^2\, v^{II} = 0$$

The solution of this differential equation becomes

$$V(x) = A_1 \cos kx + A_2 \sin kx + A_3 x + A_4$$

where A_1, A_2, A_3 and A_4 are arbitrary constants. These constants are evaluated in terms of the nodal parameters by applying the boundary conditions (B.C.). At first, a sign convention is established. The end displacements (rotations) are considered positive in the manner shown in Figure 2.1.

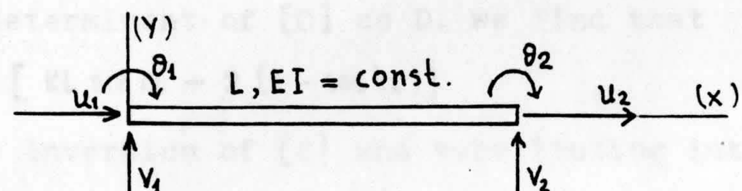


Fig. 2.1 Positive Sign Convention-Displacements (Rotations).

The boundary conditions are defined as follows:

at $x = 0$
$$\begin{cases} V(0) = V_1 & (2.1) \\ \left.\dfrac{\partial v}{\partial x}\right]_{x=0} = -\theta_1 & (2.2) \end{cases}$$

at $x = L$
$$\begin{cases} V(L) = V_2 & (2.3) \\ \left.\dfrac{\partial v}{\partial x}\right]_{x=L} = -\theta_2 & (2.4) \end{cases}$$

From B.C. (1) $\quad V_1 = A_1 + A_4$

(2) $\quad \theta_1 = -A_2 K - A_3$

(3) $\quad V_2 = A_1 \cos KL + A_2 \sin KL + A_3 L + A_4$

(4) $\quad \theta_2 = A_1 K \sin KL - A_2 K \cos KL - A_3$

Expressing relations (1), (2), (3), (4) in a matrix form, we have

$$\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -K & -1 & 0 \\ \cos KL & \sin KL & L & 1 \\ K \sin KL & -K \cos KL & -1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (2.5)$$

or in a symbolic matrix form $\qquad \{\delta\} = [C]\{A\} \qquad (2.6)$

where $\{\ \}$ denotes a column matrix and $[\ ]$ denotes a square matrix. In order to solve for matrix $[A]$ and find the coefficients A_1, A_2, A_3, A_4 in terms of the end displacements, matrix $[C]$ is inverted. Then,

$$\{A\} = [C]^{-1}\{\delta\} \qquad (2.7)$$

where $[C]^{-1}$ represents the inverse of $[C]$.

Defining the determinant of $[C]$ as D, we find that

$$D = K\left[KL \sin KL - 2(1 - \cos KL)\right]$$

Performing the inversion of $[C]$ and substituting into Equation (2.7), we have

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} (-K(1-\tilde{c})) & (\tilde{s}-KL\tilde{c}) & (K(1-\tilde{c})) & (KL-\tilde{s}) \\ (K\tilde{s}) & (1-\tilde{c}-KL\tilde{s}) & (-K\tilde{s}) & (-(1-\tilde{c})) \\ (-K^2\tilde{s}) & (K(1-\tilde{c})) & (K^2\tilde{s}) & (K(1-\tilde{c})) \\ (K[KL\tilde{s}-(1-\tilde{c})]) & (-(\tilde{s}-KL\tilde{c})) & (-K(1-\tilde{c})) & (-(KL-\tilde{s})) \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad (2.8)$$

where $\tilde{c} \equiv \cos KL$

$\tilde{s} \equiv \sin KL$

and $D = K(KL\tilde{s} - 2(1-\tilde{c}))$

The strain energy for the beam-column is given by the expression

$$U = \int_V \frac{E}{2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - y\frac{\partial^2 v}{\partial x^2}\right]^2 dV$$

Considering the area A as constant and integrating over the the area, we obtain

$$U = \int_0^L \left\{ \frac{EA}{2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right]^2 + \frac{EI}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 \right\} dx$$

The first term in the integrand of the latter equation is the square of the axial strain of the beam-column ψ^2, where

$$\psi = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

This parameter ψ is considered constant throughout the length of the beam-column. Substituting the axial strain in the strain energy expression gives

$$U = \int_0^L \left\{ \frac{EA}{2}\psi^2 + \frac{EI}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 \right\} dx$$

Considering E, A, I, and ψ as constants, we have

$$U = \frac{EAL}{2}\psi^2 + \frac{EI}{2}\int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$$

Strain energy (U) is expressed in terms of the end displacements in order to apply Castigliano's theorem to obtain end forces. The first step is the evaluation of the integral

$$I = \int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$$

which is related to the bending strain, where

$$\frac{\partial^2 v}{\partial x^2} = - A_1 K^2 \cos Kx - A_2 K^2 \sin Kx$$

or in a matrix form

$$\left\{\frac{\partial^2 v}{\partial x^2}\right\} = \left\{(-K^2 \cos Kx) \quad (-K^2 \sin Kx)\right\} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

It follows that,

$$\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right\} = \left\{A_1 \quad A_2\right\} \begin{Bmatrix} -K^2 \cos Kx \\ -K^2 \sin Kx \end{Bmatrix} \left\{(-K^2 \cos Kx) \quad (-K^2 \sin Kx)\right\} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

or

$$\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right\} = \left\{A_1 \quad A_2\right\} \begin{bmatrix} (K^4 \cos^2 Kx) & (K^4 \sin Kx \cos Kx) \\ (K^4 \sin Kx \cos Kx) & (K^4 \sin^2 Kx) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

Performing the integration yields

$$\int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx = \left\{A_1 \quad A_2\right\} \int_0^L \begin{bmatrix} (K^4 \cos^2 Kx) & (K^4 \sin Kx \cos Kx) \\ (K^4 \sin Kx \cos Kx) & (K^4 \sin^2 Kx) \end{bmatrix} dx \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

or

$$\int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx = \left\{A_1 \quad A_2\right\} \begin{bmatrix} \left(\frac{K^4 L}{2} + \frac{K^3}{2} \tilde{c}\,\tilde{s}\right) & \left(\frac{K^3}{2} \tilde{s}^2\right) \\ \left(\frac{K^3}{2} \tilde{s}^2\right) & \left(\frac{K^4 L}{2} - \frac{K^3}{2} \tilde{c}\,\tilde{s}\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \tag{2.9}$$

Applying Equation (2.8) gives

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} (-K(1-\tilde{c})) & (\tilde{s} - KL\tilde{c}) & (K(1-\tilde{c})) & (KL - \tilde{s}) \\ (K\tilde{s}) & (1-\tilde{c}-KL\tilde{s}) & (-K\tilde{s}) & (-(1-\tilde{c})) \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{2.10}$$

Transposition of Equation (2.10) yields

$$\{A_1\ A_2\} = \{V_1\ \theta_1\ V_2\ \theta_2\}\frac{1}{D}\begin{bmatrix}(-K(1-\tilde{c})) & (K\tilde{s}) \\ (\tilde{s}-KL\tilde{c}) & (1-\tilde{c}-KL\tilde{s}) \\ (K(1-\tilde{c})) & (-K\tilde{s}) \\ (KL-\tilde{s}) & (-(1-\tilde{c}))\end{bmatrix} \tag{2.11}$$

Substituting Equations (2.10) and (2.11) into Equation (2.9) and performing the matrix multiplication, one obtains

$$\int_0^L \left(\frac{\partial^2 V}{\partial x^2}\right)^2 dx = \frac{1}{D^2}\{V_1\ \theta_1\ V_2\ \theta_2\}\begin{bmatrix}K_b^{11} & K_b^{12} & K_b^{13} & K_b^{14} \\ & K_b^{22} & K_b^{23} & K_b^{24} \\ \text{SYM.} & & K_b^{33} & K_b^{34} \\ & & & K_b^{44}\end{bmatrix}\begin{Bmatrix}V_1 \\ \theta_1 \\ V_2 \\ \theta_2\end{Bmatrix} \tag{2.12a}$$

or in symbolic form $\quad I = \frac{1}{D^2}\{\delta\}^T[K_b]\{\delta\}$ (2.12b)

where:

$$K_b^{11} = K^5(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{12} = -\frac{K^5L}{2}(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{13} = -K^5(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{14} = -\frac{K^5L}{2}(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{22} = K^3(1-\tilde{c})(\tilde{s}-KL\tilde{c}-\frac{K^2L^2}{2}\tilde{s}) + \frac{K^5L^2}{2}(KL-\tilde{s})$$

$$K_b^{23} = \frac{K^5L}{2}(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{24} = K^3(KL\tilde{c}-\tilde{s})(1-\tilde{c}-\frac{K^2L^2}{2})$$

$$K_b^{33} = K^5(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{34} = \frac{K^5L}{2}(1-\tilde{c})(KL-\tilde{s})$$

$$K_b^{44} = K^3(1-\tilde{c})(\tilde{s}-KL\tilde{c}-\frac{K^2L^2}{2}\tilde{s}) + \frac{K^5L^2}{2}(KL-\tilde{s})$$

where $\{\delta\}$ is the column matrix of the nodal displacements (rotations).

The second step is the determination of the axial strain de-

defined as the ψ function in terms of the end displacements (rotations).

Noting

$$\psi = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

and thus

$$\frac{\partial u}{\partial x} = \psi - \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

Integration yields (for ψ = constant)

$$u = \psi x - \frac{1}{2}\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx + C \tag{2.13}$$

Noting

$$\frac{\partial v}{\partial x} = -A_1 K \sin Kx + A_2 K \cos Kx + A_3$$

or in matrix form

$$\left\{\frac{\partial v}{\partial x}\right\} = \left\{(-K \sin Kx) \quad (K \cos Kx) \quad (1)\right\} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

it follows that

$$\left\{\left(\frac{\partial v}{\partial x}\right)^2\right\} = \left\{A_1 \; A_2 \; A_3\right\} \begin{Bmatrix} (-K \sin Kx) \\ (K \cos Kx) \\ (1) \end{Bmatrix} \left\{(-K \sin Kx) \quad (K \cos Kx) \quad (1)\right\} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

Performing the matrix multiplication yields

$$\left\{\left(\frac{\partial v}{\partial x}\right)^2\right\} = \left\{A_1 \; A_2 \; A_3\right\} \begin{bmatrix} (K^2 \sin^2 Kx) & (-K^2 \sin Kx \cos Kx) & (-K \sin Kx) \\ & (K^2 \cos^2 Kx) & (K \cos Kx) \\ \text{SYM.} & & (1) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \tag{2.14}$$

Integration of Equation (2.14), as required in the Equation

(2.13), gives

$$\int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx = \{A_1\ A_2\ A_3\} \int_0^x \begin{bmatrix} (K^2 \sin^2 Kx) & (-\frac{K^2}{2}\sin 2Kx) & (-K\sin Kx) \\ & (K^2\cos^2 Kx) & (K\cos Kx) \\ \text{SYM} & & (1) \end{bmatrix} dx \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

or

$$\int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx = \{A_1\ A_2\ A_3\} \begin{bmatrix} \left(\frac{K^2x}{2} - \frac{K}{2}\sin Kx\cos Kx\right) & \left(-\frac{K^2}{2}\sin^2 Kx\right) & (-(1-\cos Kx)) \\ & \left(\frac{K^2x}{2} + \frac{K}{2}\sin Kx\cos Kx\right) & (\sin Kx) \\ \text{SYM} & & (x) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

Substituting into Equation (2.13) gives

$$u = \psi x - \frac{1}{2}\{A_1\ A_2\ A_3\} \begin{bmatrix} \left(\frac{K^2x}{2} - \frac{K}{2}\sin Kx\cos Kx\right) & \left(-\frac{K^2}{2}\sin^2 Kx\right) & (-(1-\cos Kx)) \\ & \left(\frac{K^2x}{2} + \frac{K}{2}\sin Kx\cos Kx\right) & (\sin Kx) \\ \text{SYM.} & & (x) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} + C$$

Applying the boundary conditions on u

1. at $x = 0$ $\quad u_{(0)} = u_1$
2. at $x = L$ $\quad u_{(L)} = u_2$

From (1)

$$u_1 = \psi \cdot 0 - \frac{1}{2}\{A_1\ A_2\ A_3\} \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \text{SYM} & & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} + C$$

or $C = u_1$

From (2) and substituting $C = u_1$

$$u_2 = \psi L - \frac{1}{2}\{A_1\ A_2\ A_3\} \begin{bmatrix} \left(\frac{K^2L}{2} - \frac{K}{2}\tilde{s}\tilde{c}\right) & \left(-\frac{K}{2}\tilde{s}^2\right) & (-(1-\tilde{c})) \\ & \left(\frac{K^2L}{2} + \frac{K}{2}\tilde{s}\tilde{c}\right) & (\tilde{s}) \\ \text{SYM} & & (L) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} + u_1$$

Solving the latter equations to find ψ, one obtains

$$\psi = \frac{u_2 - u_1}{L} + \frac{1}{2L}\{A_1\ A_2\ A_3\}\begin{bmatrix} \left(\frac{k^2 L}{2} - \frac{k}{2}\tilde{s}\tilde{c}\right) & \left(-\frac{k}{2}\tilde{s}^2\right) & \left(-(1-\tilde{c})\right) \\ & \left(\frac{k^2 L}{2} + \frac{k}{2}\tilde{s}\tilde{c}\right) & (\tilde{s}) \\ \text{SYM} & & (L) \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \tag{2.15}$$

where $\tilde{s} \equiv \sin kL$, $\tilde{c} \equiv \cos kL$

The latter expression ψ is now expressed in terms of the end displacements (rotations).

From Equation (2.8), one obtains

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \frac{1}{D}\begin{bmatrix} (-k(1-\tilde{c})) & (s - kL\tilde{c}) & (k(1-\tilde{c})) & (kL - \tilde{s}) \\ (k\tilde{s}) & (1-\tilde{c}-kL\tilde{s}) & (-k\tilde{s}) & (-(1-\tilde{c})) \\ (-k^2\tilde{s}) & (k(1-\tilde{c})) & (k^2\tilde{s}) & (k(1-\tilde{c})) \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{2.16}$$

or upon transposition gives

$$\{A_1\ A_2\ A_3\} = \{v_1\ \theta_1\ v_2\ \theta_2\}\frac{1}{D}\begin{bmatrix} (-k(1-\tilde{c})) & (k\tilde{s}) & (-k^2\tilde{s}) \\ (\tilde{s} - kL\tilde{c}) & (1-\tilde{c}-kL\tilde{s}) & (k(1-\tilde{c})) \\ (k(1-\tilde{c})) & (-k\tilde{s}) & (k^2\tilde{s}) \\ (kL-\tilde{s}) & (-(1-\tilde{c})) & (k(1-\tilde{c})) \end{bmatrix} \tag{2.17}$$

Substituting Equations (2.16) and (2.17) into Equations (2.15) and performing the matrix multiplications, yields

$$\psi = \frac{u_2 - u_1}{L} + \frac{1}{2LD^2}\{v_1\ \theta_1\ v_2\ \theta_2\}\begin{bmatrix} K_A^{11} & K_A^{12} & K_A^{13} & K_A^{14} \\ & K_A^{22} & K_A^{23} & K_A^{24} \\ & & K_A^{33} & K_A^{34} \\ \text{SYM} & & & K_A^{44} \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{2.18a}$$

or in symbolic matrix form

$$\psi = \frac{u_2 - u_1}{L} + \frac{1}{2LD^2}\{\delta\}^T[K_A]\{\delta\} \qquad (2.18b)$$

where the terms of $[K_A]$ are:

$$K_A^{11} = -k^3(1-\tilde{c})(3\tilde{s} - 2kL - kL\tilde{c})$$

$$K_A^{12} = k^2(1-\tilde{c})\left(2 - 2\tilde{c} - \frac{kL\tilde{s}}{2} - \frac{k^2L^2}{2}\right)$$

$$K_A^{13} = k^3(1-\tilde{c})(3\tilde{s} - 2kL - kL\tilde{c})$$

$$K_A^{14} = k^2(1-\tilde{c})\left(2 - 2\tilde{c} - \frac{kL\tilde{s}}{2} - \frac{k^2L^2}{2}\right)$$

$$K_A^{22} = \frac{k^3L^2}{2}(kL - 2\tilde{s} - \tilde{c}\tilde{s}) + k(1-\tilde{c})(kL - \tilde{s} + 2kL\tilde{c})$$

$$K_A^{23} = -k^2(1-\tilde{c})\left(2 - 2\tilde{c} - \frac{kL\tilde{s}}{2} - \frac{k^2L^2}{2}\right)$$

$$K_A^{24} = k(1-\tilde{c})(\tilde{s} - 3kL) + \frac{k^3L^2}{2}(3\tilde{s} - kL\tilde{c})$$

$$K_A^{33} = -k^3(1-\tilde{c})(3\tilde{s} - 2kL - kL\tilde{c})$$

$$K_A^{34} = k^2(1-\tilde{c})\left(2 - 2\tilde{c} - \frac{kL\tilde{s}}{2} - \frac{k^2L^2}{2}\right)$$

$$K_A^{44} = \frac{k^3L^2}{2}(kL - 2\tilde{s} - \tilde{c}\tilde{s}) + k(1-\tilde{c})(kL - \tilde{s} + 2kL\tilde{c})$$

Note, since

$$K_A^{13} = -K_A^{11}, \qquad K_A^{14} = K_A^{12}$$

$$K_A^{23} = -K_A^{12}, \qquad K_A^{33} = K_A^{11}$$

$$K_A^{34} = K_A^{12}, \qquad K_A^{44} = K_A^{22}$$

matrix $[K_A]$ possesses four independent terms in the (4x4) matrix.

Summarizing,

$$\psi = \frac{u_2 - u_1}{L} + \frac{1}{2LD^2}\{\delta\}^T[K_A]\{\delta\}$$

and

$$U = \frac{EAL}{2}\psi^2 + \frac{EI}{2}\{\delta\}^T[K_b]\{\delta\}$$

Recalling that the axial force N is constant throughout the length of the beam-column, it follows that the axial strain is written as

$$\psi = N/AE$$

Noting $K^2 = -\frac{N}{EI}$

one obtains

$$\psi = -\frac{K^2 L^2}{R^2} \qquad \text{and} \qquad EA\,\psi = -EI\,k^2 \tag{2.19}$$

The positive sign convention for the end forces and moments is shown in Figure 2.2.

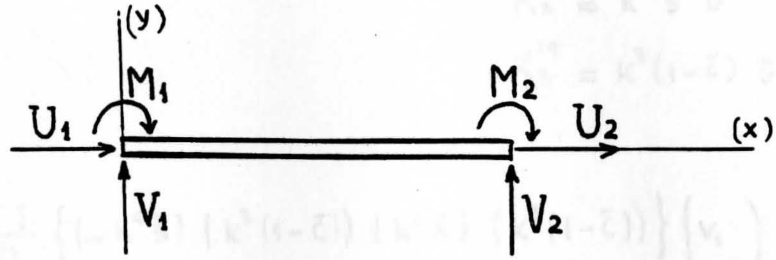


Fig. 2.2 Positive Sign Convention-Forces (Moments).

Application of Castigliano's theorem is utilized to determine the component of the stiffness matrix as follows:

$$U_1 = \frac{\partial U}{\partial u_1} \qquad \text{or} \qquad U_1 = \frac{EAL}{2}\, 2\psi \frac{\partial \psi}{\partial u_1} \qquad \text{or} \qquad U_1 = EAL\,\psi\left(-\frac{1}{L}\right)$$

or $U_1 = -EA\,\psi$ or using Equation (2.19) $U_1 = EI\,k^2$

$$U_2 = \frac{\partial U}{\partial u_2} \qquad \text{or} \qquad U_2 = \frac{EAL}{2}\, 2\psi \frac{\partial \psi}{\partial u_2} \qquad \text{or} \qquad U_2 = EAL\,\psi\,\frac{1}{L}$$

or $U_2 = EA\,\psi$ and using Equation (2.19) $U_2 = -EI\,k^2$

$$V_1 = \frac{\partial U}{\partial v_1} \qquad \text{or} \qquad V_1 = EAL\,\psi \frac{\partial \psi}{\partial v_1} + \frac{EI}{2D^2}\frac{\partial}{\partial v_1}\left[\{\delta\}^T[K_b]\{\delta\}\right]$$

or $$V_1 = EAL\,\psi\,\frac{1}{2LD^2}\frac{\partial}{\partial v_1}\left[\{\delta\}^T[K_A]\{\delta\}\right] + \frac{EI}{2D^2}\frac{\partial}{\partial v_1}\left[\{\delta\}^T[K_b]\{\delta\}\right]$$

or using Equation (2.19)

$$V_1 = -\frac{EIk^2}{D^2}\left\{K_{A1r}\right\}\{\delta\} + \frac{EI}{D^2}\left\{K_{b1r}\right\}\{\delta\}$$

where $\left\{K_{A1r}\right\}$ and $\left\{K_{b1r}\right\}$ denotes a row matrix consisting of the first row of $\left[K_A\right]$ and $\left[K_b\right]$, respectively.

Combining terms we get

$$V_1 = \frac{EI}{D^2}\left\{\{K_{b1r}\} - k^2\{K_{A1r}\}\right\}\{\delta\}$$

Calculating the matrix subtraction inside the row matrix and manipulating the terms, one obtains

$$K_s^{11} = -k^5\tilde{s}\,(kL\tilde{s} - 2(1-\tilde{c})) \quad \text{or} \quad K_s^{11} = -k^4\tilde{s}\,D$$

$$K_s^{12} = k^3(1-\tilde{c})\,D$$

$$K_s^{13} = k^4\tilde{s}\,D$$

$$K_s^{14} = k^3(1-\tilde{c})\,D$$

or finally

$$V_1 = \frac{EI}{D}\left\{(-k^4\tilde{s})\ (k^3(1-\tilde{c}))\ (k^4\tilde{s})\ (k^3(1-\tilde{c}))\right\}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

In a similar manner we obtain M_1, V_2, M_2 as

$$M_1 = \frac{EI}{D}\left\{(k^3(1-\tilde{c}))\ (k^2(kL\tilde{c}-\tilde{s}))\ (-k^3(1-\tilde{c}))\ (k^2(\tilde{s}-kL))\right\}\{\delta\}$$

$$V_2 = \frac{EI}{D}\left\{(k^4\tilde{s})\ (-k^3(1-\tilde{c}))\ (-k^4\tilde{s})\ (-k^3(1-\tilde{c}))\right\}\{\delta\}$$

$$M_2 = \frac{EI}{D}\left\{(k^3(1-\tilde{c}))\ (k^2(\tilde{s}-kL))\ (-k^3(1-\tilde{c}))\ (k^2(kL\tilde{c}-\tilde{s}))\right\}\{\delta\}$$

Combining U_1, U_2, V_1, V_2, M_1, M_2 in a single matrix equation, we have

$$\begin{Bmatrix} U_1 \\ V_1 \\ M_1 \\ U_2 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{EI}{D}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k^4\tilde{s} & k^3(1-\tilde{c}) & 0 & k^4\tilde{s} & k^3(1-\tilde{c}) \\ 0 & k^3(1-\tilde{c}) & k^2(kL\tilde{c}-\tilde{s}) & 0 & -k^3(1-\tilde{c}) & k^2(\tilde{s}-kL) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k^4\tilde{s} & -k^3(1-\tilde{c}) & 0 & -k^4\tilde{s} & -k^3(1-\tilde{c}) \\ 0 & k^3(1-\tilde{c}) & k^2(\tilde{s}-kL) & 0 & -k^3(1-\tilde{c}) & k^2(kL\tilde{c}-\tilde{s}) \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} EIk^2 \\ 0 \\ 0 \\ -EIk^2 \\ 0 \\ 0 \end{Bmatrix} \qquad (2.20a)$$

and in symbolic matrix form

$$\{q\} = [K]\{\delta\} + \{f\} \tag{2.20b}$$

In order to be able to apply the boundary conditions of the inclined beam-column one must transform the latter matrix equation to global coordinate system. The coordinates and the properly transformed end forces and displacements are shown (with the positive sign convention) in Figure 2.3.

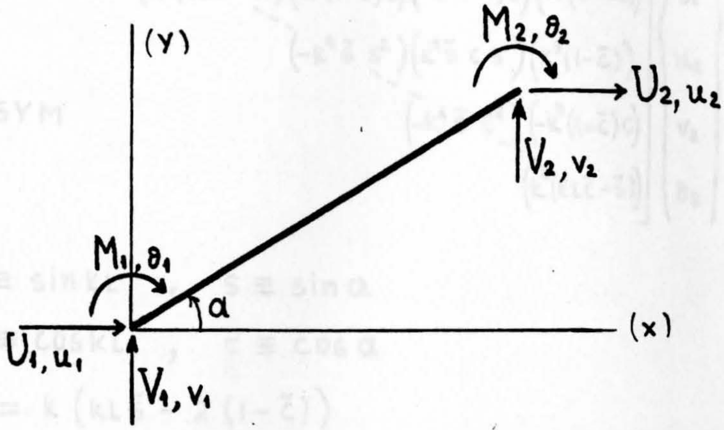


Fig. 2.3 Positive Sign Convention-Global Coordinate System.

The inclination angle a is defined from the horizontal axis. The transformation matrix $[R]$ is given as follows:

$$[R] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where} \quad \begin{array}{l} c \equiv \cos\alpha \\ s \equiv \sin\alpha \end{array}$$

Equation (2.20b) is transformed into global coordinates as

$$\{q\} = [R]^T [K] [R] \{\delta'\} + [R]^T \{f\}$$

where $[R]^T$ is the transpose of $[R]$

Performing the matrix multiplications Equation (2.20b) is written in global coordinates as

$$\{q'\} = [K']\{\delta'\} + \{f'\}$$

or in component form as

$$\begin{Bmatrix} U_1 \\ V_1 \\ M_1 \\ U_2 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{EI}{D} \begin{bmatrix} (-k^4\tilde{s}s^2) & (k^4\tilde{s}\,cs) & (-k^3(1-\tilde{c})s) & (k^4\tilde{s}\,s^2) & (-k^4\tilde{s}\,cs) & (-k^3(1-\tilde{c})s) \\ & (-k^4\tilde{s}\,c^2) & (k^3(1-\tilde{c})\,c) & (-k^4\tilde{s}\,cs) & (k^4\tilde{s}\,c^2) & (k^3(1-\tilde{c})c) \\ & & (k^2(kL\tilde{c}-\tilde{s})) & (k^3(1-\tilde{c})s) & (-k^3(1-\tilde{c})c) & (k^2(\tilde{s}-kL)) \\ & & & (-k^4\tilde{s}\,s^2) & (k^4\tilde{s}\,cs) & (k^3(1-\tilde{c})) \\ & \text{SYM} & & & (-k^4\tilde{s}\,c^2) & (-k^3(1-\tilde{c})c) \\ & & & & & (k^2(kL\tilde{c}-\tilde{s})) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} + EI \begin{Bmatrix} k^2c \\ k^2s \\ 0 \\ -k^2c \\ -k^2s \\ 0 \end{Bmatrix} \quad (2.21)$$

where $\tilde{s} \equiv \sin kL \quad , \quad s \equiv \sin\alpha$

$\tilde{c} \equiv \cos kL \quad , \quad c \equiv \cos\alpha$

and $D = k\left(kL\tilde{s} - 2(1-\tilde{c})\right)$

Equation (2.21) is nondimensionalized in end forces, end displacements, and parameters k and D as follows: setting $\hat{k} = kL$, then

$$\begin{Bmatrix} \hat{U}_1 \\ \hat{V}_1 \\ \hat{M}_1 \\ \hat{U}_2 \\ \hat{V}_2 \\ \hat{M}_2 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} (-\hat{k}^4\tilde{s}\,s^2) & (\hat{k}^4\tilde{s}\,cs) & (-\hat{k}^3(1-\tilde{c})s) & (\hat{k}^4\tilde{s}\,s^2) & (-\hat{k}^4\tilde{s}\,cs) & (-\hat{k}^3(1-\tilde{c})s) \\ & (-\hat{k}^4\tilde{s}\,c^2) & (-\hat{k}^4\tilde{s}\,c^2) & (-\hat{k}^4\tilde{s}\,cs) & (\hat{k}^4\tilde{s}\,c^2) & (\hat{k}^3(1-\tilde{c})c) \\ & & (\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})) & (\hat{k}^3(1-\tilde{c})s) & (-\hat{k}^3(1-\tilde{c})c) & (\hat{k}^2(\tilde{s}-\hat{k})) \\ & & & (-\hat{k}^4\tilde{s}\,s^2) & (\hat{k}^4\tilde{s}\,cs) & (\hat{k}^3(1-\tilde{c})s) \\ & \text{SYM} & & & (-\hat{k}^4\tilde{s}\,c^2) & (-\hat{k}^3(1-\tilde{c})c) \\ & & & & & (\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})) \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \theta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} \hat{k}^2c \\ \hat{k}^2s \\ 0 \\ -\hat{k}^2c \\ -\hat{k}^2s \\ 0 \end{Bmatrix} \quad (2.22)$$

where $\hat{U}_1 = \dfrac{U_1L^2}{EI} \;, \quad \hat{V}_1 = \dfrac{V_1L^2}{EI} \;, \quad \hat{M}_1 = \dfrac{M_1L}{EI}$

$\hat{U}_2 = \dfrac{U_2L^2}{EI} \;, \quad \hat{V}_2 = \dfrac{V_2L^2}{EI} \;, \quad \hat{M}_2 = \dfrac{M_2L}{EI}$

$$\hat{u}_1 = \frac{u_1}{L}, \quad \hat{v}_1 = \frac{v_1}{L}, \quad \hat{u}_2 = \frac{u_2}{L}, \quad \hat{v}_2 = \frac{v_2}{L}$$

$$\hat{D} = \hat{k}\left(\hat{k}\tilde{s} - 2(1-\tilde{c})\right)$$

An important note must be made here. Equation (2.20), the matrix equation relating nodal forces (moments) to nodal displacements (rotations), could have been obtained by the following sequence of steps:

A. The moment and shear forces functions are related to the displacement function v and the axial force N by the equations:

   1. $M(x) = EIv_{,xx}$
   2. $V(x) = -EIv_{,xxx} + Nv_{,x}$

   Applying boundary conditions for the nodal forces and moments, the matrix equation that relates the nodal rotations and lateral displacements to the nodal forces and moments is obtained.

B. Equating the axial nodal forces U_1 and U_2 to the axial force EIk^2 (using the proper sign), one obtains the column matrix $\{f\}$.

C. 'Patch' the two relations together into a single matrix equation.

This observation will be used in the derivation of the stiffness matrix for the dynamic problem.

CHAPTER III

APPLICATIONS TO THE STATIC INCLINED BEAM-COLUMN PROBLEM

Four different combinations of end conditions are possible and each of them is examined separately:

A. Lower end pinned - Upper end free to rotate

B. Lower end fixed - Upper end free to rotate

C. Lower end pinned - Upper end fixed against rotation

D. Lower end fixed - Upper end fixed against rotation

A. Both Ends Simply Supported

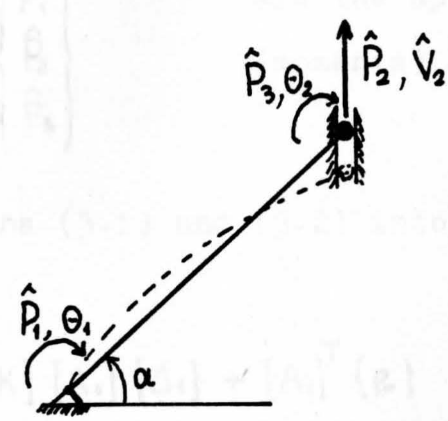


Fig. 3.1 S.S.-S.S. Beam-Column.

The end conditions, end forces and end displacements of the beam-column under consideration are shown in Figure 3.1. The matrix equation relating the nodal displacements matrix $\{\delta\}$ to the end displacements matrix $\{\Delta_1\}$ is given below.

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{3.1a}$$

or in a symbolic matrix notation

$$\{\delta\} = [A_1]\{\Delta_1\} \tag{3.1b}$$

In a similar manner, the matrix equation relating the nodal force matrix $\{q\}$ to the end force matrix $\{p\}$ is

$$\{p\} = [A_1]^T \{q\} \tag{3.2}$$

where $\{p\} = \begin{Bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{Bmatrix}$ are the applied end forces (moments) on the beam-column

Substituting Equations (3.1) and (3.2) into the Equation (2.22), one obtains

$$\{p\} = [A_1]^T [K] [A_1] \{\Delta_1\} + [A_1]^T \{p_o\} \tag{3.3}$$

For this problem the applied end forces are

$$\{p\} = \begin{Bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\hat{F} \\ 0 \end{Bmatrix} \tag{3.4}$$

where $\hat{F} = \dfrac{F L^2}{E I}$ the nondimensionalized applied force.

Performing the matrix multiplications in Equation (3.3) and utilizing Equation (3.4), one obtains

$$\begin{Bmatrix} 0 \\ -\hat{F} \\ 0 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} \left(\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})\right) & \left(-\hat{k}^3(1-\tilde{c})s\right) & \left(\hat{k}^2(\tilde{s}-\hat{k})\right) \\ & \left(-\hat{k}^4\tilde{s}\,c^2\right) & \left(-\hat{k}^3(1-\tilde{c})c\right) \\ \text{SYM} & & \left(\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \hat{V}_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\hat{k}^2 s \\ 0 \end{Bmatrix} \qquad (3.5)$$

where $\hat{D} = \hat{k}\left[\hat{k}\tilde{s} - 2(1-\tilde{c})\right]$

or

$$\frac{1}{\hat{D}} \begin{bmatrix} \left(\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})\right) & \left(-\hat{k}^3(1-\tilde{c})s\right) & \left(\hat{k}^2(\tilde{s}-\hat{k})\right) \\ & \left(-\hat{k}^4\tilde{s}\,c^2\right) & \left(-\hat{k}^3(1-\tilde{c})c\right) \\ \text{SYM} & & \left(\hat{k}^2(\hat{k}\tilde{c}-\tilde{s})\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \hat{V}_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} \qquad (3.6a)$$

or in symbolic matrix notation

$$\frac{1}{\hat{D}}\left[K_1\right]\left\{\Delta_1\right\} = \left\{f_1\right\} \qquad (3.6b)$$

In order to solve for the displacement matrix $\{\Delta_1\}$, the matrix $[K_1]$ is inverted. In a symbolic matrix form, one obtains

$$\left\{\Delta_1\right\} = \frac{\hat{D}}{d_1}\left[K_1\right]^{-1}\left\{f_1\right\}$$

where d_1 is the determinant of matrix $[K_1]$ and is given by the expression

$$d_1 = \hat{D}\,\hat{k}^7(1-\tilde{c})(\hat{k}\tilde{c} - 2\tilde{s} + \hat{k})$$

After performing the inversion, and the matrix multiplication, the unknown displacement matrix $\{\Delta_1\}$ becomes

$$\begin{Bmatrix} \theta_1 \\ \hat{V}_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{\hat{k}^2 s - \hat{F}}{\hat{k}^2 c}\right) \\ \left(-\dfrac{\hat{k}^2 s - \hat{F}}{\hat{k}^2 c^2}\right) \\ \left(\dfrac{\hat{k}^2 s - \hat{F}}{\hat{k}^2 c}\right) \end{Bmatrix} \qquad (3.7)$$

Equation (3.7) gives the relation between the actual end displacements, the applied force $\hat{F}$ and the induced axial force $\hat{k}^2$ in the inclined beam-column with inclination angle a.

From Equations (2.18a) and (2.18b) the axial strain function ψ is given in terms of the nodal displacements of the untransformed beam-column as

$$\psi = \frac{u_2' - u_1'}{L} + \frac{1}{2LD^2}\{\delta'\}^T[K_A']\{\delta'\}$$

where the prime (') denotes the untransformed (element co-ordinate) system, where

$$\psi = \left\{-\frac{1}{L} \;\; \frac{1}{L}\right\}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} + \frac{1}{2LD^2}\left\{v_1' \;\; \theta_1' \;\; v_2' \;\; \theta_2'\right\}[K_A']\begin{Bmatrix} v_1' \\ \theta_1' \\ v_2' \\ \theta_2' \end{Bmatrix} \tag{3.8}$$

The transformation relation of the nodal displacements (rotations) is given by

$$\begin{Bmatrix} u_1' \\ v_1' \\ \theta_1' \\ u_2' \\ v_2' \\ \theta_2' \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{3.9a}$$

or in symbolic matrix notation $\{\delta'\} = [R]\{\delta\}$ (3.9b)

where u_1, v_1, θ_1, u_2, v_2, θ_2 are the nodal displacements of the global system. Partitioning the matrices in Equation (3.9a), the following two relations are obtained

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vartheta_1 \\ u_2 \\ v_2 \\ \vartheta_2 \end{Bmatrix} \tag{3.10a}$$

or in a symbolic matrix form $\{\delta_1'\} = [R_1]\{\delta_1\}$ (3.10b)

and

$$\begin{Bmatrix} v_1' \\ \vartheta_1' \\ v_2' \\ \vartheta_2' \end{Bmatrix} = \begin{bmatrix} -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vartheta_1 \\ u_2 \\ v_2 \\ \vartheta_2 \end{Bmatrix} \tag{3.11a}$$

or in symbolic matrix form $\{\delta_2'\} = [R_2]\{\delta\}$ (3.11b)

The relation between the nodal displacements matrix $\{\delta\}$ and the actual end displacements of the inclined beam-column is given by Equation (3.1) as

$$\{\delta\} = [A_1]\{\Delta_1\}$$

Applying Equation (3.1) in (3.10a) and (3.11a), one obtains,

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (s)L & 0 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{3.12}$$

and

$$\begin{Bmatrix} V_1' \\ \theta_1' \\ V_2' \\ \theta_2' \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & CL & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{3.13}$$

Substituting Equations (3.12) and (3.13) into Equation (3.8) and performing the proper matrix operations, yields

$$\psi = \{0 \ S \ 0\} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} + \frac{1}{2LD^2} \{\Theta_1 \ \hat{V}_2 \ \Theta_2\} \begin{bmatrix} K_A^{22} & -K_A^{12} \cdot C & K_A^{24} \\ & K_A^{11} C^2 & -K_A^{12} C \\ SYM & & K_A^{22} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{3.14}$$

Equation (3.7) becomes

$$\begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} \hat{\lambda} \cdot C \\ -\hat{\lambda} \\ \hat{\lambda} C \end{Bmatrix} \tag{3.15a}$$

where $$\hat{\lambda} = \frac{\hat{k}^2 S - \hat{F}}{\hat{k}^2 C^2} \tag{3.15b}$$

Substituting Equation (3.15) into (3.14) and performing the calculations, yields the simplified equation

$$\psi = -\hat{\lambda} S + \frac{\hat{\lambda}^2 C^2}{2} \tag{3.16a}$$

From Equation (2.19) the relation between ψ and $\hat{k}^2$ is given as

$$\psi = -\hat{k}^2/R^2 \tag{3.16b}$$

where $R = (\frac{L}{r})$ is the slenderness ratio of the beam-column. Substituting ψ in Equation (3.15a) and $\hat{\lambda}$ in Equation (3.15b) with their equals and performing the proper calculations, the equated form of Equations (3.16a) and (3.16b) yield the following quadratic equation:

$$\hat{F}^2 R^2 + 2\hat{k}^6 c^2 - \hat{k}^4 s^2 R^2 = 0$$

or solving for $\hat{F}$

$$\hat{F} = \hat{k}^2 \left[s^2 - 2\hat{k}^2 c^2/R^2\right]^{1/2} \tag{3.17}$$

Substituting $\hat{F}$ into Equation (3.7), yields

$$\hat{V}_2 = -\frac{s - \left[s^2 - 2\hat{k}^2 c^2/R^2\right]^{1/2}}{c^2} \tag{3.18}$$

Equations (3.17) and (3.18) appear as parametric equations in $\hat{k}^2$. But, solving Equation (3.18) for $\hat{k}^2$, one obtains

$$\hat{k}^2 = -\left(\frac{\hat{V}_2^2 c^2 R^2}{2} + \hat{V}_2 s R^2\right) \tag{3.19}$$

Substituting the value $\hat{k}^2$ from Equation (3.19) into Equation (3.17), yields the cubic equation in factored form as

$$\hat{F} = -R^2 \hat{V}_2 \left(\frac{\hat{V}_2 c^2}{2} + s\right)\left(\hat{V}_2 c^2 + s\right); \quad \hat{V}_2 \leq 0 \tag{3.20}$$

Equation (3.20) is used to plot $\hat{F}$ vs. $\hat{V}_2$, for certain values of R and a. First, $\hat{F}$ is normalized to the Euler critical buckling load for the given case which is π^2. The normalized applied force then, denoted by $\hat{F}_n$, is related to $\hat{V}_2$ as follows:

$$\hat{F}_n = -\left(\frac{R}{\pi}\right)^2 \hat{V}_2 \left(\frac{\hat{V}_2 c^2}{2} + s\right)\left(\hat{V}_2 c^2 + s\right); \quad \hat{V}_2 \leq 0 \tag{3.21}$$

Recall that R is the slenderness ratio of the beam-column, c = cosa, s = sina, and a is the angle of the inclined beam-column with the horizontal. A plot of $\hat{F}_n$ vs. $\hat{V}_2$, given by the cubic Equation (3.21), is shown in Figure 3.2. Also, the straight lines $\hat{F}_n = \hat{V}_2 c^2 + s$ and $\hat{F}_n = \frac{\hat{V}_2 c^2}{2} + s$ are plotted, each linear factors of the cubic equation.

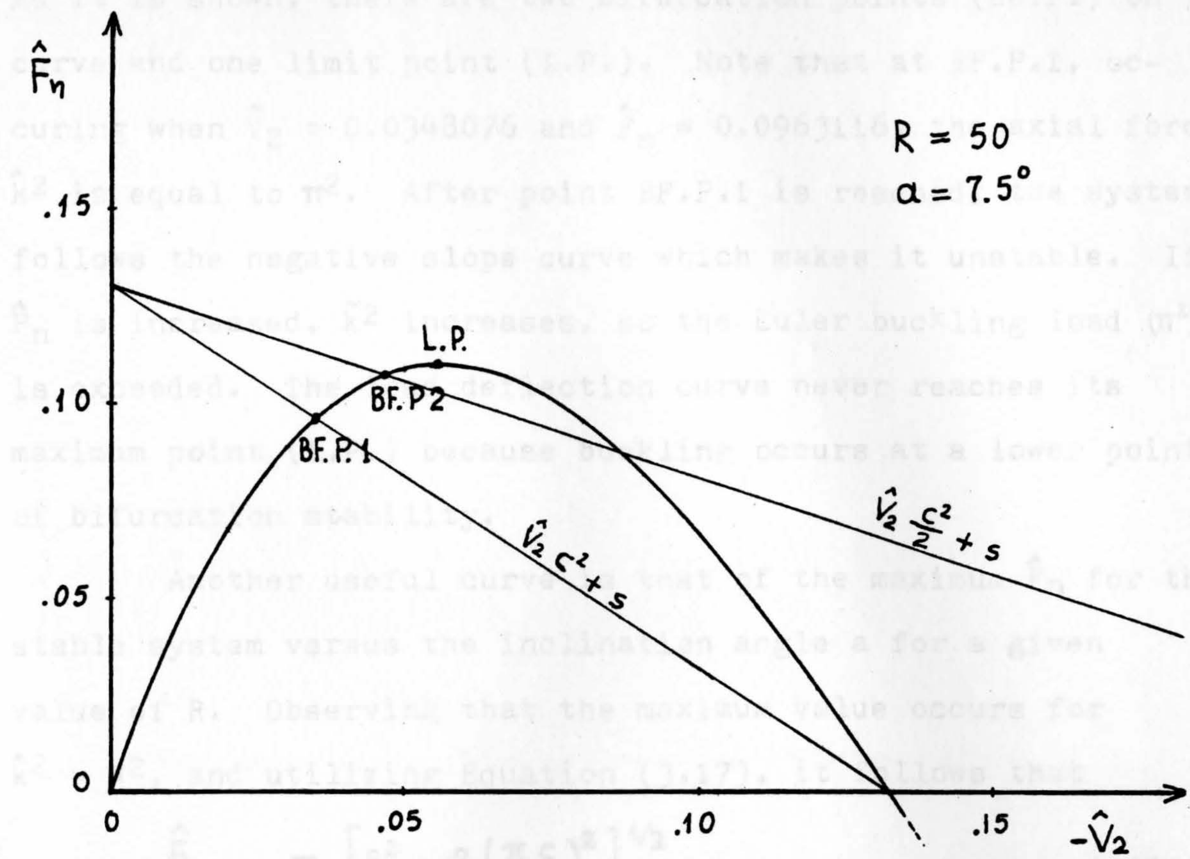


Fig. 3.2 $\hat{F}_n$ vs. $\hat{V}_2$ for S.S.-S.S.

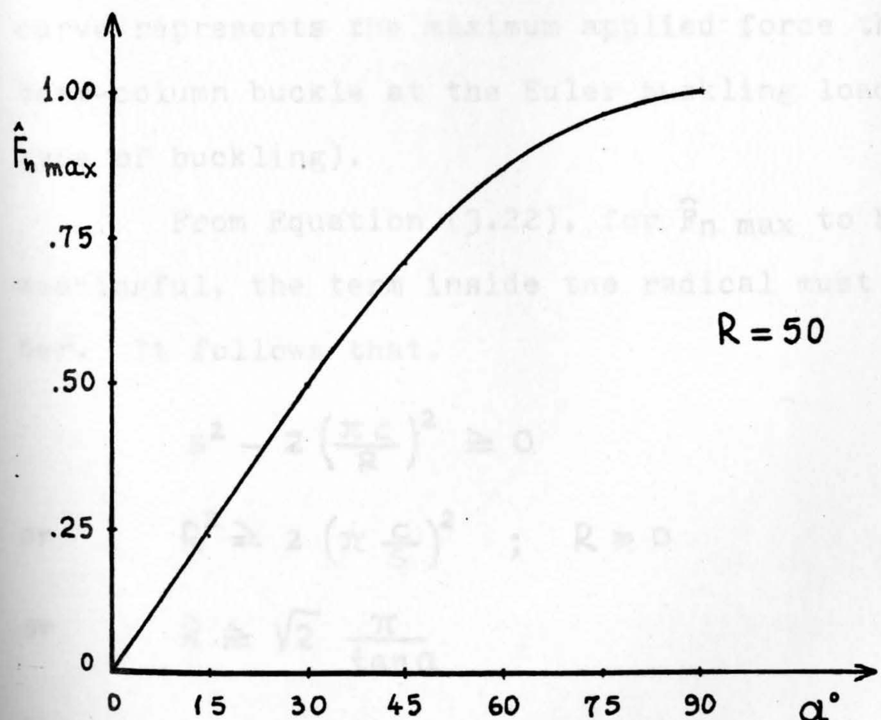


Fig. 3.3 $\hat{F}_{n\ max}$ vs. a curve - S.S-S.S.

As it is shown, there are two bifurcation points (BF.P.) on the curve and one limit point (L.P.). Note that at BF.P.1, occuring when $\hat{V}_2 = 0.0348076$ and $\hat{F}_n = 0.0963116$, the axial force $\hat{k}^2$ is equal to π^2. After point BF.P.1 is reached, the system follows the negative slope curve which makes it unstable. If $\hat{F}_n$ is increased, $\hat{k}^2$ increases, so the Euler buckling load (π^2) is exceeded. The load deflection curve never reaches its maximum point (L.P.) because buckling occurs at a lower point of bifurcation stability.

Another useful curve is that of the maximum $\hat{F}_n$ for the stable system versus the inclination angle a for a given value of R. Observing that the maximum value occurs for $\hat{k}^2 = \pi^2$, and utilizing Equation (3.17), it follows that

$$\hat{F}_{n\,max} = \left[s^2 - 2\left(\frac{\pi c}{R}\right)^2\right]^{1/2} \tag{3.22}$$

The plot of this curve is shown in Figure 3.3. $\hat{F}_{n\ max}$ on the curve represents the maximum applied force that will make the beam-column buckle at the Euler buckling load (bifurcation type of buckling).

From Equation (3.22), for $\hat{F}_{n\ max}$ to be physically meaningful, the term inside the radical must be positive number. It follows that,

$$s^2 - 2\left(\frac{\pi c}{R}\right)^2 \geq 0$$

or

$$R^2 \geq 2\left(\pi \frac{c}{s}\right)^2 \quad ; \quad R \geqslant 0$$

or

$$R \geq \sqrt{2}\ \frac{\pi}{\tan \alpha} \tag{3.23}$$

This relation between R and a is plotted in Figure 3.4.

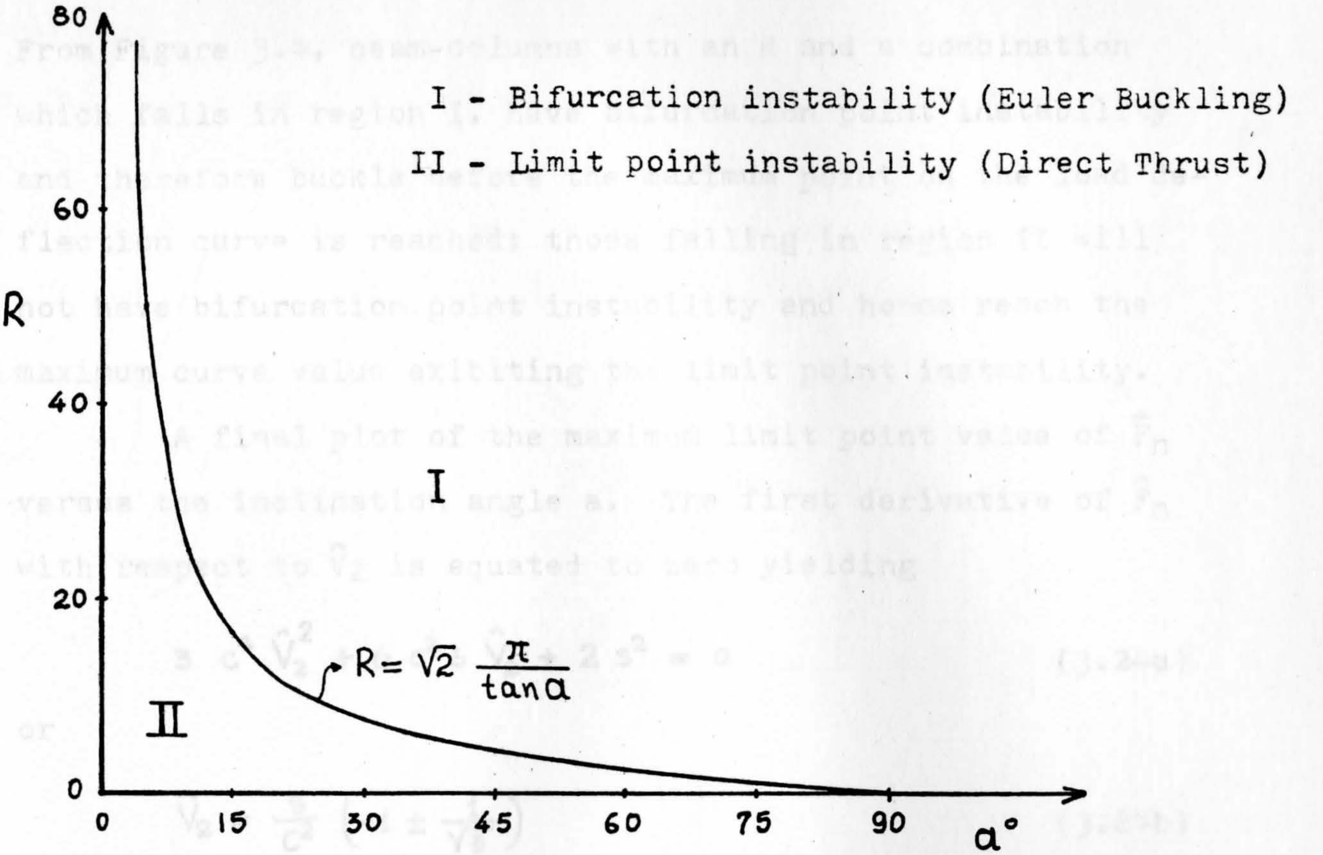


Fig. 3.4 Bifurcation-Limit Point Zones - S.S.-S.S.

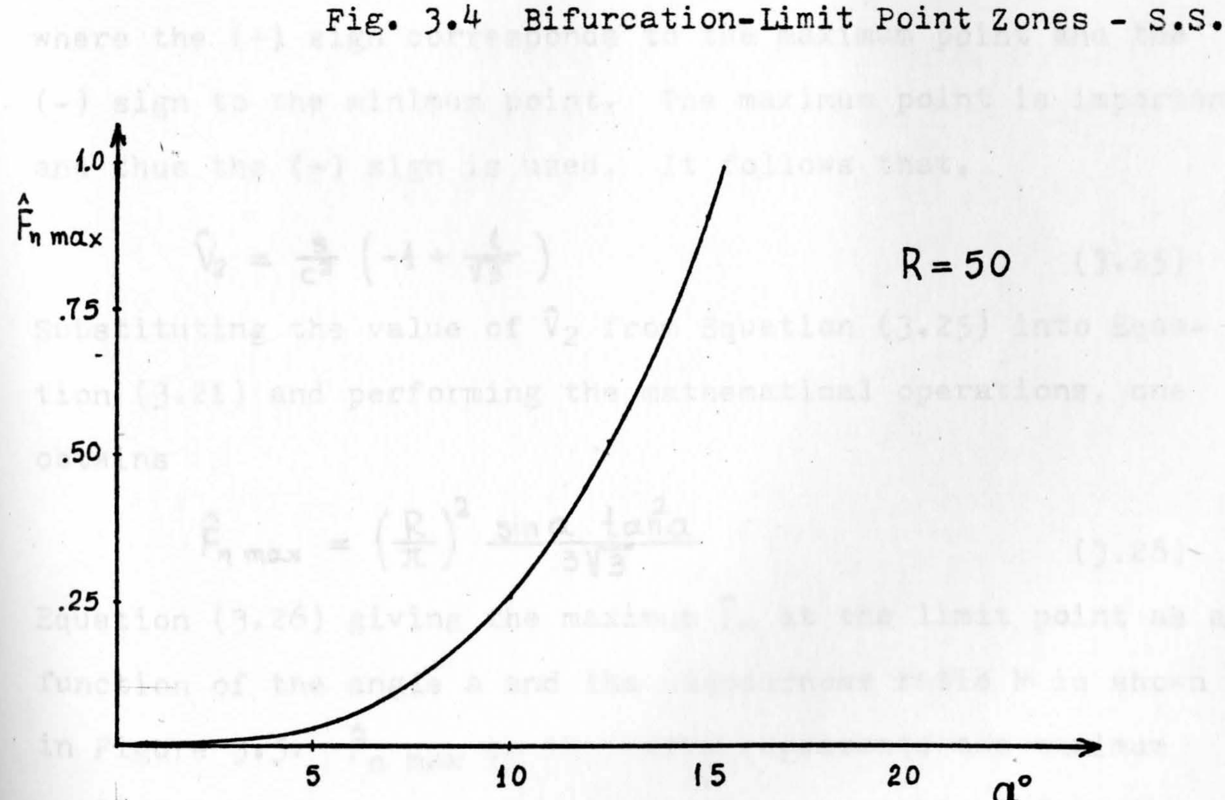


Fig. 3.5 $\hat{F}_{n\,max}$ vs. a curve, S.S.-S.S.

From Figure 3.4, beam-columns with an R and a combination which falls in region I, have bifurcation point instability and therefore buckle before the maximum point on the load deflection curve is reached; those falling in region II will not have bifurcation point instability and hence reach the maximum curve value exibiting the limit point instability.

A final plot of the maximum limit point value of $\hat{F}_n$ versus the inclination angle a. The first derivative of $\hat{F}_n$ with respect to $\hat{V}_2$ is equated to zero yielding

$$3\, c^4\, \hat{V}_2^2 + 6\, c^2 s\, \hat{V}_2 + 2\, s^2 = 0 \tag{3.24a}$$

or

$$\hat{V}_2 = \frac{s}{c^2}\left(-1 \pm \frac{1}{\sqrt{3}}\right) \tag{3.24b}$$

where the (+) sign corresponds to the maximum point and the (-) sign to the minimum point. The maximum point is important and thus the (+) sign is used. It follows that,

$$\hat{V}_2 = \frac{s}{c^2}\left(-1 + \frac{1}{\sqrt{3}}\right) \tag{3.25}$$

Substituting the value of $\hat{V}_2$ from Equation (3.25) into Equation (3.21) and performing the mathematical operations, one obtains

$$\hat{F}_{n\,max} = \left(\frac{R}{\pi}\right)^2 \frac{\sin a\, \tan^2 a}{3\sqrt{3}} \tag{3.26}$$

Equation (3.26) giving the maximum $\hat{F}_n$ at the limit point as a function of the angle a and the slenderness ratio R is shown in Figure 3.5. $\hat{F}_{n\,max}$ on the curve represents the maximum applied load that makes the beam-column unstable before the Euler's buckling load is reached (buckling due to direct thrust).

The bifurcation is a phenomenon occuring only in case A, the simply supported beam-column. For the remaining cases no bifurcation instability occurs which implies that the beam-columns become unstable before the axial force reaches Euler's buckling load for the particular support conditions.

B. Lower End Fixed-Upper End Simply Supported (free to rotate)

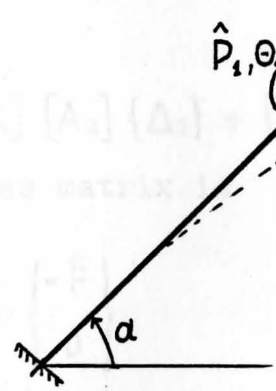


Fig. 3.6 Fixed-S.S. Beam-Column

In Figure 3.6 the beam-column under consideration, the end forces and end displacements are shown. The matrix equation relating the nodal displacements matrix $\{\delta\}$ with the beam-column actual end displacements $\{\Delta_2\}$, is given by

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{3.27a}$$

or in symbolic matrix notation

$$\{\delta\} = [A_2]\{\Delta_2\} \tag{3.27b}$$

Similarly the relation between the nodal forces matrix $\{q\}$ with the applied end forces matrix $\{p\}$, is

$$\{p\} = [A_2]^T \{q\} \tag{3.28}$$

where
$$\{p\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix}$$

Substituting Equations (3.27) and (3.28) into Equation (2.22) one obtains

$$\{p\} = [A_2]^T [K] [A_2] \{\Delta_2\} + [A_2]^T \{P_0\} \tag{3.29}$$

The applied end forces matrix is

$$\{p\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix} = \begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix} \tag{3.30}$$

Performing the matrix multiplications in Equation (3.29) and substituting Equation (3.30), yields

$$\begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} (-\hat{k}^4 \tilde{s} c^2) & (-\hat{k}^3 (1-\tilde{c}) c) \\ \mathrm{SYM} & (\hat{k}^2 (\hat{k}\tilde{c} - \tilde{s})) \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} -\hat{k}^2 s \\ 0 \end{Bmatrix} \tag{3.31}$$

It follows that

$$\begin{Bmatrix} \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} (-\hat{k}^4 \tilde{s} c^2) & (-\hat{k}^3 (1-\tilde{c}) c) \\ \mathrm{SYM} & (\hat{k}^2 (\hat{k}\tilde{c} - \tilde{s})) \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_2 \end{Bmatrix} \tag{3.32a}$$

or in symbolic matrix form

$$\{f_2\} = \frac{1}{\hat{D}} [K_2] \{\Delta_2\} \tag{3.32b}$$

Solving Equation (3.32a) for displacements, yields

$$\{\Delta_2\} = \frac{\hat{D}}{d_2} [K_2]^{-1} \{f_2\} \tag{3.33}$$

where d_2 is the determinant of the matrix $[K_2]$ given by the expression

$$d_2 = -\hat{D}\ \hat{k}^5\ \tilde{c}\ c^2$$

Performing the inversion and then the matrix multiplications, the following equation of the displacements is obtained

$$\begin{Bmatrix} \hat{V}_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{(\hat{k}^2 s - \hat{F})(\hat{k}\tilde{c} - \tilde{s})}{\hat{k}^3\ \tilde{c}\ c^2} \\ -\dfrac{(\hat{k}^2 s - \hat{F})\ \hat{k}\ (1-\tilde{c})\ c}{\hat{k}^3\ \tilde{c}\ c^2} \end{Bmatrix} \tag{3.34}$$

Equation (3.34) gives the end displacements in terms of the applied force $\hat{F}$, and the induced axial force $\hat{k}^2$, for a beam-column with an inclination angle a.

From Equation (3.8) the axial strain ψ in terms of the nodal displacements for the untransformed beam-column is given by

$$\psi = \left\{ -\frac{1}{L} \quad \frac{1}{L} \right\} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} + \frac{1}{2LD^2} \{ v_1' \ \vartheta_1' \ v_2' \ \vartheta_2' \} [K_A'] \begin{Bmatrix} v_1' \\ \vartheta_1' \\ v_2' \\ \vartheta_2' \end{Bmatrix}$$

where prime (') denotes the untransformed system.

The transformation of the nodal displacements is given by the Equation (3.9) as

$$\{\delta'\} = [R]\{\delta\}$$

or after partitioning by Equations (3.10) and (3.11)

$$\{\delta_1'\} = [R_1]\{\delta\}$$

and $$\{\delta_2'\} = [R_2]\{\delta\}$$

The relation between the nodal displacements and the beam-column end displacements is given by Equation (3.27) as

$$\{\delta\} = [A_2]\{\Delta_2\} \tag{3.27}$$

Substituting Equations (3.27) in Equations (3.10) and (3.11), substituting these results in Equation (3.8), and performing the proper matrix operations, yields the following expression for ψ:

$$\psi = \{s \quad 0\}\begin{Bmatrix}\hat{V}_2 \\ \theta_2\end{Bmatrix} + \frac{1}{2\hat{D}^2}\{\hat{V}_2 \quad \theta_2\}\begin{bmatrix}(K_A^{11} c^2) & (-K_A^{12} c) \\ \text{SYM} & (K_A^{22})\end{bmatrix}\begin{Bmatrix}\hat{V}_2 \\ \theta_2\end{Bmatrix} \tag{3.35}$$

where the terms of matrix $[K_A]$ are given on page 19. From Equation (3.34), one obtains

$$\begin{Bmatrix}\hat{V}_2 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix}\hat{\lambda}\,(\hat{k}\,\bar{c} - \bar{s}) \\ \hat{\lambda}\,\hat{k}\,(1-\bar{c})\,c\end{Bmatrix} \tag{3.36a}$$

where $$\hat{\lambda} = -\frac{(\hat{k}^2 s - \hat{F})}{\hat{k}^3\,\bar{c}\,c^2} \tag{3.36b}$$

Substituting Equation (3.36) into Equation (3.35) and performing the matrix multiplications, yields

$$\psi = \hat{V}_2\, s + \frac{\hat{\lambda}^2 c^2}{4}\left[3\left(\hat{k}^2 \bar{c}^2 - \hat{k}\,\bar{c}\,\bar{s}\right) + \hat{k}^2 \bar{s}^2\right]$$

or substituting $\hat{\lambda}$ in terms or $\hat{V}_2$ $\left(\hat{\lambda} = \frac{\hat{V}_2}{\hat{k}\bar{c} - \bar{s}}\right)$, one obtains

$$\psi = \hat{V}_2\, s + \frac{\hat{V}_2\, c^2}{4\,(\hat{k}\,\bar{c} - \bar{s})^2}\left[3\left(\hat{k}^2 \bar{c}^2 - \hat{k}\,\bar{c}\,\bar{s}\right) + \hat{k}^2 \bar{s}^2\right] \tag{3.37}$$

or

$$\psi = \hat{V}_2\, s + \frac{\hat{V}_2\, c^2}{\left(1 - \frac{\tan\hat{k}}{\hat{k}}\right)^2}\,\frac{1}{4}\left[3\left(1 - \frac{\tan\hat{k}}{\hat{k}}\right) + \tan^2\hat{k}\right] \tag{3.38}$$

For convinience in calculations, the following substitutions are made:

$$T = 1 - \frac{\tan\hat{k}}{\hat{k}}, \text{ and } \quad S = \frac{1}{4}\left[3T + \tan^2\hat{k}\right] \tag{3.39}$$

Substitution of Equation (3.39) into (3.38), yields

$$\psi = \hat{V}_2 s + \frac{\hat{V}_2^2 c^2 S}{T^2}$$

or by utilizing Equation (2.19)

$$-\frac{\hat{k}^2}{R^2} = \hat{V}_2 s + \hat{V}_2^2 c^2 \frac{S}{T^2}$$

or $$(c^2 S)\,\hat{V}_2^2 + (T^2 s)\,\hat{V}_2 + \frac{\hat{k}^2 T^2}{R^2} = 0 \qquad (3.40)$$

Solving the quadratic equation for $\hat{V}_2$, one obtains

$$\hat{V}_2 = \frac{-T^2 s \pm \left[T^4 s^2 - 4 c^2 \hat{k}^2 S T^2/R^2\right]^{1/2}}{2 c^2 S} \qquad (3.41)$$

From the two possible solutions for $\hat{V}_2$ the most important is the one with the positive sign in front of the radical because, as it has been found from the load-deflection curve, using the positive sign produces the lowest maximum point of the curve.

From Equation (3.34) the $\hat{V}_2$ expression becomes

$$\hat{V}_2 = -\frac{\hat{k}^2 s - \hat{F}}{\hat{k}^2 c^2}\left(1 - \frac{\tan \hat{k}}{\hat{k}}\right)$$

or

$$\hat{V}_2 = -\frac{\hat{k}^2 s - \hat{F}}{\hat{k}^2 c^2}\, T \qquad (3.42)$$

Solution of the equation above for $\hat{F}$ yields

$$\hat{F} = \hat{k}^2\left(s + \frac{\hat{V}_2 c^2}{T}\right) \qquad (3.43)$$

Normalizing $\hat{F}$ to the Euler's buckling load for the case examined ($\beta_o = 4.49340946$), one obtains

$$\hat{F}_\eta = \left(\frac{\hat{k}}{\beta_o}\right)^2\left(s + \frac{\hat{V}_2 c^2}{T}\right) \qquad (3.44)$$

A plot of the load-deflection curve is constructed using two parametric equations (parametric on $\hat{k}$) in the form

$$\hat{V}_2 = \frac{-T^2 s + \left[T^4 s^2 - 4 c^2 \hat{k}^2 S T^2/R^2\right]^{1/2}}{2 c^2 S} \qquad (3.45a)$$

$$\hat{F}_n = \left(\frac{\hat{k}}{\beta_o}\right)^2 \left(s + \frac{\hat{V}_2 c^2}{T}\right) \qquad (3.45b)$$

where $c = \cos a$, $s = \sin a$, $\beta_o = 4.49340946$

R = slenderness ratio , a = inclination angle

$T = 1 - \frac{\tan \hat{k}}{\hat{k}}$

$S = \frac{1}{4} \left(3T + \tan^2 \hat{k}\right)$

In order to plot the load-deflection curve for a beam-column with the known R and a, a value for $\hat{k}$ is choosen and then $\hat{V}_2$ and $\hat{F}_n$ are calculated. For the plot a high speed digital computer is utilized. The computer program used is listed in the Appendix A. In the program both values for $\hat{V}_2$ (plus and minus) are used. From the output it is determined that the first maximum point always occurs for $\hat{V}_2$ with the positive sign in front of the radical in Equation (3.41). Another observation is that this maximum point always occurs for the values of $\hat{k}^2$ less than the Euler's buckling load β_o^2 which means that there is no bifurcation point on the curve and the beam-column buckles due to direct thrust only.
In Figure 3.7 the plot of the load-deflection curve for various inclination angles is shown.

Another useful plot is the curve of the maximum $\hat{F}_n$ versus the angle a. Such a plot is shown in Figure 3.8, with points on curve found from the load-deflection curve for each a.

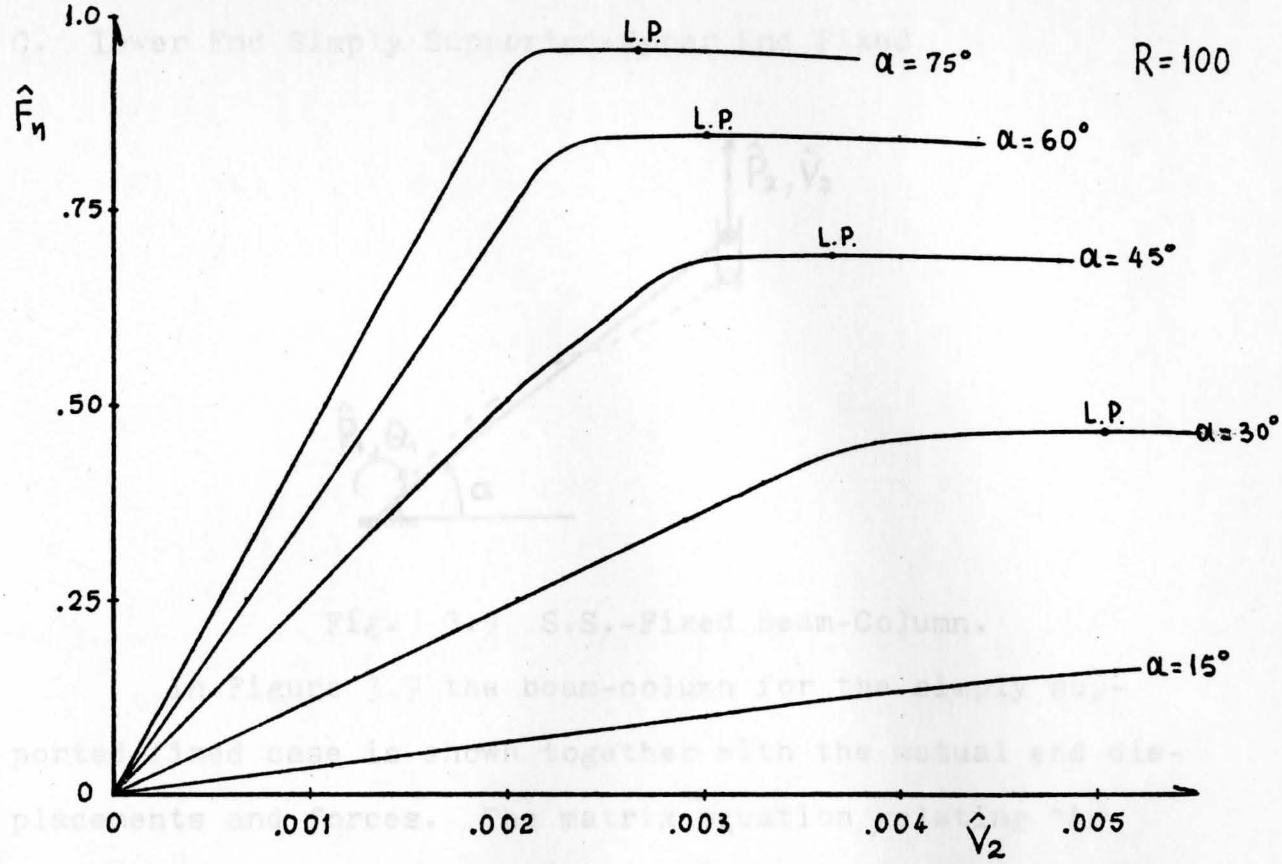


Fig. 3.7 Load-Deflection Curves - Fixed-S.S.

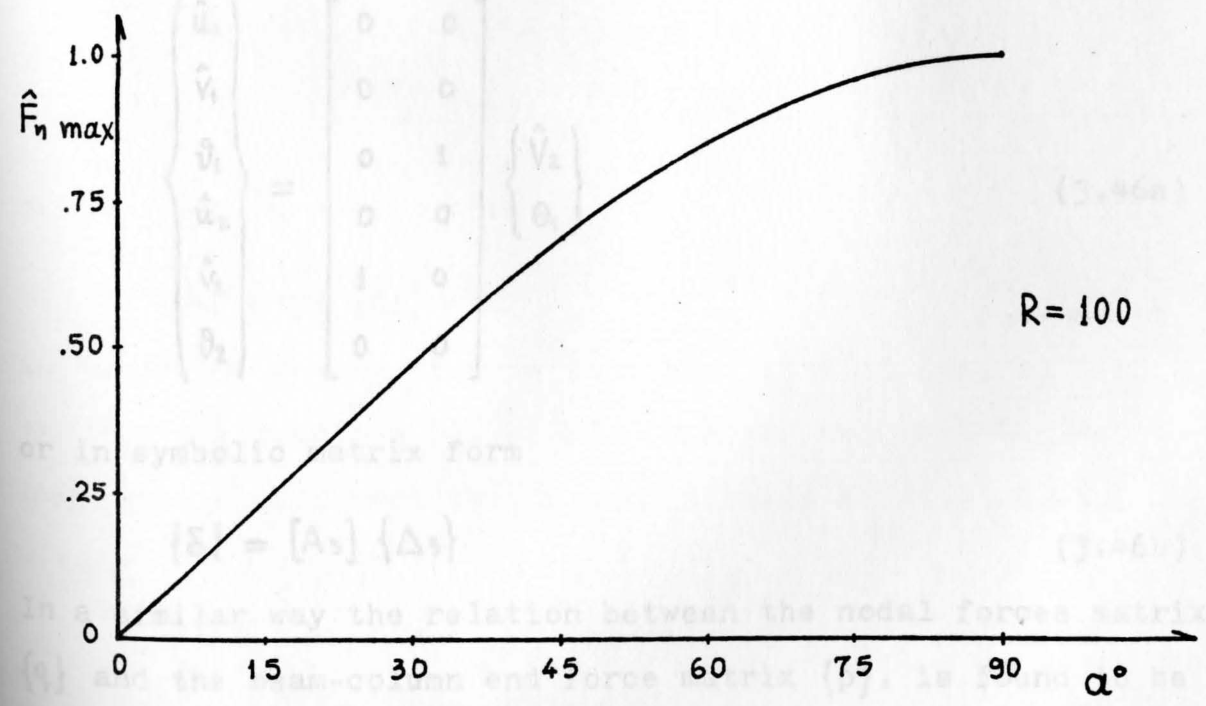


Fig. 3.8 $\hat{F}_{n\ max}$ vs. a Curve - Fixed-S.S.

C. Lower End Simply Supported-Upper End Fixed

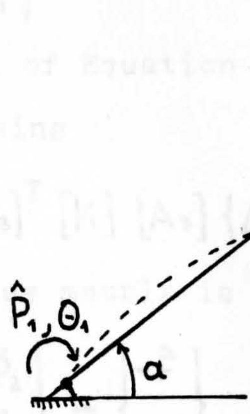


Fig. 3.9 S.S.-Fixed Beam-Column.

In Figure 3.9 the beam-column for the simply supported-fixed case is shown together with the actual end displacements and forces. The matrix equation relating the nodal displacements matrix $\{\delta\}$ with the end displacements matrix becomes

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \Theta_1 \end{Bmatrix} \tag{3.46a}$$

or in symbolic matrix form

$$\{\delta\} = [A_3]\,\{\Delta_3\} \tag{3.46b}$$

In a similar way the relation between the nodal forces matrix $\{q\}$ and the beam-column end force matrix $\{p\}$, is found to be

$$\{p\} = [A_3]^T \{q\} \tag{3.47}$$

where

$$\{p\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix} \tag{3.48}$$

By substitution of Equation (3.47) and (3.46) into Equation (2.22), one obtains

$$\{p\} = [A_3]^T [K] [A_3] \{\Delta_3\} + [A_3]^T \{p_o\} \tag{3.49}$$

The applied forces matrix is related with the end forces by

$$\{p\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix} = \begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix} \tag{3.50}$$

Performing the matrix multiplications in Equation (3.49) and substituting Equation (3.50) into (3.49), one obtains

$$\begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} (-\hat{k}^4 \tilde{s}\, c^2) & (-\hat{k}^3 (1-\tilde{c})\, c) \\ \text{SYM} & (\hat{k}^2 (\hat{k}\tilde{c} - \tilde{s})) \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_1 \end{Bmatrix} + \begin{Bmatrix} -\hat{k}^2 s \\ 0 \end{Bmatrix} \tag{3.51}$$

It follows that

$$\begin{Bmatrix} \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} (-\hat{k}^4 \tilde{s}\, c^2) & (-\hat{k}^3 (1-\tilde{c})\, c) \\ \text{SYM} & (\hat{k}^2 (\hat{k}\tilde{c} - \tilde{s})) \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_1 \end{Bmatrix} \tag{3.52a}$$

or in symbolic matrix form

$$\{f_2\} = \frac{1}{\hat{D}} [K_2] \{A_3\} \tag{3.52b}$$

An important observation is that Equations (3.52a, b) are the same as Equation (3.32) although the matrices $\{\Delta_2\}$ and $\{\Delta_3\}$ possess different components. Solving Equation (3.52) yields

$$\begin{Bmatrix} \hat{V}_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{(\hat{k}^2 s - \hat{F})(\hat{k}\tilde{c} - \tilde{s})}{\hat{k}^3 \tilde{c}\, c^2} \\ -\dfrac{(\hat{k}^2 s - \hat{F})(1-\tilde{c})\,\hat{k}\, c}{\hat{k}^3 \tilde{c}\, c^2} \end{Bmatrix} \tag{3.53}$$

Comparing Equation (3.53) with Equation (3.34), it can be seen that the deflection $\hat{V}_2$ is the same for both cases, while the rotations ϑ_1 and ϑ_2 have the same values.

Proceeding in a similar manner as in case B, the ψ function becomes

$$\psi = \hat{V}_2 s + \frac{\hat{V}_2^2 c^2}{4(\hat{k}\bar{c} - \bar{s})^2}\left[3\left(\hat{k}^2 \bar{c}^2 - \hat{k}\bar{c}\bar{s}\right) + \hat{k}^2 \bar{s}^2\right] \tag{3.54}$$

This equation is the same as Equation (3.37). Therefore, both cases have the same equations relating $\hat{V}_2$, $\hat{F}$ and $\hat{k}$ and hence the same load-deflection curves. This implies that for case C the buckling load $\hat{F}_n$ is the same as in case B although the deflected shapes of the beam-column are not the same for both cases. It is actually seen that one shape is symmetric to the other with respect to a point.

D. Both Ends Fixed

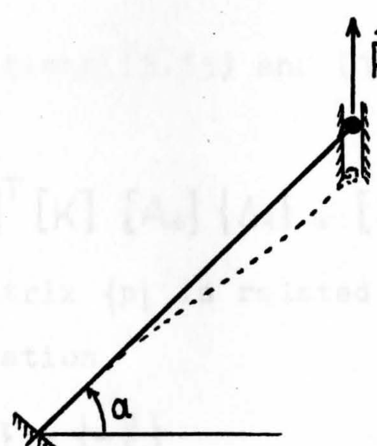


Fig. 3.10 Fixed-Fixed Beam-Column.

In Figure 3.10 the beam-column under consideration and its end forces and displacements are shown. The nodal displacements matrix $\{\delta\}$ is related with the end displacements matrix $\{\Delta_4\}$ by the matrix equation

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \{\hat{V}_2\} \tag{3.55a}$$

or in symbolic matrix form

$$\{\delta\} = [A_4]\{\Delta_4\} \tag{3.55b}$$

Similarly, the nodal forces matrix $\{q\}$ is related to the end forces matrix $\{p\}$ by the equation

$$\{p\} = [A_4]^T\{q\} \tag{3.56}$$

where $\{p\} = \{\hat{P}_2\}$ (3.57)

Substituting Equations (3.55) and (3.56) into Equation (2.22), one obtains

$$\{p\} = [A_4]^T[K]\ [A_4]\{\Delta_4\} + [A_4]^T\{p_0\} \tag{3.58}$$

The end forces matrix $\{p\}$ is related to the applied end forces by the equation

$$\{p\} = \{\hat{P}_2\} = \{-\hat{F}\} \tag{3.59}$$

After performing the matrix multiplications in Equation (3.58), applying Equation (3.59) and manipulating the resulting matrix equation, one obtains

$$\{\hat{k}^2 s - \hat{F}\} = \frac{1}{\hat{D}}\left[-\hat{k}^4\ \mathfrak{Z}\ c^2\right]\{\hat{V}_2\} \tag{3.60}$$

Solving the above equation, one obtains

$$\hat{V}_2 = \left\{ - \frac{(\hat{k}^2 s - \hat{F})\ \hat{D}}{\hat{k}^4\ \tilde{s}\ c^2} \right\} \tag{3.61}$$

Proceeding as in case B the equation of the axial strain function ψ is found to be

$$\psi = \hat{V}_2\ s + \frac{\hat{V}_2^{\,2}\ c^2}{2\ \hat{D}^2}\ \hat{k}^3\ (1-\tilde{c})(2\hat{k} + \hat{k}\ \tilde{c} - 3\ \tilde{s}) \tag{3.62}$$

Substituting ψ and $\hat{D}$ with their equals and making proper manipulations, one obtains

$$\hat{V}_2\ s + \hat{V}_2^{\,2}\ \frac{\hat{k}\ (1-\tilde{c})(2\hat{k} + \hat{k}\tilde{c} - 3\tilde{s})\ c^2}{2\left[\hat{k}\ \tilde{s} - 2\ (1-\tilde{c})\right]^2} + \frac{\hat{k}^2}{R^2} = 0 \tag{3.63}$$

Defining $\quad T = \hat{k}\ \tilde{s} - 2\ (1-\tilde{c})$

and $\quad S = \hat{k}\ (1-\tilde{c})(2\hat{k} + \hat{k}\tilde{c} - 3\tilde{s})$

it follows from Equation (3.63) that

$$(S\ c^2)\ \hat{V}_2^{\,2} + (2T^2\ s)\ \hat{V}_2 + \frac{2\ \hat{k}^2\ T^2}{R^2} = 0 \tag{3.64}$$

Solving the above quadratic equation, yields

$$\hat{V}_2 = \frac{-T^2 s \pm \left[T^4 s^2 - 2\hat{k}^2 c^2\ S\ T^2/R^2\right]^{1/2}}{S\ c^2} \tag{3.65}$$

As in cases B and C the positive sign for the radical is the most important since again the first maximum of the load-deflection curve is found by using the positive sign. Solving Equation (3.61), one obtains

$$\hat{F} = \hat{k}^2 \left(s + \hat{V}_2\ \frac{\hat{k}\ \tilde{s}\ c^2}{T} \right)$$

Normalizing $\hat{F}$ to the Euler's buckling load for this case (β_o = 8.98682), yields

$$\hat{F}_n = \left(\frac{\hat{k}}{\beta_o}\right)^2 \left(s + \hat{V}_2\ \frac{\hat{k}\ \tilde{s}\ c^2}{T} \right) \tag{3.66}$$

Utilizing Equations (3.66) and (3.65), the load-deflection curve is found for a given beam-column.

The equations obtained are given as

$$\hat{V}_2 = \frac{-T^2 s + \left[T^4 s^2 - 2\hat{k}^2 c^2 S\, T^2/R^2\right]^{1/2}}{S\, c^2} \tag{3.67a}$$

$$\hat{F}_n = \left(\frac{\hat{k}}{\beta_o}\right)^2 \left(s + \hat{V}_2\, \frac{\hat{k}\,\tilde{s}\, c^2}{T}\right) \tag{3.67b}$$

where R = the slenderness ratio, a = the slope angle

$c = \cos a$, $s = \sin a$, $\tilde{s} = \sin\hat{k}$, $\tilde{c} = \cos\hat{k}$

$T = \hat{k}\,\tilde{s} - 2\,(1 - \tilde{c})$

$S = \hat{k}\,(1 - \tilde{c})\,(2\,\hat{k} + \hat{k}\,\tilde{c} - 3\,\tilde{s})$

The plot, for given R and a of the beam-column, is constructed by selecting a value of $\hat{k}$, and calculating $\hat{V}_2$ and $\hat{F}_n$ from Equations (3.67a) and (3.67b). Again, a high speed digital computer is utilized for the plot and the computer program used is given in Appendix A. For this case there is no bifurcation point since the value of $\hat{k}^2$ is less than the Euler's buckling load when the maximum point of the load-deflection curve (point of instability) is reached. Again, the beam-column buckles due to direct thrust only.

In Figure 3.11 the load-deflection curve is shown for different angles of inclination a. A very important finding in drawing this curve is that when $\beta_o = 2\pi$ (the Euler's buckling load for the first mode shape) the value of $\hat{F}_n$ exceeds 1 while when $\hat{F}$ is normalized to the Euler's buckling load for the second mode shape ($\beta_o = 8.98682$) the value of $\hat{F}_n$ never exceeds 1. This means that the prevailing

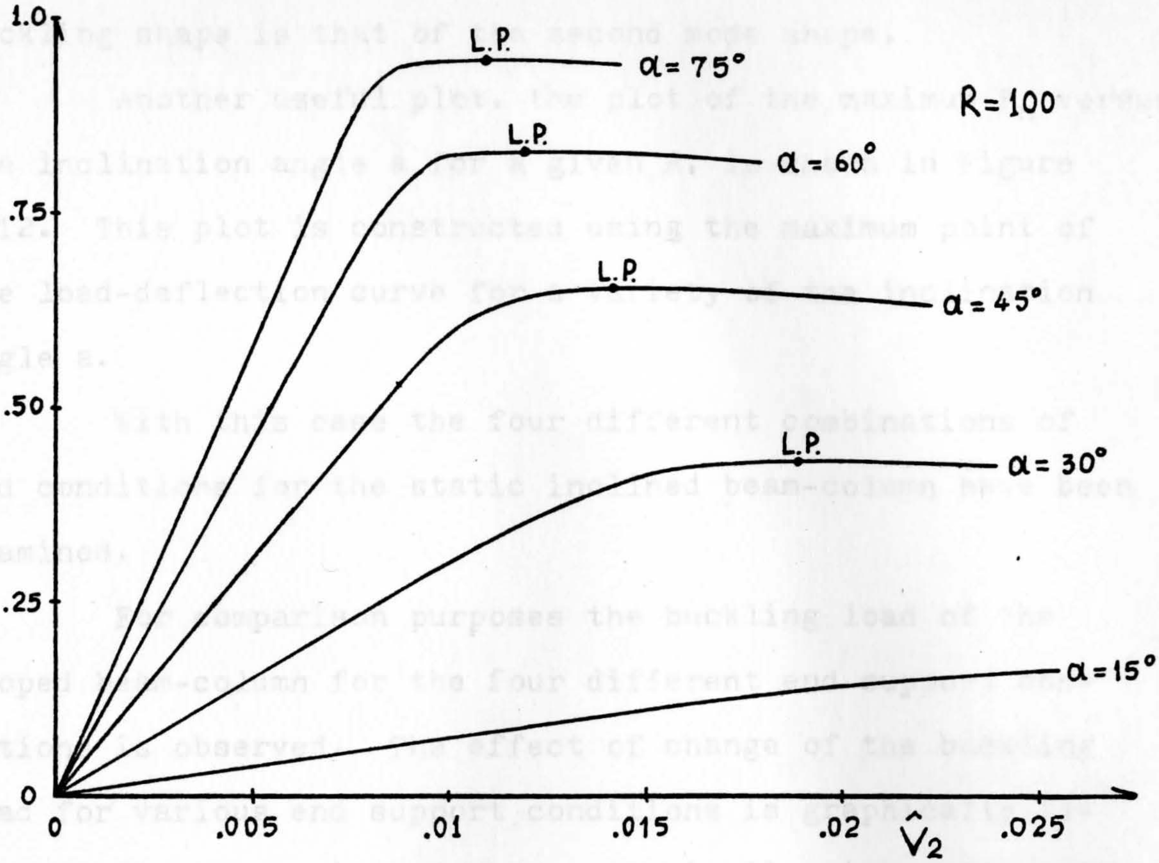


Fig. 3.11 Load-Deflection Curves - Fixed-Fixed.

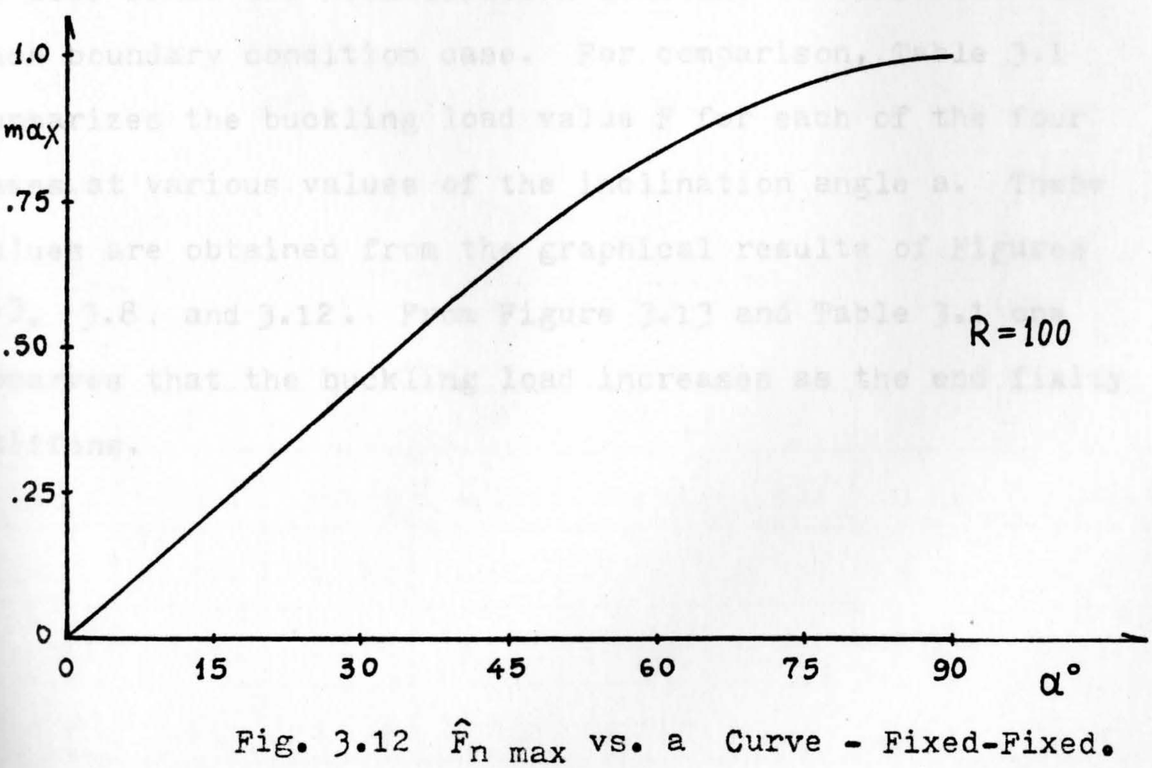


Fig. 3.12 $\hat{F}_{n\ max}$ vs. a Curve - Fixed-Fixed.

buckling shape is that of the second mode shape.

Another useful plot, the plot of the maximum F_n versus the inclination angle a for a given R, is shown in Figure 3.12. This plot is constructed using the maximum point of the load-deflection curve for a variety of the inclination angle a.

With this case the four different combinations of end conditions for the static inclined beam-column have been examined.

For comparison purposes the buckling load of the sloped beam-column for the four different end support conditions is observed. The effect of change of the buckling load for various end support conditions is graphically illustrated. The maximum F versus the inclination angle a is plotted in Figure 3.13. Note that $\hat{F}$ and not $\hat{F}_n$ (normalized) is used since the normalization constant is different for each boundary condition case. For comparison, Table 3.1 summarizes the buckling load value F for each of the four cases at various values of the inclination angle a. These values are obtained from the graphical results of Figures 3.3, 3.8, and 3.12. From Figure 3.13 and Table 3.1 one observes that the buckling load increases as the end fixity stiffens.

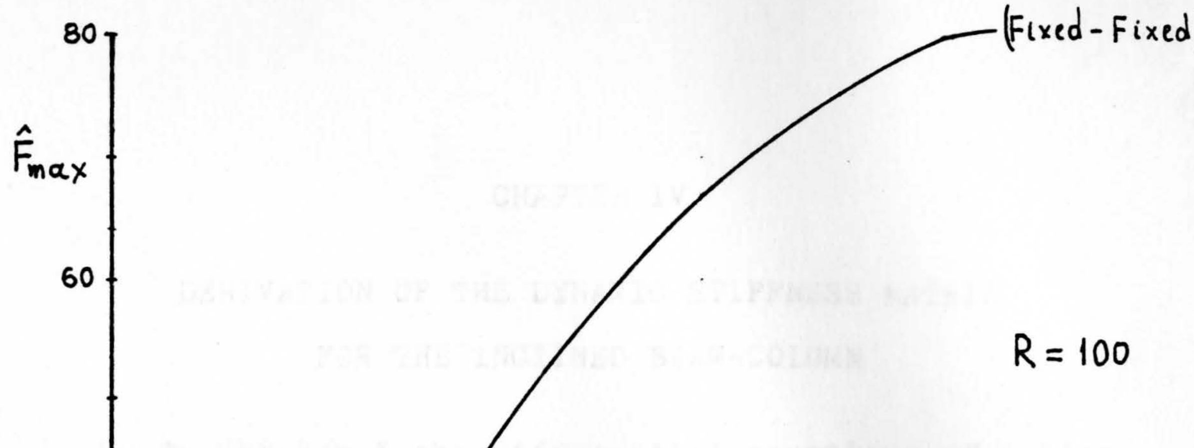


Fig. 3.13 $\hat{F}_{max}$ vs. a Curves for Each Case.

TABLE 3.1

CRITICAL BUCKLING LOAD VALUES FOR COMBINATIONS OF END CONDITIONS, A VARIETY OF INCLINATION ANGLES AND R = 100

	5°	15°	30°	45°	60°	75°	90°
S.S.-S.S.	0.741	2.519	4.920	6.972	8.544	9.533	9.869
S.S.-F.	0.854	4.435	9.436	13.756	17.125	19.319	20.191
F.-S.S.	0.854	4.435	9.436	13.756	17.125	19.319	20.191
F.-F.	1.410	14.025	34.755	52.752	66.974	76.520	80.763

CHAPTER IV

DERIVATION OF THE DYNAMIC STIFFNESS MATRIX FOR THE INCLINED BEAM-COLUMN

In Chapter I the differential equations of motion for a beam-column are given by Equations (1.22) and (1.23) as

$$EI\, v_{,xxxx} - \left(EA\left(u_{,x} + \tfrac{1}{2} v_{,x}^2\right) v_{,x}\right)_{,x} + \rho A \ddot{v} = 0 \qquad (1.22)$$

and $$EA\left(u_{,x} + \tfrac{1}{2} v_{,x}^2\right)_{,x} - \rho A \ddot{u} = 0 \qquad (1.23)$$

Because the equations are nonlinearly coupled both statically and dynamically, a first approximation is made assuming that the axial inertial term $\rho A\ddot{u}$ is relatively small. Hence,

$$\rho A\ddot{u} = 0$$

this uncouples the equations dynamically. Equation (1.23) becomes

$$EA\left(u_{,x} + \frac{1}{2} v_{,x}^2\right) = \text{Constant} \qquad (4.1)$$

The term $(u_{,x} + \frac{1}{2} v^2_{,x})$ is the axial strain of the beam-column previously defined as ψ. Also, the axial force N is defined as

$$N = EA\left(u_{,x} + \tfrac{1}{2} v_{,x}^2\right) \qquad (4.2)$$

and from Equation (4.1) it is constant.

Substituting Equation (4.2) into Equation (1.22), yields

$$EI\, v_{,xxxx} - N\, v_{,xx} + \rho A \ddot{v} = 0$$

or dividing through by EI

$$V_{,xxxx} - \frac{N}{EI} V_{,xx} + \frac{\rho A}{EI} \ddot{V} = 0$$

Denoting $\quad k^2 = -\frac{N}{EI}$

and $\quad \bar{m} = \frac{\rho A}{EI}$

the above equation yields

$$V_{,xxxx} + k^2 V_{,xx} + \bar{m} \ddot{V} = 0 \tag{4.3}$$

Recall that dot (·) and subscript after comma denote appropriate time or length derivative, respectively. The term v is the transverse deflection function and is both position and time dependent ($v_{(x,t)}$).

The function $v_{(x,t)}$ must be harmonic with finite bound as the time t increases. Hence,

$$V_{(x,t)} = V_{(x)} \, e^{i\omega t} \tag{4.4}$$

Using Equation (4.4), Equation (4.3) becomes

$$\left(V_{(x),xxxx} + k^2 V_{(x),xx} - \bar{m} \omega^2 V_{(x)}\right) e^{i\omega t} = 0$$

or

$$V_{(x),xxxx} + k^2 V_{(x),xx} - \bar{m} \omega^2 V_{(x)} = 0 \tag{4.5}$$

Solution of Equation (4.5) is obtained assuming $V_{(x)} = A\, e^{\lambda x}$, then the characteristic equation becomes

$$\lambda^4 + k^2 \lambda^2 - \bar{m} \omega^2 = 0 \tag{4.6}$$

Solution of the Equation (4.6), yields the following four roots :

$$\lambda_1 = \left[\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\, \omega^2\right]^{1/2} - \frac{k^2}{2}\right]^{1/2}$$

$$\lambda_2 = \left[-\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\,\omega^2\right]^{1/2} - \frac{k^2}{2}\right]^{1/2}$$

$$\lambda_3 = -\left[\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\,\omega^2\right]^{1/2} - \frac{k^2}{2}\right]^{1/2}$$

$$\lambda_4 = -\left[-\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\,\omega^2\right]^{1/2} - \frac{k^2}{2}\right]^{1/2}$$

Defining $$\alpha = \left[\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\,\omega^2\right]^{1/2} + \frac{k^2}{2}\right] \qquad (4.7a)$$

and $$\rho = \left[\left[\left(\frac{k^2}{2}\right)^2 + \bar{m}\,\omega^2\right]^{1/2} - \frac{k^2}{2}\right] \qquad (4.7b)$$

the four roots of the equation become

$$\lambda_1 = \rho$$
$$\lambda_2 = i\alpha$$
$$\lambda_3 = -\rho$$
$$\lambda_4 = -i\alpha$$

The solution of the Equation (4.5) after proper mathematical operations is given as

$$V_{(x)} = A_1 \sin\alpha x + A_2 \cos\alpha x + A_3 \sinh\rho x + A_4 \cosh\rho x \qquad (4.8)$$

where A_1, A_2, A_3 and A_4 are constants to be evaluated. The evaluation of the A_i's is determined in terms of the nodal displacements (rotations) by applying the boundary conditions. At first, the positive sign convention of the displacements (rotations) is established in the manner shown in Figure 4.1.

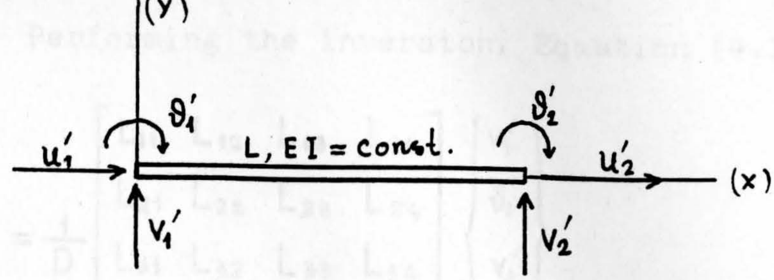


Fig. 4.1 Positive Sign Convention-Displacements (Rotations).

The boundary conditions are defined as follows:

at $x = 0$ $\quad V(x) = V_1'$ (4.9a)

$V(x)_{,x} = -\vartheta_1'$ (4.9b)

at $x = L$ $\quad V(x) = V_2'$ (4.9c)

$V(x)_{,x} = -\vartheta_2'$ (4.9d)

It follows that,

$$V_1' = A_2 + A_4$$
$$\vartheta_1' = -A_1\alpha - A_3\rho$$
$$V_2' = A_1\tilde{s} + A_2\tilde{c} + A_3 S + A_4 C$$
$$\vartheta_2' = -A_1\alpha\tilde{c} + A_2\alpha\tilde{s} - A_3\rho C - A_4\rho S$$

where $\tilde{c} = \cos\alpha l$, $\tilde{s} = \sin\alpha l$, $C = \cosh\rho l$, $S = \sinh\rho l$.

The above equations are expressed in a matrix form as follows

$$\begin{Bmatrix} V_1' \\ \vartheta_1' \\ V_2' \\ \vartheta_2' \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\alpha & 0 & -\rho & 0 \\ \tilde{s} & \tilde{c} & S & C \\ -\alpha\tilde{c} & \alpha\tilde{s} & -\rho C & -\rho S \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (4.10a)$$

or in symbolic matrix notation as

$$\{\delta'\} = [\hat{L}]\{A\} \qquad (4.10b)$$

In order to solve for the unknown constants, matrix $[\hat{L}]$ must be inverted. Performing the inversion, Equation (4.10) yields

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{Bmatrix} v_1' \\ \vartheta_1' \\ v_2' \\ \vartheta_2' \end{Bmatrix} \tag{4.11a}$$

or in symbolic matrix notation

$$\{A\} = [\hat{L}]^{-1} \{\delta'\} \tag{4.11b}$$

where $D = 2\alpha\rho(1-\bar{c}C) + (\rho^2-\alpha^2)\bar{s}S$ (4.12)

and the components of the $[\hat{L}]$ matrix are defined as follows:

$$L_{11} = -\rho(\rho\bar{c}S + \alpha\bar{s}C), \qquad L_{12} = -[\rho(1-\bar{c}C) - \alpha\bar{s}S]$$

$$L_{13} = \rho(\alpha\bar{s} + \rho S), \qquad L_{14} = \rho(C-\bar{c})$$

$$L_{21} = \rho[\alpha(1-\bar{c}C) + \rho\bar{s}S], \qquad L_{22} = \alpha\bar{c}S - \rho\bar{s}C$$

$$L_{23} = -\alpha\rho(C-\bar{c}), \qquad L_{24} = \rho\bar{s} - \alpha S$$

$$L_{31} = \alpha(\rho\bar{c}S + \alpha\bar{s}C), \qquad L_{32} = -[\alpha(1-\bar{c}C) + \rho\bar{s}S]$$

$$L_{33} = -\alpha(\rho S + \alpha\bar{s}), \qquad L_{34} = -\alpha(C-\bar{c})$$

$$L_{41} = \alpha[\rho(1-\bar{c}C) - \alpha\bar{s}S], \qquad L_{42} = -(\alpha\bar{c}S - \rho\bar{s}C)$$

$$L_{43} = \alpha\rho(C-\bar{c}), \qquad L_{44} = \alpha S - \rho\bar{s}$$

The relation between the nodal forces and nodal displacements are now derived. First, a positive sign convention for the nodal forces (moments) is established as shown in Figure 4.2.

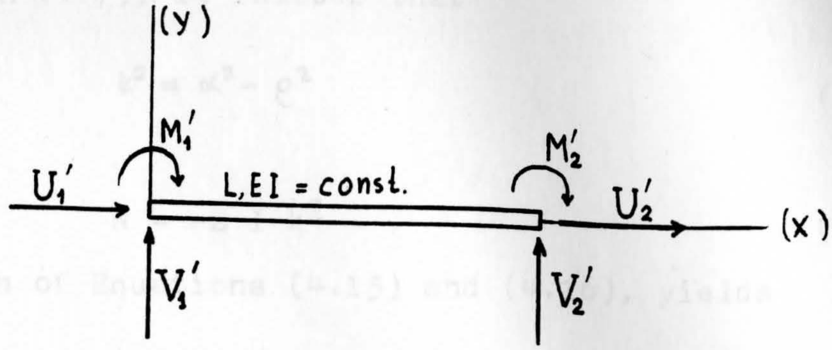


Fig. 4.2 Positive Sign Convention-Nodal Forces (Moments).

The equations relating the moment and shear force functions, to the displacement function and the axial force, are as follows:

$$M(x) = EI\, v_{,xx} \tag{4.13a}$$

$$V(x) = -EI\, v_{,xxx} + N\, v_{,x} \tag{4.13b}$$

The boundary conditions are defined as follows:

at $x = 0$ $\quad M(x) = M_1'$

$\qquad\qquad\quad V(x) = -V_1'$

at $x = l$ $\quad M(x) = -M_2'$

$\qquad\qquad\quad V(x) = V_2'$

It follows that,

$$M_1' = EI\left(-A_2\,\alpha^2 + A_4\,\rho^2\right) \tag{4.14a}$$

$$M_2' = EI\left(A_1\,\alpha^2\,\tilde{s} + A_2\,\alpha^2\,\tilde{c} - A_3\,\rho^2\,S - A_4\,\rho^2\,C\right) \tag{4.14b}$$

$$V_1' = EI\left(-A_1\,\alpha^3 + A_3\,\rho^3\right) - N\left(A_1\,\alpha + A_3\,\rho\right) \tag{4.14c}$$

$$V_2' = EI\left(A_1\,\alpha^3\,\tilde{c} - A_2\,\alpha^3\,\tilde{s} - A_3\,\rho^3\,C - A_4\,\rho^3\,S\right) - \tag{4.14d}$$
$$-N\left(-A_1\,\alpha\,\tilde{c} + A_2\,\alpha\,\tilde{s} - A_3\,\rho\,C - A_4\,\rho\,S\right)$$

From Equation (4.7), it follows that

$$k^2 = \alpha^2 - \rho^2 \tag{4.15}$$

with

$$N = -E\,I\,k^2 \tag{4.16}$$

Consideration of Equations (4.15) and (4.16), yields

$$N = EI\,(\rho^2 - \alpha^2) \tag{4.17}$$

Substituting Equation (4.17) into Equation (4.14), one obtains

$$M_1' = EI\left(-A_2\,\alpha^2 + A_4\,\rho^2\right) \tag{4.18a}$$

$$M_2' = EI\left(A_1\,\alpha^2\,\tilde{s} + A_2\,\alpha^2\,\tilde{c} - A_3\,\rho^2\,S - A_4\,\rho^2\,C\right) \tag{4.18b}$$

$$V_1' = EI\left(-A_1\,\alpha\,\rho^2 + A_3\,\alpha^2\,\rho\right) \tag{4.18c}$$

$$V_2' = EI\left(A_1\,\alpha\rho^2\,\tilde{c} - A_2\,\alpha\,\rho^2\,\tilde{s} - A_3\,\alpha^2\rho\,C - A_4\alpha^2\rho\,S\right) \tag{4.18d}$$

Expressing Equation (4.18) in matrix form, yields

$$\begin{Bmatrix} V_1' \\ M_1' \\ V_2' \\ M_2' \end{Bmatrix} = EI \begin{bmatrix} (-\alpha\rho^2) & (0) & (\alpha^2\rho) & (0) \\ (0) & (-\alpha^2) & (0) & (\rho^2) \\ (\alpha\rho^2\tilde{c}) & (-\alpha\rho^2\tilde{s}) & (-\alpha^2\rho\,C) & (-\alpha^2\rho\,S) \\ (\alpha^2\tilde{s}) & (\alpha^2\tilde{c}) & (-\rho^2 S) & (-\rho^2 C) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \tag{4.19a}$$

or in symbolic matrix notation

$$\{G\} = EI\,[N]\,\{A\} \tag{4.19b}$$

Substituting Equation (4.11) into (4.19), one obtains

$$\{G\} = EI\,[N]\,[\hat{L}]^{-1}\,\{\delta'\} \tag{4.20}$$

Performing the matrix multiplications in Equation (4.20), yields

$$\begin{Bmatrix} V_1' \\ M_1' \\ V_2' \\ M_2' \end{Bmatrix} = \frac{EI}{D} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ & Q_{22} & Q_{23} & Q_{24} \\ & & Q_{33} & Q_{34} \\ \text{SYM} & & & Q_{44} \end{bmatrix} \begin{Bmatrix} v_1' \\ \vartheta_1' \\ v_2' \\ \vartheta_2' \end{Bmatrix} \tag{4.21}$$

where the terms Q_{ij} are defined as follows

$$Q_{11} = \alpha\rho(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C), \quad Q_{12} = \alpha\rho\left[(\rho^2-\alpha^2)(1-\bar{c}C) - 2\alpha\rho\bar{s}S\right]$$

$$Q_{13} = -\alpha\rho(\alpha^2+\rho^2)(\alpha\bar{s} + \rho S), \quad Q_{14} = -\alpha\rho(\alpha^2+\rho^2)(C-\bar{c})$$

$$Q_{22} = (\alpha^2+\rho^2)(\rho\bar{s}C - \alpha\bar{c}S), \quad Q_{23} = -Q_{14}$$

$$Q_{24} = (\alpha^2+\rho^2)(\alpha S - \rho\bar{s}), \quad Q_{33} = Q_{11}$$

$$Q_{34} = -Q_{12}, \quad Q_{44} = Q_{22}$$

Note that there are six (6) independent terms in $[Q]$. Utilizing the observation made at the end of Chapter II, the matrix equation relating the nodal forces (moments) to the nodal displacements (rotations) takes the following form:

$$\begin{Bmatrix} U_1' \\ V_1' \\ M_1' \\ U_2' \\ V_2' \\ M_2' \end{Bmatrix} = \frac{EI}{D} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{11} & Q_{12} & 0 & Q_{13} & Q_{14} \\ 0 & Q_{12} & Q_{22} & 0 & Q_{23} & Q_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{13} & Q_{23} & 0 & Q_{33} & Q_{34} \\ 0 & Q_{14} & Q_{24} & 0 & Q_{34} & Q_{44} \end{bmatrix} \begin{Bmatrix} u_1' \\ v_1' \\ \vartheta_1' \\ u_2' \\ v_2' \\ \vartheta_2' \end{Bmatrix} + \begin{Bmatrix} EIk^2 \\ 0 \\ 0 \\ -EIk^2 \\ 0 \\ 0 \end{Bmatrix} \tag{4.22a}$$

or in symbolic matrix notation

$$\{q'\} = [K']\{\delta'\} + \{P_o'\} \tag{4.22b}$$

The application of the boundary conditions for the beam-column requires that the latter matrix equation be transformed to a global coordinate system. The positive sign convention for the nodal forces and displacements and for the coordinates of the global system is established and is shown in Figure 4.3.

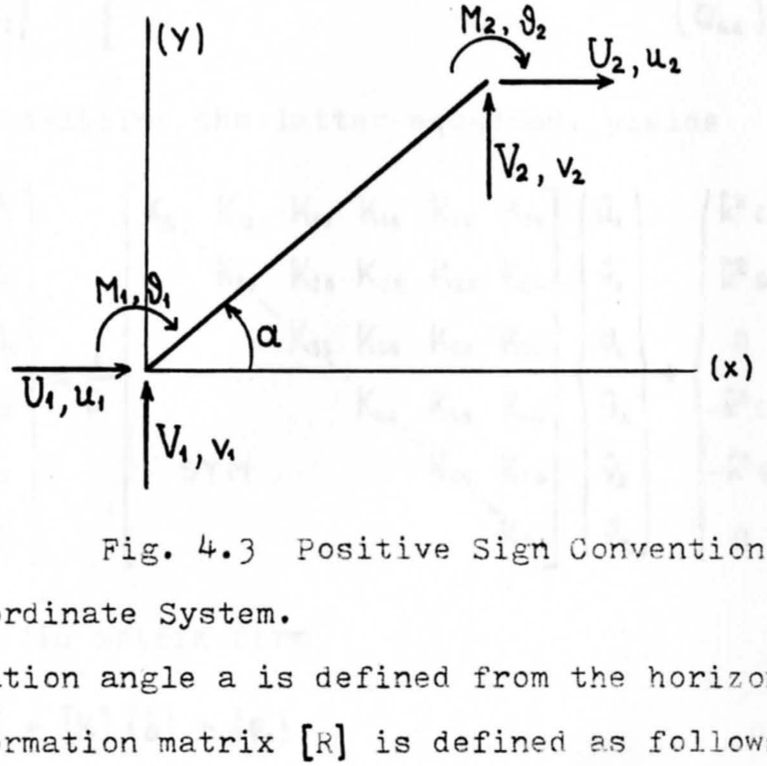


Fig. 4.3 Positive Sign Convention -Global Coordinate System.

The inclination angle a is defined from the horizontal axis. The transformation matrix [R] is defined as follows:

$$[R] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{where} \quad c = \cos a, \quad s = \sin a$$

Equation (4.22b) is transformed into global coordinates as

$$\{q\} = [R]^T [K'] [R] \{\delta\} + [R]^T \{p_o'\} \qquad (4.23)$$

Performing the matrix multiplications Equation (4.23) becomes

$$\begin{Bmatrix} U_1 \\ V_1 \\ M_1 \\ U_2 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{EI}{D} \begin{bmatrix} (Q_{11} s^2) & (-Q_{11} cs) & (-Q_{12} s) & (Q_{13} s^2) & (-Q_{13} cs) & (-Q_{14} s) \\ & (Q_{11} c^2) & (Q_{12} c) & (-Q_{13} cs) & (Q_{13} c^2) & (Q_{14} c) \\ & & (Q_{22}) & (-Q_{23} s) & (Q_{23} c) & (Q_{24}) \\ & & & (Q_{33} s^2) & (-Q_{33} cs) & (-Q_{34} s) \\ & \text{SYM} & & & (Q_{33} c^2) & (Q_{34} c) \\ & & & & & (Q_{44}) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vartheta_1 \\ u_2 \\ v_2 \\ \vartheta_2 \end{Bmatrix} + \begin{Bmatrix} EIk^2 c \\ EIk^2 s \\ 0 \\ -EIk^2 c \\ -EIk^2 s \\ 0 \end{Bmatrix}$$

Nondimensionalizing the latter equation, yields

$$\begin{Bmatrix} \hat{U}_1 \\ \hat{V}_1 \\ \hat{M}_1 \\ \hat{U}_2 \\ \hat{V}_2 \\ \hat{M}_2 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & & & K_{44} & K_{45} & K_{46} \\ & \text{SYM} & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} + \begin{Bmatrix} \hat{k}^2 c \\ \hat{k}^2 s \\ 0 \\ -\hat{k}^2 c \\ -\hat{k}^2 s \\ 0 \end{Bmatrix} \qquad (4.24a)$$

or in symbolic matrix form

$$\{q\} = [K] \{\delta\} + \{p_o\} \qquad (4.24b)$$

where

$$K_{11} = \alpha\rho\,(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C)\,s^2$$

$$K_{12} = -\alpha\rho\,(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C)\,cs$$

$$K_{13} = -\alpha\rho\left[(\rho^2-\alpha^2)(1-\bar{c}C) - 2\alpha\rho\bar{s}S\right] s$$

$$K_{14} = -\alpha\rho\,(\alpha^2+\rho^2)(\alpha\bar{s} + \rho S)\,s^2$$

$$K_{15} = \alpha\rho\,(\alpha^2+\rho^2)(\alpha\bar{s} + \rho S)\,cs$$

$$K_{16} = \alpha\rho\,(\alpha^2+\rho^2)(C - \bar{c})\,s$$

$$K_{22} = \alpha\rho\,(\alpha^2+\rho^2)(\rho\bar{c}S+\alpha\bar{s}C)\,c^2$$

$$K_{23} = \alpha\rho\left[(\rho^2-\alpha^2)(1-\bar{c}C)-2\alpha\rho\bar{s}S\right]c$$

$$K_{24} = K_{15}$$

$$K_{25} = -\alpha\rho\,(\alpha^2+\rho^2)(\alpha\bar{s}+\rho S)\,c^2$$

$$K_{26} = -\alpha\rho\,(\alpha^2+\rho^2)(C-\bar{c})\,c$$

$$K_{33} = (\alpha^2+\rho^2)(\rho\bar{s}C-\alpha\bar{c}S)$$

$$K_{34} = -K_{16}$$

$$K_{35} = -K_{26}$$

$$K_{36} = (\alpha^2+\rho^2)(\alpha S-\rho\bar{s})$$

$$K_{44} = K_{11}$$

$$K_{45} = K_{12}$$

$$K_{46} = -K_{13}$$

$$K_{55} = K_{22}$$

$$K_{56} = -K_{23}$$

$$K_{66} = K_{33}$$

and

$$\alpha = \left[\left[\left(\frac{\hat{k}^2}{2}\right)^2+\hat{m}\,\omega^2\right]^{1/2}+\frac{\hat{k}^2}{2}\right]^{1/2}$$

$$\rho = \left[\left[\left(\frac{\hat{k}^2}{2}\right)^2+\hat{m}\,\omega^2\right]^{1/2}-\frac{\hat{k}^2}{2}\right]^{1/2}$$

$$\hat{k}^2 = -\frac{NL^2}{EI}\,,\qquad \hat{m} = \frac{\rho A L^4}{EI}$$

$$\bar{c} = \cos\alpha\,,\quad \bar{s} = \sin\alpha\,,\quad C = \cosh\rho\,,\quad S = \sinh\rho$$

$$c = \cos a\,,\quad s = \sin a$$

Also,

$$\hat{U}_1 = \frac{U_1 L^2}{EI}\,,\quad \hat{V}_1 = \frac{V_1 L^2}{EI}\,,\quad \hat{M}_1 = \frac{M_1 L}{EI}$$

$$\hat{U}_2 = \frac{U_2 L^2}{EI}\,,\quad \hat{V}_2 = \frac{V_2 L^2}{EI}\,,\quad \hat{M}_2 = \frac{M_2 L}{EI}$$

$$\hat{u}_1 = \frac{u_1}{L}\,,\quad \hat{u}_2 = \frac{u_2}{L}\,,\quad \hat{v}_1 = \frac{v_1}{L}\,,\quad \hat{v}_2 = \frac{v_2}{L}$$

and finally, $D = 2\alpha\rho\,(1-\bar{c}\,C) + (\rho^2-\alpha^2)\,\bar{s}\,S$

The stiffness matrix has been developed without using the ψ function like in the static problem. The ψ function, however, is necessary for the solution of the dynamic problem. The axial strain function ψ is given by the relation

$$\psi = u_{,x} + \frac{1}{2}\, v_{,x}^2$$

which is constant. It follows that

$$u_{,x} = \psi - \frac{1}{2}\, v_{,x}^2$$

Integration of the latter equation, yields

$$u = \psi\, x - \frac{1}{2}\int_0^x v_{,x}^2\, dx \qquad (4.25)$$

Noting $v_{,x}$ as equal to

$$v_{,x} = \left\{ (\alpha\bar{c})\ (-\alpha\bar{s})\ (\rho C)\ (\rho S) \right\} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}$$

it follows that,

$$v_{,x}^2 = \left\{ A_1\ A_2\ A_3\ A_4 \right\} \begin{bmatrix} \alpha^2\bar{c}^2 & -\alpha^2\bar{c}\bar{s} & \alpha\rho\bar{c}C & \alpha\rho\bar{c}S \\ & \alpha^2\bar{s}^2 & -\alpha\rho\bar{s}C & -\alpha\rho\bar{s}S \\ & & \rho^2C^2 & \rho^2CS \\ \text{SYM} & & & \rho^2S^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (4.26a)$$

or in symbolic matrix notation

$$v_{,x}^2 = \{A\}^T \left[K_{Ax}\right] \{A\} \qquad (4.26b)$$

Equation (4.25) becomes

$$u = \psi x - \frac{1}{2}\int_0^x \{A\}^T [K'_{Ax}] \{A\}\, dx$$

or

$$u = \psi x - \frac{1}{2}\{A\}^T \int_0^x [K'_{Ax}]\, dx \;\{A\}$$

Performing the integration, applying the boundary conditions

that at $x = 0$ $u = u_1$

and at $x = L$ $u = u_2$

and solving for ψ, one obtains

$$\psi = \frac{u'_2 - u'_1}{L} + \frac{1}{2L}\{A\}^T [K'_A] \{A\} \tag{4.27}$$

where $[K_A]$ is a symmetric matrix with components given as follows:

$$K'_{A_{11}} = \frac{\alpha^2 L}{2} + \frac{\alpha}{2}\tilde{c}\tilde{s}\,, \qquad K'_{A_{12}} = -\frac{\alpha}{2}\tilde{s}^2$$

$$K'_{A_{13}} = \frac{\alpha e}{\alpha^2+e^2}\left[e\tilde{c}S + \alpha\tilde{s}C\right], \qquad K'_{A_{14}} = \frac{\alpha e}{\alpha^2+e^2}\left[\alpha\tilde{s}S - e(1-\tilde{c}C)\right]$$

$$K'_{A_{22}} = \frac{\alpha^2 L}{2} - \frac{\alpha}{2}\tilde{c}\tilde{s}\,, \qquad K'_{A_{23}} = -\frac{\alpha e}{\alpha^2+e^2}\left[e\tilde{s}S + \alpha(1-\tilde{c}C)\right]$$

$$K'_{A_{24}} = -\frac{\alpha e}{\alpha^2+e^2}\left[e\tilde{s}C - \alpha\tilde{c}S\right], \qquad K'_{A_{33}} = \frac{e^2 L}{2} + \frac{e}{2}CS$$

$$K'_{A_{34}} = \frac{e}{2}S^2, \qquad K'_{A_{44}} = -\left(\frac{e^2}{2}L - \frac{e}{2}CS\right)$$

In order to obtain ψ in terms of the nodal displacements (rotations), Equation (4.11) is substituted into Equation (4.27) yielding

$$\psi = \frac{u'_2 - u'_1}{L} + \frac{1}{2LD^2}\{\delta'\}^T [\hat{L}]^{-T} [K'_A] [\hat{L}]^{-1} \{\delta'\} \tag{4.28}$$

or

$$\psi = \frac{u_2' - u_1'}{L} + \frac{1}{2LD^2}\{\delta'\}^T[K_A]\{\delta'\} \tag{4.29}$$

where $[K_A] = [\hat{L}]^{-T}[K_A'][\hat{L}]^{-1}$

The terms of matrix $[K_A]$ are algebrically enormous and for this paper they are not calculated. For the problem of an applied dynamic load $F(t)$ of frequency $\bar{\omega}$ they are needed since, for any case the manipulation of Equation (4.24) yields an equation containing three unknowns $\hat{F}$, $\hat{k}$ and $\hat{V}_2$ and the dynamic load-deflection curve maynot be plotted without the use of the ψ function.

CHAPTER V

APPLICATIONS TO THE DYNAMIC INCLINED BEAM-COLUMN PROBLEM

The same four different combinations of end-support conditions examined in the static problem, are considered for the dynamic problem. Recall that only the free vibration problem will be examined herein.

A. Both Ends Simply Supported

Fig. 5.1 S.S.-S.S. Beam-Column.

The end forces, and end displacements are shown in Figure 5.1 for a beam-column with both ends simply supported. The relation between the nodal displacements $\{\delta\}$ and the end displacements matrix $\{\Delta_1\}$ is given by

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{5.1a}$$

or in symbolic matrix notation

$$\{\delta\} = [A_1]\{\Delta_1\} \tag{5.1b}$$

The relation between nodal forces and end forces is given in a similar manner as

$$\{P\} = [A_1]^T\{q\} \tag{5.2}$$

where

$$\{P\} = \begin{Bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{Bmatrix}$$

The relation between end forces and applied end forces is given here as

$$\{P\} = \begin{Bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\hat{F} \\ 0 \end{Bmatrix} \tag{5.3}$$

Substituting Equations (5.2) and (5.1) into Equation (4.24), one obtains

$$\{P\} = [A_1]^T[K]\,[A_1]\{\Delta_1\} + [A_1]^T\{P_0\} \tag{5.4}$$

Substituting Equation (5.3) into the latter equation, performing the matrix multiplications and combinations, yields

$$\begin{Bmatrix} 0 \\ \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} K_{33} & K_{35} & K_{36} \\ & K_{55} & K_{56} \\ \text{SYM} & & K_{66} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} \tag{5.5a}$$

or in symbolic matrix notation

$$\{f_1\} = \frac{1}{D} [K_1] \{\Delta_1\} \tag{5.5b}$$

Solving for the displacements $\{\Delta_1\}$, one obtains

$$\{\Delta_1\} = \frac{D}{d_1} [K_1]^{-1} \{f_1\} \tag{5.6}$$

where $\quad d_1 = \alpha\rho\, D\, (\alpha^2+\rho^2)(\rho^3 \tilde{c}\, S - \alpha^3 \tilde{s}\, C)\; c^2$

Performing the inversion of $[K_1]$ and then multiplying the matrices, yields

$$\begin{Bmatrix} \Theta_1 \\ \hat{V}_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{(\alpha \tilde{s} + \rho S)(\hat{k}^2 s - \hat{F})}{(\alpha^3 \tilde{s} C - \rho^3 \tilde{c} S)\, c} \\[2ex] -\dfrac{(\alpha^2+\rho^2)\, \tilde{s} S\, (\hat{k}^2 s - \hat{F})}{\alpha\rho\, (\alpha^3 \tilde{s} C - \rho^3 \tilde{c} S)\, c^2} \\[2ex] \dfrac{(\alpha \tilde{s} C + \rho \tilde{c} S)(\hat{k}^2 s - \hat{F})}{(\alpha^3 \tilde{s} C - \rho^3 \tilde{c} S)\, c} \end{Bmatrix} \tag{5.7a}$$

or for $\hat{V}_2$ only

$$\hat{V}_2 = -\frac{(\alpha^2+\rho^2)\, \tilde{s} S\, (\hat{k}^2 s - \hat{F})}{\alpha\rho\, (\alpha^3 \tilde{s} C - \rho^3 \tilde{c} S)\, c^2} \tag{5.7b}$$

Since the free vibration problem is examined, it is assumed that the applied force $\hat{F}$ is neglected in the analysis. Then, manipulating Equation (5.7b), it follows that

$$\hat{V}_2\, \alpha\rho\, (\alpha^3 \tilde{s}\, C - \rho^3 \tilde{c}\, S)\, c^2 = -\, \hat{k}^2\, s\, (\alpha^2+\rho^2)\, \tilde{s}\, S \tag{5.8}$$

The ψ function is given by Equation (4.29). Considering the nonlinear term $\frac{1}{2}\, v_{,x}^2$ very small in comparison with the linear term $u_{,x}$, the ψ function is given by

$$\psi = \frac{u_2' - u_1'}{L} = \hat{u}_2' - \hat{u}_1'$$

or
$$\psi = \{-1 \quad 1\} \begin{Bmatrix} \hat{u}_1' \\ \hat{u}_2' \end{Bmatrix} \tag{5.9}$$

where the displacements $\hat{u}_1'$ and $\hat{u}_2'$ are the nodal axial displacements in the untransformed system. Their relation with the nodal displacements of the global system is given by Equation (3.10a), as

$$\begin{Bmatrix} \hat{u}_1' \\ \hat{u}_2^2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vartheta_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \vartheta_2 \end{Bmatrix}$$

or in symbolic matrix notation by Equation (3.10b), as

$$\{\delta_1'\} = [R_1]\{\delta\}$$

Applying the latter equation, Equation (5.9) becomes

$$\psi = \{-1 \quad 1\}\,[R_1]\,\{\delta\}$$

Then, applying Equation (5.1b) the latter equation yields

$$\psi = \{-1 \quad 1\}\,[R_1]\,[A_1]\,\{\Delta_1\} \tag{5.10}$$

Performing the matrix multiplications Equation (5.10) yields

$$\psi = \{0 \quad S \quad 0\} \begin{Bmatrix} \theta_1 \\ \hat{V}_2 \\ \theta_2 \end{Bmatrix}$$

or $$\psi = \hat{V}_2 \, s \tag{5.11}$$

Equation (5.11) reveals that, without considering the non-linear term, the axial strain is given by the component of $\hat{V}_2$ in the direction of the undeformed beam-column. From Equation (5.11) using the relation between ψ and $\hat{k}$, one obtains

$$-\hat{k}^2 = \hat{V}_2 R^2 s \tag{5.12}$$

Substituting this value of $\hat{k}^2$ into Equation (5.8), yields

$$\hat{V}_2 \, \alpha \rho \left(\alpha^3 \bar{s} \, C - \rho^3 \bar{c} \, S\right) c^2 = \hat{V}_2 \, R^2 s^2 \left(\alpha^2 + \rho^2\right) \bar{s} \, S$$

or $$\hat{V}_2 \left(\alpha \rho \left(\alpha^3 \bar{s} \, C - \rho^3 \bar{c} \, S\right) c^2 - R^2 s^2 \left(\alpha^2 + \rho^2\right) \bar{s} \, S\right) = 0 \tag{5.13}$$

For the free vibration problem at the resonant frequency $\hat{V}_2$ must be arbitrary. Therefore, the coefficient of $\hat{V}_2$ in Equation (5.13) must be zero. It follows that,

$$\alpha \rho \left(\alpha^3 \bar{s} \, C - \rho^3 \bar{c} \, S\right) c^2 - R^2 s^2 \left(\alpha^2 + \rho^2\right) \bar{s} \, S = 0 \tag{5.14}$$

The latter equation is the frequency equation for the simply supported inclined beam-column. It is a trancentental equation containing the squared frequency term $\hat{m}\,\omega^2$ and the induced axial force term $\hat{k}^2$ in relation with the inclination angle a and the slenderness ratio R. A plot of axial force versus the fundamental frequency is constructed for a variety of inclination angles a and a given R ratio. The computer program used to determine the roots of Equation (5.14) is given in Appendix A; program Number 3. This program reads various values of a and R and prints out the $\hat{k}$ versus the lowest $\hat{m}\,\omega^2$ curve as shown in Figure 5.2.

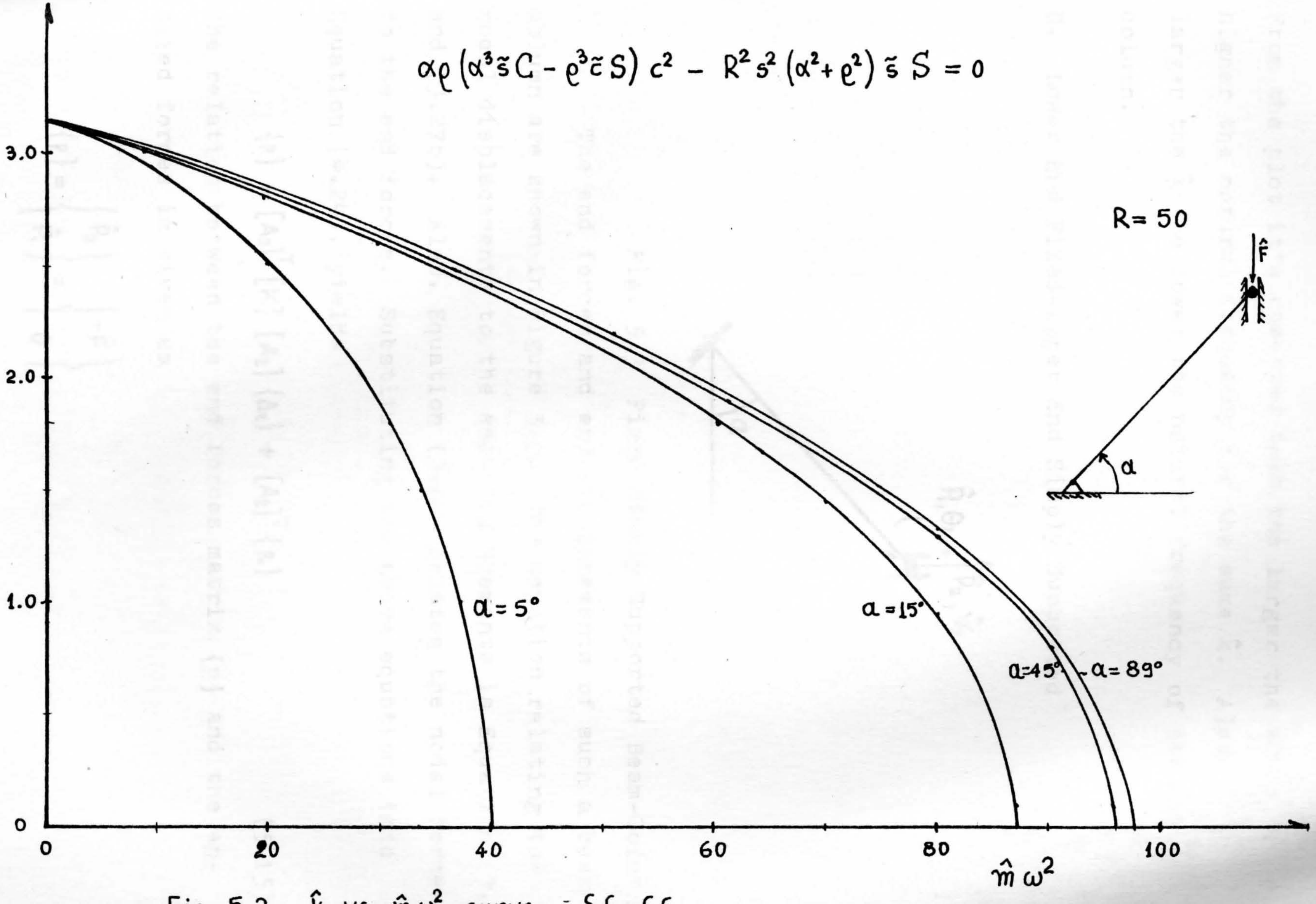


Fig. 5.2 $\hat{k}$ vs. $\hat{m}\omega^2$ curve – S.S.-S.S.

From the plot it's observed that the larger the angle a the higher the natural frequency for the same $\hat{k}$. Also, the larger the $\hat{k}$ the lower the natural frequency of the beam-column.

B. Lower End Fixed-Upper End Simply Supported

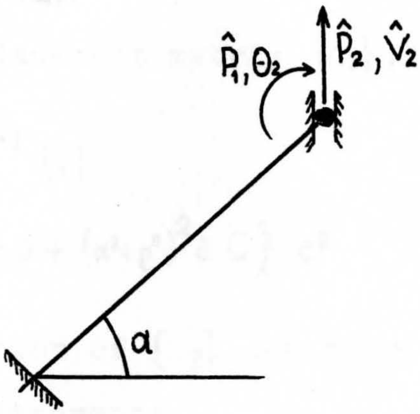


Fig. 5.3 Fixed-Simply Supported Beam-Column.

The end forces and end displacements of such a beam-column are shown in Figure 5.3. The equation relating the nodal displacements to the end displacements is Equation (3.27a) and (3.27b). Also, Equation (3.28) relates the nodal forces to the end forces. Substituting the above equations into Equation (4.24), yields

$$\{p\} = [A_2]^T [K] [A_2] \{\Delta_2\} + [A_2]^T \{p_o\} \tag{5.15}$$

The relation between the end forces matrix $\{p\}$ and the applied forces is given as

$$\{p\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix} = \begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix}$$

Applying the latter equation into Equation (5.15), performing the matrix multiplications and manipulating properly, yields

$$\begin{Bmatrix} \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} K_{55} & K_{56} \\ \text{SYM} & K_{66} \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_2 \end{Bmatrix} \tag{5.16a}$$

or in symbolic matrix form

$$\{f\} = \frac{1}{D} [K_2] \{\Delta_2\} \tag{5.16b}$$

Solving for the displacement matrix $\{\Delta_2\}$, yields

$$\{\Delta_2\} = \frac{D}{d_2} [K_2]^{-1} \{f\} \tag{5.17}$$

where $d_2 = \alpha \rho D \left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c} C\right) \cdot c^2$

Performing the inversion of $[K_2]$ and then multiplying, one obtains for the displacements

$$\begin{Bmatrix} \hat{V}_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{(\hat{k}^2 s - \hat{F})(\alpha^2+\rho^2)(\alpha \tilde{c} S - \rho \tilde{s} C)}{\alpha\rho\left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c} C\right) c^2} \\ -\dfrac{(\hat{k}^2 s - \hat{F})\left[(\alpha^2-\rho^2)(1-\tilde{c} C) + 2\alpha\rho \tilde{s} S\right]}{\left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c} C\right) c} \end{Bmatrix} \tag{5.18a}$$

or for $\hat{V}_2$ only

$$\hat{V}_2 = -\frac{(\hat{k}^2 s - \hat{F})(\alpha^2+\rho^2)(\alpha \tilde{c} S - \rho \tilde{s} C)}{\alpha\rho\left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c} C\right) c^2} \tag{5.18b}$$

For $\hat{F}$ very small, rearranging Equation (5.18b), one obtains

$$\hat{V}_2 \alpha\rho \left[\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c} C\right] c^2 = -\hat{k}^2 s (\alpha^2+\rho^2)(\alpha \tilde{c} S - \rho \tilde{s} C) \tag{5.19}$$

From the ψ function, assuming that the nonlinear term is small in comparison to the linear one, transforming the resulting ψ equation in a similar manner as in case A, yields

$$\psi = \{-1 \quad 1\} [R_1] [A_2] \{\Delta_2\}$$

where $[R_1]$ is defined in Equation (3.10a) and $[A_2]$ in Equation (3.27a). Performing the matrix multiplications the latter equation becomes

$$\psi = \hat{V}_2 \, s$$

which is the same as Equation (5.1). This means that ψ is a function of $\hat{V}_2$ only and not of any other end displacement (rotation). It follows that

$$-\hat{k}^2 = \hat{V}_2 \, R^2 \, s$$

Substituting the value of $\hat{k}^2$ from the latter equation into Equation (5.19) and rearranging, yields

$$\hat{V}_2 \left[\alpha\rho\left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c}\, C\right)c^2 - R^2 s^2 (\alpha^2+\rho^2)(\alpha \tilde{c} S - \rho \bar{s} C)\right] = 0 \qquad (5.20)$$

For the free vibration problem at the resonant frequency, $\hat{V}_2$ must be arbitrary. Therefore, the following frequency equation is obtained:

$$\alpha\rho\left(\alpha\rho D + (\alpha^2+\rho^2)^2 \tilde{c}\, C\right)c^2 - R^2 s^2 (\alpha^2+\rho^2)(\alpha \tilde{c} S - \rho \bar{s} C) = 0 \qquad (5.21)$$

Again, the plot of the induced axial force $\hat{k}$ versus the square of the frequency $\hat{m}\,\omega^2$ curve is constructed. The use of a high-speed digital computer is needed for the plot of such a curve. The program used is the same as the one used in case A and is given in the Appendix A; program No. 3. The only difference is that the function given in Equation (5.21) is utilized. A plot of the results is shown in Figure 5.4.

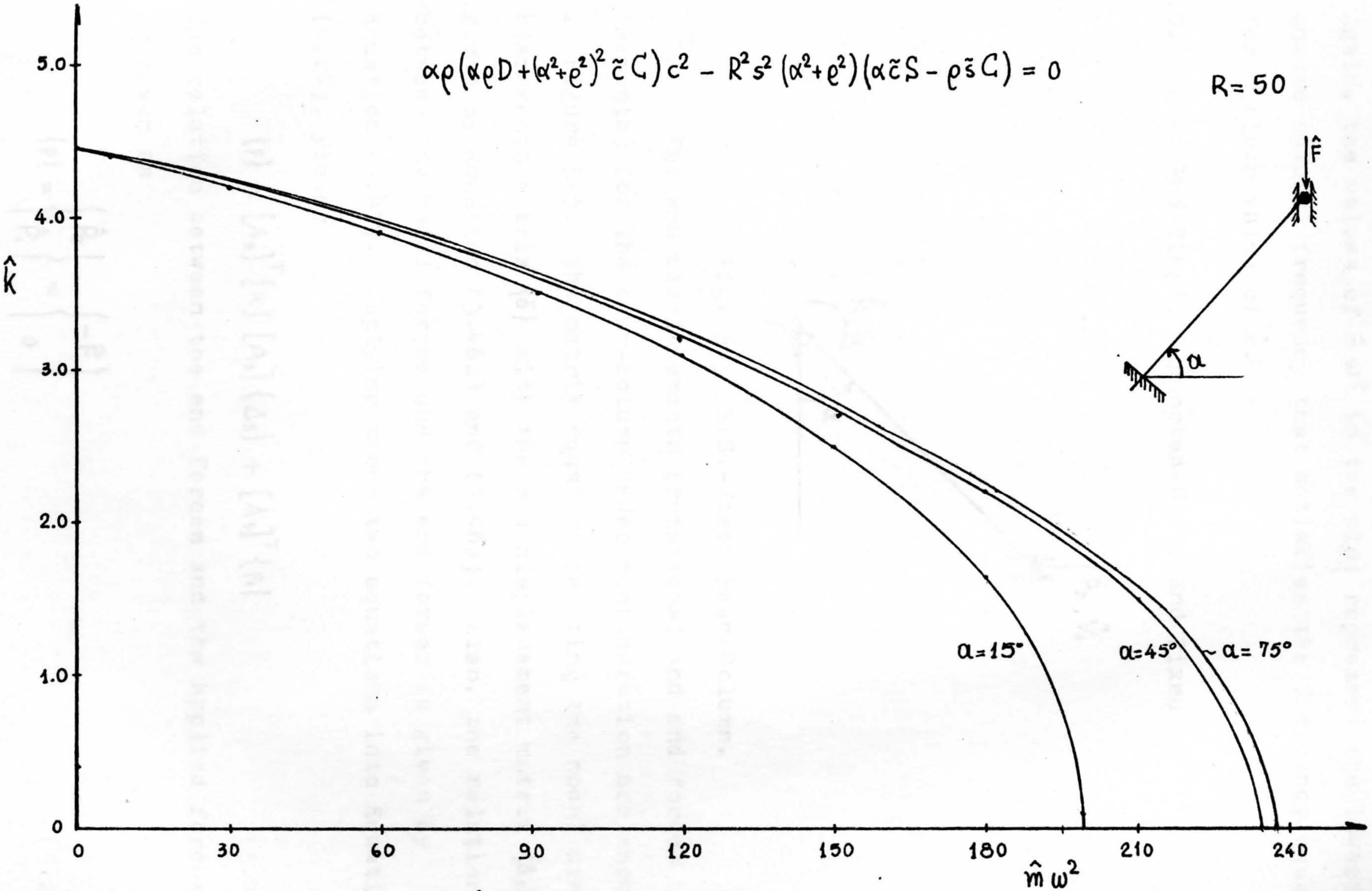


Fig. 5.4 $\hat{k}$ vs. $\hat{m}\omega^2$ curve — Fixed-S.S.

Again, the values of $\hat{m}\,\omega^2$ in the plot represent the lowest square natural frequency that satisfies the frequency equation for a given value of $\hat{k}$.

C. Lower End Simply Supported-Upper End Fixed

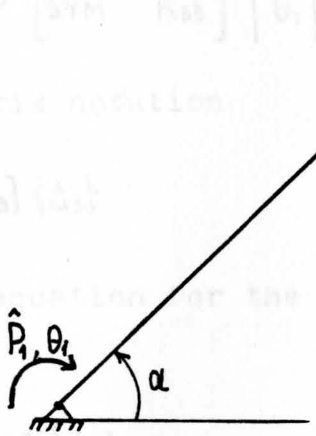


Fig. 5.5 S.S.-Fixed Beam-Column.

The end displacements (rotations) and end forces (moments) for the beam-column under consideration are shown in Figure 5.5. The matrix equation relating the nodal displacements matrix $\{\delta\}$ with the end displacement matrix $\{\Delta_3\}$ is given by Equation (3.46a) and (3.46b). Also, the relation between the nodal forces and the end forces is given by Equation (3.47). Applying those two equations into Equation (4.24), yields

$$\{P\} = [A_3]^T [K] [A_3] \{\Delta_3\} + [A_3]^T \{P_0\} \tag{5.22}$$

The relation between the end forces and the applied forces is given as

$$\{P\} = \begin{Bmatrix} \hat{P}_2 \\ \hat{P}_1 \end{Bmatrix} = \begin{Bmatrix} -\hat{F} \\ 0 \end{Bmatrix} \tag{5.23}$$

Performing the matrix multiplications in Equation (5.22), substituting into Equation (5.23), and after manipulations, one obtains,

$$\begin{Bmatrix} \hat{k}^2 s - \hat{F} \\ 0 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} K_{55} & K_{35} \\ SYM & K_{33} \end{bmatrix} \begin{Bmatrix} \hat{V}_2 \\ \theta_1 \end{Bmatrix} \tag{5.24a}$$

or in symbolic matrix notation

$$\{f\} = \frac{1}{D} [K_3] \{\Delta_3\} \tag{5.24b}$$

Solving the above equation for the displacement matrix $\{\Delta_3\}$, yields

$$\{\Delta_3\} = \frac{D}{d_3} [K_3]^{-1} \{f\} \tag{5.25}$$

where $d_3 = \alpha \varrho \, D \, (\alpha^2 + \varrho^2)^2 \, \tilde{c} \, C \; c^2$

Performing the matrix inversion and the resulting matrix multiplication in Equation (5.25), yields

$$\begin{Bmatrix} \hat{V}_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{(\hat{k}^2 s - \hat{F})(\alpha \tilde{c} S - \varrho \tilde{s} C)}{\alpha \varrho (\alpha^2 + \varrho^2) \tilde{c} C \; c^2} \\ -\dfrac{(\hat{k}^2 s - \hat{F})(C - \tilde{c})}{(\alpha^2 + \varrho^2) \tilde{c} C \; c} \end{Bmatrix} \tag{5.26a}$$

or for $\hat{V}_2$ only,

$$\hat{V}_2 = -\frac{(\hat{k}^2 s - \hat{F})(\alpha \tilde{c} S - \varrho \tilde{s} C)}{\alpha \varrho (\alpha^2 + \varrho^2) \tilde{c} C \; c^2} \tag{5.26b}$$

Neglecting $\hat{F}$, which is assumed to be small, Equation (5.26b) yields

$$\hat{V}_2 \, \alpha \varrho (\alpha^2 + \varrho^2) \tilde{c} C \; c^2 = -\hat{k}^2 s \, (\alpha \tilde{c} S - \varrho \tilde{s} C) \tag{5.27}$$

Observing from the previous cases that the ψ function, neglecting the nonlinear term $\frac{1}{2} v_{,x}^2$, is only a function of $\hat{V}_2$, it follows that

$$\psi = \hat{V}_2 \, s$$

or similarly to the previous cases

$$-\hat{k}^2 = \hat{V}_2 \, R^2 s$$

Substituting the latter equation into Equation (5.27), one obtains

$$\hat{V}_2 \left[\alpha \rho (\alpha^2 + \rho^2) \tilde{c} \, C \, c^2 - R^2 s^2 (\alpha \tilde{c} \, S - \rho \tilde{s} \, C)\right] = 0 \qquad (5.28)$$

From Equation (5.28) one obtains the frequency equation for the free vibration problem, as

$$\alpha \rho \left(\alpha^2 + \rho^2\right) \tilde{c} \, C \, c^2 - R^2 s^2 \left(\alpha \tilde{c} \, S - \rho \tilde{s} \, C\right) = 0 \qquad (5.29)$$

Similar to the previous cases, the $\hat{k}$ versus $\hat{m}\,\omega^2$ plot is constructed. The computer program used for this case is the same as in previous cases with the exception of frequency function given in Equation (5.29). The plot of the resulting curves are shown in Figure 5.6.

D. Both Ends Fixed

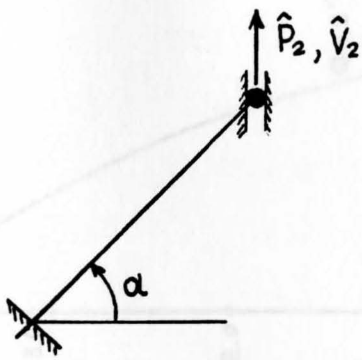


Fig. 5.7 Fixed-Fixed Beam-Column

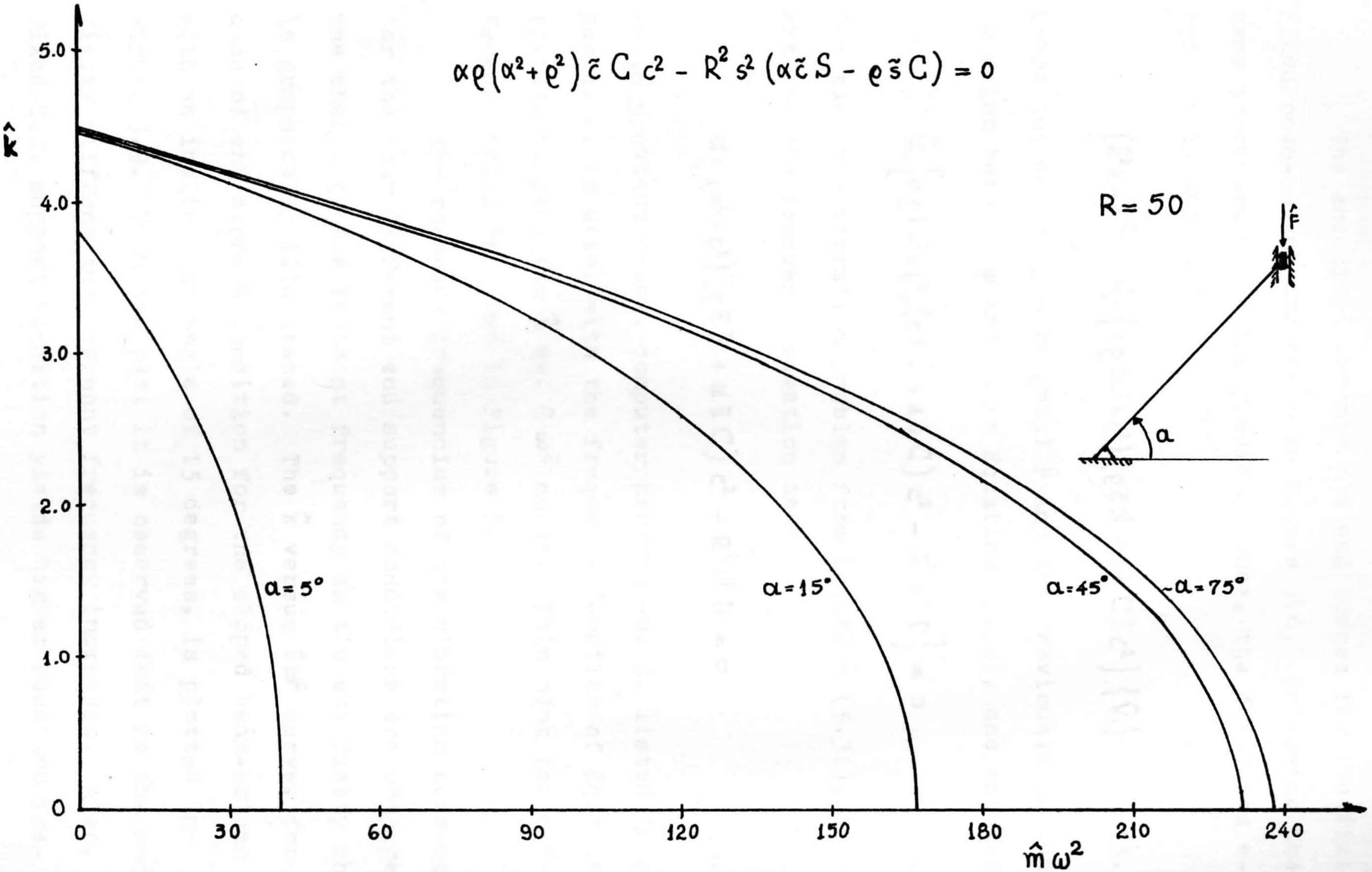


Fig. 5.6 $\hat{k}$ vs. $\hat{m}\omega^2$ curves – S.S.-Fixed.

The end displacements and end forces for the fixed-fixed beam-column are shown in Figure 5.6. Following the same procedure as in the previous cases, the following equation is obtained:

$$\{\hat{k}^2 s - \hat{F}\} = \frac{1}{D}\{\alpha\rho(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C)c^2\}\{\hat{V}_2\} \qquad (5.30)$$

Using the assumption of small $\hat{F}$ and the previously derived relation between ψ and $\hat{V}_2$ in Equation (5.30), one obtains

$$\hat{V}_2\left[\alpha\rho(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C)c^2 - R^2 s^2 D\right] = 0 \qquad (5.31)$$

For the free vibration problem from Equation (5.31), one obtains the frequency equation as

$$\alpha\rho(\alpha^2+\rho^2)(\rho\bar{c}S + \alpha\bar{s}C)c^2 - R^2 s^2 D = 0 \qquad (5.32)$$

As in previous cases, computer program No. 3, listed in Appendix A, is used, with the frequency function of Equation (5.12), to plot the $\hat{k}$ vs. $\hat{m}\,\omega^2$ curve. This plot for different angles is shown in Figure 5.8.

The resonant frequencies of the vibrating beam-column for the four different end support conditions are compared. The change of the resonant frequency as the end fixity changes is graphically illustrated. The $\hat{k}$ versus $\hat{m}\omega^2$ curve, for each case of end support condition for the sloped beam-column with an inclination angle of 15 degrees, is plotted in Figure 5.9. From the plot it is observed that as the end fixity stiffens the resonant frequency increases. Also, the Fixed-S.S. support condition yields higher resonant fre-

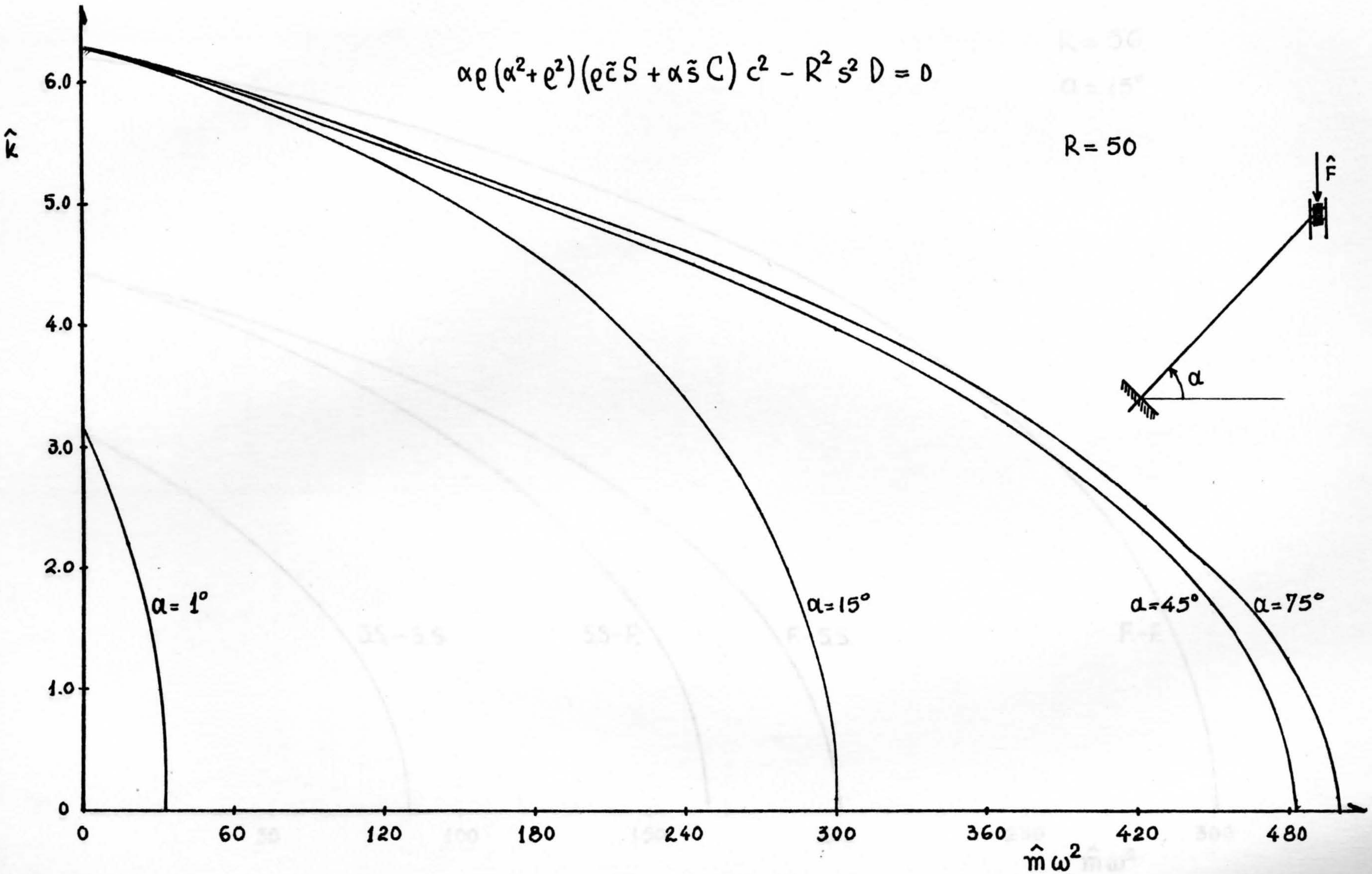


Fig. 5.8 $\hat{k}$ vs. $\hat{m}\omega^2$ curves – Fixed-Fixed.

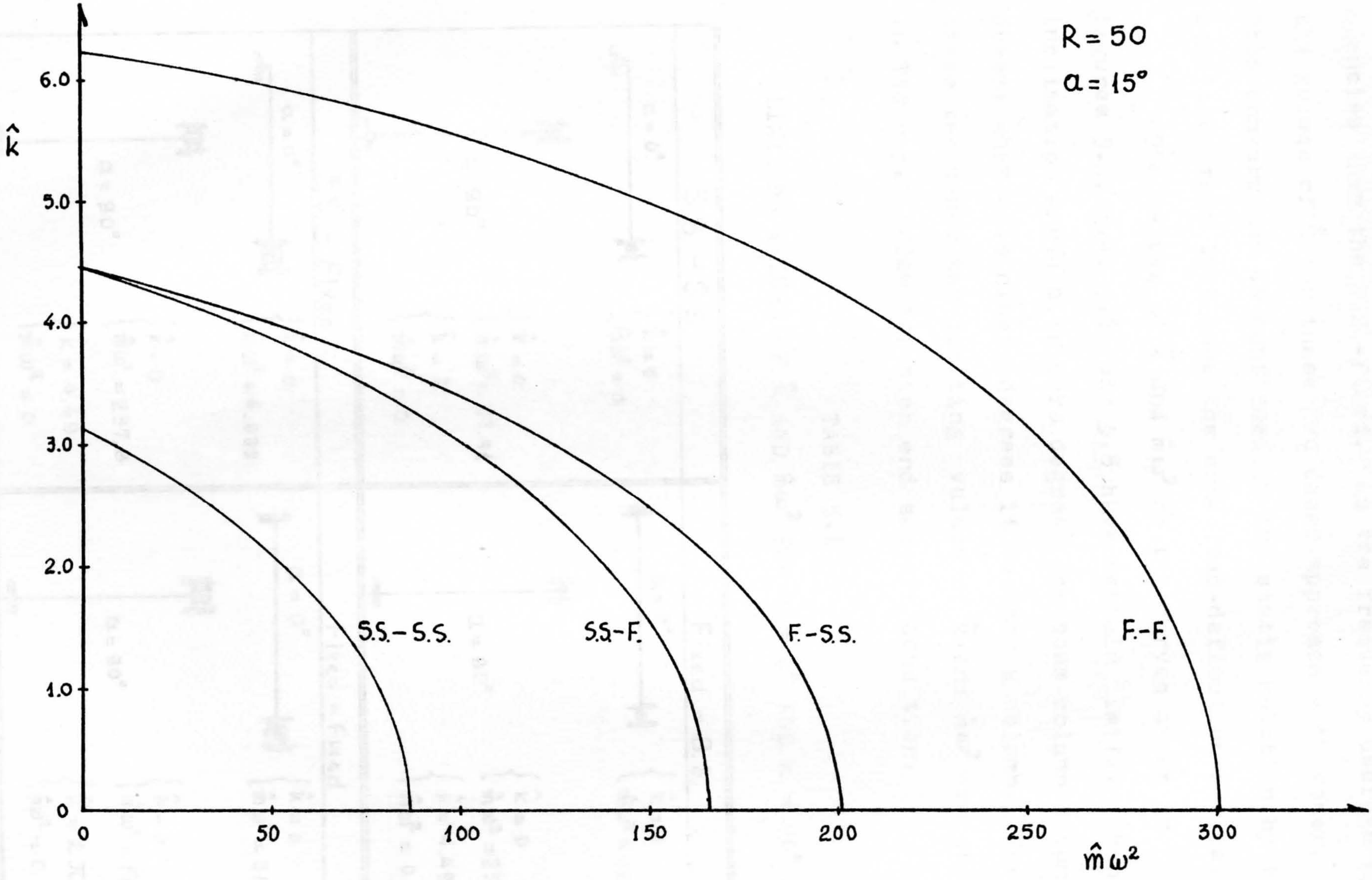


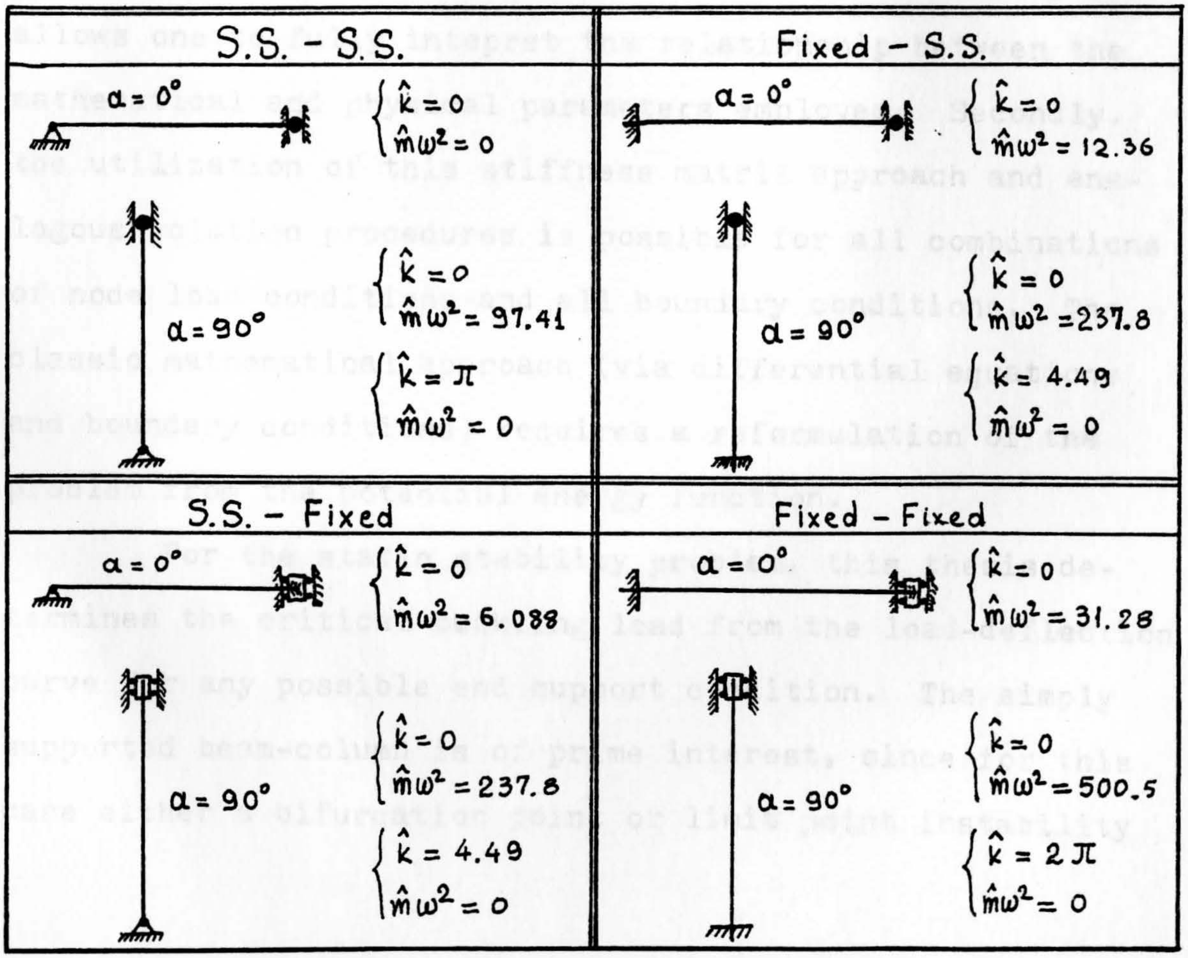
Fig. 5.9 $\hat{k}$ vs. $\hat{m}\omega^2$ curves for Each Case.

quencies than the S.S.-Fixed. As the frequency decreases, the values of $\hat{k}$ for these two cases approach each other. This convergence is confirmed in the static solution by the condition that they have the same load-deflection curves.

The values of $\hat{k}$ and $\hat{m}\omega^2$ on the curves shown in Figures 5.2, 5.4, 5.6, and 5.8 have certain limits. When the inclination angle a is zero degrees the beam-column becomes abeam; when a is ninety degrees it becomes a column. For these two cases the limiting values of $\hat{k}$ and $\hat{m}\omega^2$ are shown in Table 5.1 below for each end support condition.

TABLE 5.1

LIMITING VALUES OF $\hat{k}$ AND $\hat{m}\omega^2$ FOR a = 0° AND a = 90°



CHAPTER VI

DISCUSSION AND CONCLUSIONS

This thesis solves the static stability problem and the resonant frequency dynamic problem of an inclined beam-column using an approximate large deflection theory. The solutions are obtained utilizing modern matrix techniques. The stiffness matrices, for both problems, are derived using Castigliano's Theorem in a rather unique way, utilizing the solutions of the associated differential equations.

The developed solution of the static stability problem, allows one to fully intepret the relationship between the mathematical and physical parameters employed. Secondly, the utilization of this stiffness matrix approach and analogous solution procedures is possible for all combinations of node load conditions and all boundary conditions. The classic mathematical approach (via differential equations and boundary conditions) requires a reformulation of the problem from the potential energy function.

For the static stability problem, this thesis determines the critical buckling load from the load-deflection curve for any possible end support condition. The simply supported beam-column is of prime interest, since for this case either a bifurcation point or limit point instability may occur. The limit point instability or direct thrust type of buckling occurs whenever the maximum point of the load-deflection curve is reached without the induced axial compressive force exceeding the Euler's buckling load of the vertical column. This occurs for certain combinations of the slenderness ratio R and inclination angle a (measured from the horizontal line). It usually occurs for very small angles a ($0^\circ < a < 5^\circ$), when the beam-column is closer to the beam position (horizontal). The bifurcation type of buckling occurs whenever the Euler's buckling load is reached before the corresponding load-deflection curve has reached its maximum point. It usually occurs for slanted beam-columns with a medium-to-steep angle a ($25^\circ < a < 90^\circ$), when it is closer to a column. The physical meaning of the bifurcation point is that although the mathematical equations are following the load-deflection curve, the induced axial force may never exceed Euler's buckling load.

For the fixed-simply supported and the simply supported-fixed cases the respective load-deflection curves are the same. There is no bifurcation point for either case. The instability occurs as direct thrust buckling. For the case of both ends fixed (no rotation), direct thrust buckling controls. A very interesting observation is made in this case. When the applied force $\hat{F}$ is normalized to the Euler's buckling load for the first mode shape $\beta_o = 2\pi$, the normalized values of $\hat{F}_n$ for part of the curve exceed the value of one; when the function is normalized to the second mode shape value $\beta_o = 8.98682$, it never exceeds the value of one. This means that the deflection function of the beam-column is that determined by the second mode shape. For all four cases it is observed that as the inclination angle a increases, the critical buckling load ($\hat{F}_n$) increases. Also, as the end fixity increases, the critical buckling load increases.

In order to solve the dynamic problem, two necessary assumptions are made. First, the assumption that the externally applied load F is relatively small. The second assumption is that the nonlinear term in the axial strain expression is relatively small in comparison to the linear term. This assumption leads to a classical type formulation of a transcendental equation for determination of natural frequency. These two assumptions are made after the derivation of the dynamic stiffness matrix and the application of the end conditions. The second assumption is valid for large values of the inclination angle a. Thus, the values obtained relating $\hat{k}$ and $\hat{m}\omega^2$ for complete range of values $\hat{k}$ are more accurate for large values of inclination angle a. The values of the natural frequency for small values $\hat{k}$ and for angle a close to 0° or to 90° are the same as the values found in the literature for such cases. Thus, these results are accurate over the complete range of a ($0° < a < 90°$). From the curves it is observed that the natural frequency decreases as the axial force increases and it becomes zero when the axial force is equal to the Euler's buckling load for each case. Also, it is observed that as the end fixity increases the natural frequency increases.

The following conclusions are drawn from this study:

1. The matrix approach lends itself to an efficient mathematical process for obtaining solutions to the inclined beam-column problem.
2. Solutions obtained in the static stability problem are more mathematically compact and more easily formulated and intepreted than in the classical differential equations approach.
3. Some solutions of the natural frequency problem have inherint errors due to the linearization techniques required by the solution procedures. The combination of large axial force and small angle of inclination yields the largest degree of inaccuracy.
4. Recommendations for future work performed on this topic take the following directions:
   a. An attempt should be made to incorporate the nonlinear axial strain term for the determination of natural frequency. This may possibly be performed by utilizing the exact nonlinear strain energy for the problem [6] and incorporating a matrix series formulation to obtain a solution.
   b. An alternate procedure to the above may be to utilize a modern perturbation technique on the exact energy function - sometimes referred to as the "eigenvalue method" in its linearized form. Since the problem is highly nonlinear, at least second order perturbations would be required.

For either of the latter recommendations the solutions would require some knowledge of nonlinear vibrations

APPENDIX A

Computer Programs

Computer Program Number 1 - Fixed-S.S.

Solution of the parametric equations:

$$\hat{V}_2 = \frac{-T^2 s \pm \left[T^4 s^2 - 4c^2 \hat{k}^2 S T^2/R^2\right]^{1/2}}{2c^2 S} \tag{3.41}$$

$$\hat{F}_n = \left(\frac{\hat{k}}{\beta_o}\right)^2 \left(s + \frac{\hat{V}_2 c^2}{T}\right) \tag{3.44}$$

```
      DOUBLE PRECISION PI,HK,A,DTAN,DSIN,DCOS,DSQRT,R,S,T,
     1V2HP,V2HN,FHNP,FHNN,TIRT,BO
      READ(5,100) AD,R
      PI=3.141592654D0
      BO=4.4934094579D0
      WRITE(6,101)
      HK=0.1D0
      A=AD*PI/180.
    5 TNK=DTAN(HK)
      T=1.-(TNK/HK)
      S=(3.*T+(TNK)**2)/4.
      TIRT=(T**4)*(DSIN(A)**2)-(4.*(DCOS(A)**2)*(HK**2)*S*
     1(T**2))/(R**2)
      IF(TIRT)7,6,6
    6 ROOT=DSQRT(TIRT)
      V2HP=(-(T**2)*DSIN(A)+ROOT)/(2.*(DCOS(A)**2)*S)
      V2HN=(-(T**2)*DSIN(A)-ROOT)/(2.*(DCOS(A)**2)*S)
      FHNP=((HK/BO)**2)*(DSIN(A)+((DCOS(A)**2)*V2HP/T))
      FHNN=((HK/BO)**2)*(DSIN(A)+((DCOS(A)**2)*V2HN/T))
      WRITE(6,102) HK,FHNP,V2HP,FHNN,V2HN
  100 FORMAT(F10.1,F10.1)
  101 FORMAT('1',5X,'AXIAL FORCE',10X,'APPLIED FORCE',7X,'VER
     1TICAL DEFL',10X,'APPLIED FORCE',7X,'VERTICAL DEFL')
  102 FORMAT(6X,F8.5,12X,F12.9,8X,F12.9,11X,F12.9,8X,F12.9)
    7 HK=HK+0.1
      IF(HK-2.*PI)5,8,8
    8 CONTINUE
      STOP
      END
```

Computer Program Number 2 - Fixed-Fixed.

Solution of the parametric equations:

$$\hat{V}_2 = \frac{-T^2 s \pm \left[T^4 s^2 - 2\hat{k}^2 c^2 S T^2/R^2\right]^{1/2}}{S c^2} \tag{3.65}$$

$$\hat{F}_n = \left(\frac{\hat{k}}{\beta_0}\right)^2 \left(s + \hat{V}_2 \frac{\hat{k}\,\tilde{s}\,c^2}{T}\right) \tag{3.66}$$

```
      DOUBLE PRECISION PI,BO,HK,A,DCOS,TIRT,DSQRT,V2HN,V2HP,
     1DSIN,FHNP,FHNN,S,T
      READ(5,100) AD,R
      PI=3.141592654D0
      HK=0.1D0
      BO=2.*4.4934094579D0
      WRITE(6,101)
      A=AD*PI/180.
    5 CHK=DCOS(HK)
      SHK=DSIN(HK)
      T=HK*SHK-2.*(1.-CHK)
      S=HK*(1.-CHK)*(2.*HK+HK*CHK-3.*SHK)
      TIRT=(T**4)*(DSIN(A)**2)-2.*(HK**2)*(DCOS(A)**2)*S*
     1(T**2)/(R**2)
      IF(TIRT)7,6,6
    6 ROOT=DSQRT(TIRT)
      V2HP=(-(T**2)*DSIN(A)+ROOT)/(S*(DCOS(A)**2))
      V2HN=(-(T**2)*DSIN(A)-ROOT)/(S*(DCOS(A)**2))
      FHNP=((HK/BO)**2)*(DSIN(A)+(V2HP*HK*SHK*(DCOS(A)**2)/T))
      FHNN=((HK/BO)**2)*(DSIN(A)+(V2HN*HK*SHK*(DCOS(A)**2)/T))
      WRITE(6,102) HK,FHNP,V2HP,FHNN,V2HN
    7 HK=HK+0.1
      IF(HK-3.*PI) 5,8,8
  100 FORMAT(F10.1,F10.1)
  101 FORMAT('1',5X,'AXIAL FORCE',10X,'APPLIED FORCE',7X,'VER
     1TICAL DEFL',10X,'APPLIED FORCE',7X,'VERTICAL DEFL')
  102 FORMAT(6X,F12.9,12X,F12.9,8X,F12.9,11X,F12.9,8X,F12.9)
    8 CONTINUE
      STOP
      END
```

Computer Program Number 3

Solution of the transcendental equations:

$$\alpha\rho\left(\alpha^3 \tilde{s} C - \rho^3 \tilde{c} S\right) c^2 - R^2 s^2 \left(\alpha^2+\rho^2\right) \tilde{s} S = 0 \qquad (5.14)$$

$$\alpha\rho\left(\alpha\rho D + \left(\alpha^2+\rho^2\right)^2 \tilde{c} C\right) c^2 - R^2 s^2 \left(\alpha^2+\rho^2\right)\left(\alpha \tilde{c} S - \rho \tilde{s} C\right) = 0 \qquad (5.21)$$

$$\alpha\rho\left(\alpha^2+\rho^2\right) \tilde{c} C c^2 - R^2 s^2 \left(\alpha \tilde{c} S - \rho \tilde{s} C\right) = 0 \qquad (5.29)$$

$$\alpha\rho\left(\alpha^2+\rho^2\right)\left(\rho \tilde{c} S + \alpha \tilde{s} C\right) c^2 - R^2 s^2 D = 0 \qquad (5.32)$$

```
      DOUBLE PRECISION DSQRT,A,PI,C,SQHK,RT,DEXP,SH,CH,D,
     1ALPHA,RO,DSIN,DCOS,FUNC,HK,SQ,FREQ
      BO=4.4934094579D0
      READ(5,101) AD,R
      WRITE(6,100) AD
      PI=3.141592654D0
      A=AD*PI/180.
      WRITE(6,102)
      HK=0.1D0
    2 FREQ=1.D0
   43 C=1.D0
      L=0
      SQHK=(HK**2)/2.
    3 IF(FREQ) 20,4,4
    4 RT=DSQRT((SQHK**2)+FREQ)
      ALPHA=DSQRT(RT+SQHK)
      RO=DSQRT(RT-SQHK)
      ERO=DEXP(RO)
      SH=(ERO-(1./ERO))/2.
      CH=(ERO+(1./ERO))/2.
      D=(RO**2-ALPHA**2)*DSIN(ALPHA)*SH+2.*ALPHA*RO(1.-
     1DCOS(ALPHA)*CH)
      SQ=ALPHA**2+RO**2
      FUNC=ALPHA*RO*SQ*(RO*DCOS(ALPHA)*SH+ALPHA*DSIN(ALPHA)
     1*CH)*(DCOS(A)**2)+D*(R**2)*(DSIN(A)**2)
      FUNC=FUNC/1000.
      IFUNC=FUNC
      FFUNC=FUNC-IFUNC
      AFFUNC=ABS(FFUNC)
      IF(AFFUNC-0.001) 31,31,32
   31 FFUNC=0.
   32 CFUNC=IFUNC+FFUNC
      IF(L-1) 18,17,18
   17 TFUNC=FUNC
   18 IF(TFUNC) 5,10,6
```

```
  5   IF(CFUNC) 12,10,8
  6   IF(CFUNC) 8,10,12
 12   FREQ=FREQ+(10./C)
      GO TO 3
  8   FREQ=FREQ-(10./C)
      C=C*10.
      GO TO 3
 10   WRITE(6,103) HK,FREQ
  7   HK=HK+0.1
      IF(HK-7.) 2,2,20
100   FORMAT('1','THE INCLINATION ANGLE IS ',F5.1,
     1' DEGREES')
101   FORMAT(F10.1,F10.1)
102   FORMAT(///,10X,'AXIAL FORCE',10X,'NATURAL FREQUENCY')
103   FORMAT(12X,F6.2,15X,F9.5)
 20   CONTINUE
      STOP
      END
```

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