TOPOLOGICAL VARIANTS OF THE CLOSED GRAPH THEOREM

by

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ABSTRACT

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The Closed Graph Theorem plays a fundamental role in functional analysis, particularly in the study of Banach spaces.

This thesis examines the extent to which the linearity of the Closed Graph Theorem have been replaced by topological conditions imposed on the domain, codomain and/or the function.

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CONTENTS

***H**istorical **O**utline of **T**opology*

Considerations of a topological nature, depending on concepts of limit and continuity, originated together with the oldest problems of geometry and mechanics, such as the calculation of areas and the movement of figures.

In the hands of eminent mathematicians and for a long period, infinite series were a tool used in an entirely formal manner, that is without regard for convergence considerations. Gauß seems to have been the first to think about the legitimacy of the use of infinite processes, such as the series expansion of Newton's binomial with an arbitrary exponent, which at times led to surpassing absurdities. The credit, however, is due to Abel and Cauchy for having defined the concepts of a convergent series and sequence and the concept of continuous function with the rigor that is so familiar to us today.

The first mathematician who attempted to isolate the idea of a **topological space** and who sensed its far reaching importance was Riemann. However, in order for the expansion of topology in this direction to become possible, it was indispensable that this new discipline should have at its disposal experience and information concerning important particular cases.

Then came Cantor's investigations in 1874, meeting with opposition from many of his contemporaries because of their complete novelty. These investigations were in part inspired by the desire to analyze the difficult questions concerning the convergence of the Fourier series. Simultaneously, the theory of real numbers was erected on a solid foundation by Dedekind and Cantor. The systematic study of the concept of a set, of an accumulation point, etc. are linked to the work of Cantor.

Parallel to the investigations on the topology of the line and of p-dimensional Euclidean space, it was attempted to make use of the same methods, not only with the respect to point sets in the sense of elementary geometry but also to sets whose elements were curves, surfaces, and, above all, functions. The pioneers in this period of infancy of functional analysis were Ascoli, Pincherle, and principally Volterra. To the latter we owe a systematic study (1887) of line functions (or functionals according to the terminology adopted since Hadamard) and of the infinitesimal calculus of functionals.

An epoch-making step of progress was achieved, at the beginning of our century, by the introduction of the so-called Hilbert spaces, later defined axiomatically by von Neumann (1927). These spaces are, without doubt, the most important and fertile example of topological spaces of an infinite number of dimensions among all the examples of such spaces known today. By their rich structure which includes, the concept of the sum of vectors, the product of a scalar and a vector, and the scalar product of two vectors, these Hilbert spaces unite with their geometrical elegance an impressive variety of possible analytical applications.

The existence of so many examples of spaces like the Euclidean spaces and their subspaces and the various function spaces in which topological considerations find natural applications gave rise to the desire, or rather the necessity, to synthesize an approach which would permit the study of the properties held simultaneously by all these spaces and would, consequently, bring about a better comprehension of the particular aspects of each one of them.

Thus, general topology originated with the introduction, in1906, of metric spaces by Fréchet and with the elaboration of an autonomous theory of abstract topological spaces by Hausdorff in 1914; the merits of Hausdorff's achievements are recognized by the association of his name with the so-called Hausdorff spaces. Later on Kuratowski formulated the axioms of the closure operator and showed that the axioms are equivalent to topology axioms. From this time on, the steps of progress of the new discipline followed rapidly one upon the other.

During the period in which the topology of the line developed, the discovery of the compactness criteria of Bolzano-Weierstraß and Borel-Lebesgue stood out at once. To this group of results there belongs a theorem, due to Weierstraß, according to which every continuous function on a bounded and closed interval attains a minimum there. Weirstraß observation that the application of an analogous principle in function spaces is not always valid but must be based on a previous justification met with response in Hilbert's proof of the existence of a minimum for the integral and the subsequent solution of the classical Dirichlet problem concerning harmonic functions. This remark of Weierstraß is one of the sources from which the concepts of semi-continuity and of compactness in function spaces draw their interest. Semi-continuity was discovered by Baire in the case of real variables and was later utilized by Tonelli in the calculus of variations. We owe to Fréchet the formulation of the idea of compactness in metric spaces in the sequential manner of Bolzano-Weierstraß and the recognition of its equivalence, in this case, to the property of Borel-Lebesgue. The concept of a compact space as it is now considered in topology became the object of systematic study, based on the criterion of Borel-Lebesgue, as recently as 1929 and originated with Alexandroff and Urysohn.

Normal spaces, the importance of which derive, to a considerable extent, from the extension theorem for continuous real-valued functions, were introduced by Tietze. First established by Lebesgue for the case of functions defined on a subset of the plane and by Tietze for the case of functions defined on a metric space, this theorem attained a general form in a basic paper by Urysohn. The related category of completely regular spaces was brought out by Tychonoff 's work on the compactification of topological spaces and includes, according to a theorem of Pontrjagin, all topological groups.

Continuity is a purely local phenomenon; the corresponding global phenomenon we call uniform continuity today. The first trace of the idea of uniform space in mathematics is found in Cauchy's general criterion for the convergence of a series or sequence. Under the influence of Weirstraß and Heine, the ideas of a uniformly convergent series and of a uniformly continuous function entered the domain of mathematical analysis.

To Fréchet and Hausdorff, we owe the concept of a complete metric space (one in which Cachy's convergence criterion is satisfied) and the concept of a uniformly continuous function on a metric space and the possibility of completing every metric space by a construction analogous to that employed by Cantor in order to define the real numbers on the basis of rational numbers. One of the fruits of this order of ideas is the Riesz-Fischer theorem according to which the space of square integrable functions in the sense of Lebesgue is complete.

With the definition of topological groups by Leja in 1925 and of compact spaces by Alexandroff and Urysohn in 1929, the concept of a uniformly continuous function came to have significance in a greater number of cases. Finally, in 1940, Tukey introduced uniform spaces thus encompassing in one single theory various aspects common to the theories of metric spaces, topological groups, and compact spaces.

In the panorama of the hystorical roots of topology just outlined, we did not mention the idea which, originating in Riemann's work, was subsequently developed by Betti and Poincaré, leading to the analysis situs or algebraic topology of today. The reason is that these ideas belong to a direction of research distinct from that in which we shall be interested in the present monograph.

*G*eneral *R*emarks

Given a function f:X →Y where X and Y are arbitrary topological spaces, the fundamental question arises:

When is f continuous?

In the framework of topological vector spaces where f is linear and X and Y are Banach spaces, it is sufficient for continuity of f that f has a closed graph. Consequently, the Closed Graph Theorem has long been recognized as a major tool in Functional Analysis. Until recently, very little was known in the framework of General Topology with regards to the function with the closed graph being continuous. In recent years, however, interest has grown in this topic simultaneously with an increasing interest in more general subjects of non-continuous functions.

The Closed Graph Theorem was originally proven by Stefan Banach [1] in the early 30's and was subsequently published in his famous "Théorie des opérations linéaires." It was used in the proof of the Open Mapping Theorem and as such is one of the most important theorems of Abstract Analysis.

It is easy to see that for bounded real valued functions, a function having a closed graph is equivalent to its continuity. [18]

J. L. Kelley [10] extends this theorem to Lindelöf spaces. It was later shown by Husain [9] that every almost continuous closed-graph function into a locally compact Hausdorff space is continuous.

By a topological version of Closed Graph Theorem, we mean any result corresponding to the Banach Closed Graph Theorem where the **linearity** is replaced by **almost continuity**.

Chapter 1

FUNCTIONAL ANALYSIS

1.1 Closed Graph Theorem in Functional Analysis

The graph of a function $f : X \to Y$ denoted by $gr\, f$ is defined as follows:

$$gr\, f = \{(x, f(x)) : x \in X\}.$$

To explain what is meant by the closed graph, we start with notion that given a set S of X, where X is a metric space, we say that a point x belongs to the closure of S, denoted Cl(S), if and only if there exists a sequence $\{x_n\}$ of points from S such that $\lim_{n\to\infty} x_n = x$. A set S is closed if S = Cl(S).

Finally, the function $f : X \to Y$ has a closed graph, $gr(f)$, if the graph of the function f is a closed subset of the product X × Y.

We will illustrate this notion by looking at the function which does not have a closed graph. Consider the function :

$$f(x) = \begin{cases} 2x, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$$

Fig. 1.1.1.

The set $gr\, f$ is not closed in the X × Y plane because it does not contain all of its limit points. In particular, the point (1,2) is the limit point of the set $gr(f)$, but it does not belong to the graph. In addition, there is an alternative characterization of the Closed Graph Property.

Chapter 1

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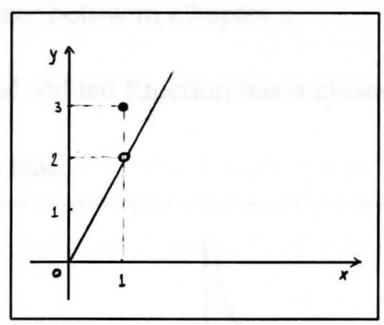

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The set gr(f) is not closed in the $X \times Y$ plane because it does not contain all of its limit points. In particular, the point (1,2) is the limit point of the set gr(f), but it does not belong to the graph. In addition, there is an alternative characterization of the Closed Graph Property.

First, recall that a sequence $\{(x,\tau)\}$ converges to a point x in X provided that for each neighborhood U of x there is a natural number N such that $x_i \in U$ for all $i \geq N$. Consider the following:

Lemma 1.1.1. Let $f: X \to Y$ be a function. Let $\{x_n\}$ be any sequence of elements of the domain of f such that $\{x_n\} \to x$. If $\{(f(x_n))\} \to y$, then the function f has a closed graph if $f(x) = y$.

Applying this to the previous function, the sequence $\{1+ (1/n)\}$ converges to 1, $\{f(1+ (1/n)\}$ converges to 2 but $f(1) = 3$, which is not equal to 2. Thus, this function does not have a closed graph.

Heine's Condition of Continuity: A function $f: \mathbf{R} \to \mathbf{R}$ is continuous if and only if for any x and for any sequence $\{x_n\}$, if $\{x_n\}$ converges to x, then $\{f(x_n)\}$ converges to $f(x)$.

We shall provide the proof of the statement below in Chapter 2.

Proposition 1.1.1. Every continuous real-valued function has a closed graph.

But the converse of the above proposition is not true.

Consider the function:

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

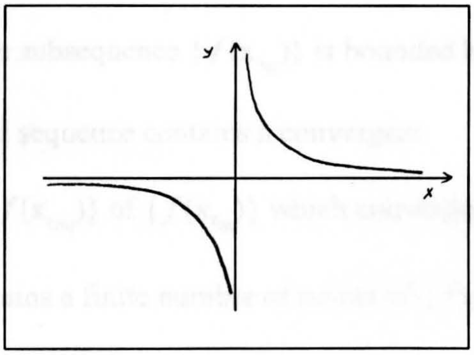


Fig. 1.1.2.

Function f is not continuous at 0 but it has a closed graph.

Using Heine's condition of continuity the sequence $\{1/n\}$ converges to 0, but $\{f(1/n)\} = \{ n \}$ does not converge to $f(0) = 0$, so f is discontinuous at the origin. To show that the function f has a closed graph, we need to show that if $\{x_n\}$ is any sequence of elements of the domain of f such that $\{x_n\} \to x$ and if $\{f(x_n)\} \to y$, then $f(x) = y$. $\{f(x_n)\}$ does not converge even though $\{1/n\}$ does. f does not have a limit at 0. In fact, $\lim_{n\to\infty} f(1/n) = \infty$. Since the limit does not exist, the closed graph condition is automatically satisfied.

Proposition 1.1.2. Let f: $\mathbf{R} \to \mathbf{R}$ be bounded and suppose that gr(f) is closed in the $X \times Y$ plane. Then f is continuous.

Proof: Let x be any point in the domain of f. Suppose that $\{ x_n \}$ is any sequence of points in the domain of f converging to x. If we show that $\{f(x_n)\}$ converges to $f(x)$, then by Heine's condition, f is continuous. Assume that $\{f(x_n)\}$ does not converge to $f(x)$. Let $K \subset \mathbf{R}$ be any open subset of the range of f such that K contains $f(x)$. Since $\{f(x_n)\}$ does not converge to $f(x)$, K does not contain all but a finite number of points of $\{f(x_n)\}$. Thus, there exists a subsequence $\{f(x_{n_m})\}$ of $\{f(x_n)\}$ such that none of the elements of $\{f(x_{n_m})\}$ are contained in K. The subsequence $\{f(x_{n_m})\}$ is bounded because f is bounded. And we know that this bounded sequence contains a convergent subsequence, so there exists a subsequence $\{f(x_{n_{m_i}})\}$ of $\{f(x_{n_m})\}$ which converges to some k. If $f(x) = k$ then $k \in K$ which contains a finite number of points of $\{f(x_{n_{m_i}})\}$, which is a contradiction. So $f(x) \neq k$.

Since all subsequences of a convergent sequence of real numbers converge to the same limit point as the original sequence, and since $\{ x_{n_{m_i}} \}$ is a subsequence of $\{ x_{n_{m_i}} \}$

and hence of $\{x_n\}$, then $\{x_n\}$converges to x implies sequence $\{x_{n_{m_i}}\}$ also converges to x. However, this is a contradiction of the Closed Graph Property since $\{x_{n_{m_i}}\}$ converges to x and $\{f(x_{n_{m_i}})\}$ converges to k, but $f(x) \neq k$. Hence, $\{f(x_n)\}$ converges to $f(x)$, and f is continuous by Heine's condition. ■

Definition 1.1.1. We say that X(K) is a **linear space** over the field K of scalars if:

i) $(X, \oplus)$ is a **commutative group**

ii) For any scalars $k, j \in K$ and $u, v \in X$:

a) $ku \in X$

b) $k(u \oplus v) = ku \oplus kv$

c) $(k + j)u = ku \oplus ju$

d) $k(ju) = (kj)(u)$

e) $1u = u$.

Definition 1.1.2. A **norm** $\| . \|$ over X(K) is a function $\| . \| : X(K) \to \mathbf{R}_+ \cup \{0\}$ such that for any $k \in K$ and $u, v \in X$ we have:

i) $\| ku \| = | k | \| u \|$

ii) $\| u \oplus v \| \leq \| u \| + \| v \|$

A linear space with a norm is called **normed space.**

Definition 1.1.3. A space $\{X, \| . \|\}$ is called a **Banach space**, if the metric space $\{X, d\}$ is complete, where $d(x,y) = \| x - y \|$.

Definition 1.1.4. Let $T : X \to Y$ be a transformation from a linear space X(K) into linear spaces Y(K), where K is the field of real or complex numbers. A transformation T is called a **linear operator** if:

$T(x_1 + x_2) = T(x_1) + T(x_2)$, and

$T(\alpha x) = \alpha T(x)$, for any $x, x_1, x_2 \in X$, and $\alpha \in K$.

If $Y = K$, then a linear operator $T : X \rightarrow K$ is called a **linear functional.**

Lemma 1.1.2. (Open Mapping Theorem) Let T be a continuous linear operator from a Banach space X onto a Banach space Y, that is TX = Y. Then T sends X-open subsets of X onto Y-open subsets of Y.

Lemma 1.1.3. (Inverse Mapping Theorem) If T is a continuous, linear, and bijective (≡ "1-1" and "onto") operator from a Banach space X onto a Banach space Y, then the inverse mapping T^{-1} is a continuous linear operator from Y onto X.

Definition 1.1.5. Let T be a transformation from a Banach space X into a Banach space Y. A transformation T is called a **closed graph transformation** if the set $Z = \{ (x, Tx): x \in X \}$ is closed in $X \times Y$. The set Z is called the *graph* of T.

Theorem 1.1.1. (Closed Graph Theorem) A closed graph linear operator from a Banach space X into a Banach space Y is continuous.

Proof: Observe that the product of two Banach spaces $\{X, \| \|_X\}$ and $\{Y, \| \|_Y\}$ is a Banach space with the norm $\|(x,y)\| = \| x \|_X + \| y \|_Y$. The graph Z of the operator T is a linear subspace of $X \times Y$; since it is closed it is a Banach space.

Now, we shall show that the transformation T_1 given by: $T_1(x, Tx) = x$, from the Banach space Z onto Banach space X is an injective, continuous, and, linear operator. Let, $x_1, x_2, x \in X, \alpha \in K$, then taking $z_1 = (x_1, Tx_1)$, $z_2 = (x_2, Tx_2)$, $z = (x, Tx)$ we have:

$T_1(z_1 + z_2) = T_1(x_1 + x_2, T x_1 + T x_2) = T_1(x_1 + x_2, T(x_1 + x_2)) = x_1 + x_2 = T_1 z_1 + T_1 z_2$.

Similarly,

$$T_1(\alpha z) = T_1(\alpha x, T\alpha x) = \alpha x = \alpha T_1 z .$$

Furthermore, if $z_k = (x_k, T x_k) \to 0$ (in Z), then $x_k \to 0$ (in X), hence $T_1 z_k \to 0$. Finally, bijectivity of T_1 is obvious. Hence, we can apply the Inverse Mapping Theorem to T_1, which gives that T_1^{-1} is continuous.

Now, let $x_k \to 0$ (in X), then $T_{x_k}^{-1} = (x_k, T_{x_k}) \to 0$ (in Z), so $\| x_k \| \; \| T_{x_k} \| \to_{k\to\infty} 0$. In particular, $T_{x_k} \to_{k\to\infty} 0$ (in Y). This proves that T is continuous at 0. We now show that T is, then, continuous on the whole space X. The following standard arguments hold for any linear topological spaces X and Y.

Let T be continuous at 0, and let $y_0 = Tx_0$. Pick an arbitrary neighborhood U_{y_0} of $y_0 \in Y$, then $U_0 = (U_{y_0} - y_0)$ is a neighborhood of 0_Y in Y. There is a neighborhood V_0 of 0_X such that if $x \in V_0$, then $Tx \in U_0$. Let us denote $V_{x_0} = (V_0 + x_0)$ and let $x \in V_{x_0}$. Then $x = x_1 + x_0$, where $x_1 \in V_0$; hence $T x_1 \in U_0$. Thus, $Tx = T x_1 + y_0 \in U_0 + y_0 = U_{y_0}$. This shows that T is continuous at x_0, which also finishes the proof of the Closed Graph Theorem. ■

Chapter 2

TOPOLOGY

2.1 Compactness of the Range Space and Generalization

The concept of compactness is a generalization of a very important theorem in classical analysis, the Heine-Borel Theorem. The Heine-Borel Theorem is named in honor of Eduard Heine (1821-1881) and Emile Borel (1871-1956), and states that if a and b are real numbers with a < b and $\mathcal{O}$ is a collection of open intervals such that $[a,b] \subseteq \cup\{O : O \in \mathcal{O}\}$, then there is a finite subset $\{O_1, O_2, \ldots, O_n\}$ of $\mathcal{O}$ such that $[a,b] \subseteq \cup_{i=1}^{n} O_i$.

Recall that a collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{T})$ is a cover of $B \subseteq X$ provided $B \subseteq \cup\mathcal{A}$, and the cover $\mathcal{A}$ of B is an open cover of B provided each member of $\mathcal{A}$ is open.

Definition 2.1.1. A subset A of a topological space $(X, \mathcal{T})$ is **compact** provided every open cover of A has a finite subcover.

Definition 2.1.2. A topological space $(X, \mathcal{T})$ is locally compact at a point $p \in X$, provided there is an open set U and a compact subspace K of X such that $p \in U \subseteq K$. A topological space is locally compact if it is locally compact at each of its points.

Denote the set $\{(x, y) \in X \times X : x = y\}$ by Δ, called the diagonal of X.

Chapter 2

2.1 Compactness of the Range Space and Generalization

The concept of compactness is a generalization of a very important theorem in classical analysis , the Heine-Borel Theorem. The Heine-Borel Theorem is named in honor of Eduard Heine (1821-1881) and Emile Borel (1871-1956), and asserts that if a and b are real numbers with a < b and $\boldsymbol{O}$ is a collection of open intervals such that $[a,b] \subseteq \cup\{O:O \in \boldsymbol{O}\}$, then there is a finite subset $\{O_1, O_2,..., O_n\}$ of $\boldsymbol{O}$ such that $[a,b] \subseteq \cup^{N}_{n=1} O_n$.

Recall that a collection $\boldsymbol{A}$ of the subsets of a topological space (X, τ) is a **cover** of $B \subset X$ provided $B \subset \cup_{A\in \boldsymbol{A}} A$, and the cover $\boldsymbol{A}$ of B is an **open cover** of B provided each member of $\boldsymbol{A}$ is open.

Definition 2.1.1. A subset A of a topological space (X,τ) is **compact** provided every open cover of A has a finite subcover.

Definition 2.1.2. A topological space, (X, τ), is locally compact at a point, $p \in X$, provided there is an open set U and a compact subspace K of X such that $p \in U \subset K$. A topological space is **locally compact** if it is locally compact at each of its points.

Consider the set $\Delta = \{ (x,y) \in X \times X: x = y \}$. Δ is called the **diagonal** of X.

Remark 2.1.1. We claim that if (X, τ) is a topological space, if U denotes the product topology on $X \times X$, and if U_Δ is the subspace topology on Δ determined by U, then (X, τ) is homeomorphic to (Δ, U_Δ).

Let h: $X \rightarrow \Delta$ be given by $h(x) = (x,x)$. Clearly, h is continuous, '1-1' and 'onto' ($h(x_1) = h(x_2) \Rightarrow x_1 = x_2$ since $(x_1, x_1) = (x_2, x_2) \Rightarrow x_1 = x_2$;), and h^{-1} is continuous, which proves the above claim.

Further,

Theorem 2.1.1. If X is Hausdorff, then Δ is closed (in $X \times X$).

Proof:

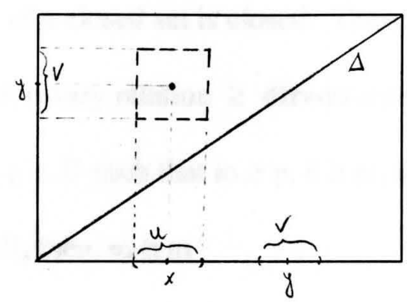


Fig. 2.1.1.

We shall show that the complement of Δ, namely $(X \times X) \setminus \Delta$, is open. Let $(x, y) \in (X \times X) \setminus \Delta$. Since X is a T_2 (Hausdorff) space, there are two open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \varnothing$. In other words $(x, y) \in U \times V \subset (X \times X) \setminus \Delta$ which is open since $U \times V$ is open (since x and y are arbitrary). Hence, Δ is closed in $X \times X$. ■

Remark 2.1.2. If Δ is replaced by the graph of a continuous function f, the set $gr(f) = \{(x,y): y = f(x)\}$ is homeomorphic to X. To show that this holds consider the following:

Let $h : X \to gr(f)$ be given by $h(x) = (x, f(x))$ where f is a continuous function. Then, similarly as in the Remark 2.1.1., h is homeomorphism. Hence, $gr(f) \approx X$.

Now, if we assume that Y is Hausdorff space, then we obtain the following result:

Theorem 2.1.2. Let $f: X \to Y$ be function and Y is Hausdorff space. If f is continuous, then f has a closed graph.

Proof: Let $g(x,y) = (f(x),y)$. Clearly, $g(x,y) \in \Delta \Leftrightarrow y = f(x) \Leftrightarrow (x, y) \in gr(f)$. Hence, $gr(f) = g^{-1}(\Delta)$. Now, g is continuous, and Δ is closed, and $gr(f)$ is also closed because the inverse image of a closed set is closed. Therefore, f has a closed graph. ■

Definition 2.1.3. A binary relation $\geq$ **directs** a set D if D is not empty and:

a) if m, n, p $\in$ D such that $m \geq p$, $n \geq m$, then $n \geq p$

b) if m $\in$ D, then $m \geq m$

and,

c) if m,n $\in$ D, then there is p $\in$ D such that $p \geq m$, $p \geq n$.

Definition 2.1.4. A directed set is a pair $(D, \geq)$, such that $\geq$ directs D.

Definition 2.1.5. A **net** is a pair $(S, \geq)$, such that S is a function and $\geq$ directs the domain of S.

Definition 2.1.6. Let $f: X \to Y$ and $x \in X$. The <u>cluster set of f at x</u>, denoted $C(f; x)$, is defined as the set of all points $y \in Y$ such that there exist a net $x_\alpha \in X$ with $\lim x_\alpha = x$ and $\lim f(x_\alpha) = y$.

Definition 2.1.7. A family $\mathcal{A}$ of subsets of a set has the **finite intersection property (FIP)** provided that if $\mathcal{A}'$ is a finite subcollection of $\mathcal{A}$, then, $\cap\{A: A \in \mathcal{A}'\} \neq \emptyset$.

The converse of the Theorem 2.1.2. holds when Y is compact.

Weston's Lemma Let Y be compact and let $f: X \to Y$. Then, f is continuous at x_0 if and only if $C(f; x_0) = f(x_0)$.

Lemma 2.1.1. Let $f: X \to Y$ be a function where Y is compact space. If gr(f) is closed then f is continuous.

Proof:

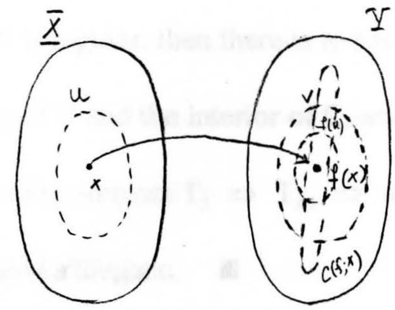


Fig. 2.1.2.

Let $x \in X$ and let V be an open neighborhood of $f(x)$ in Y. Since f has a closed graph, we need to show that there must exist a neighborhood U of x such that $f(U) \subset V$. Indeed, otherwise $\{Cl(f(U) \cap (Y\setminus V) : U \in N(x)\}$, where N(x) is the collection of all neighborhoods of x, is a collection of closed sets in the compact space Y\V which satisfies FIP. This implies:

$$C(f;x) \cap (Y\setminus V) \neq \emptyset$$

which is, by the Weston's Lemma, a contradiction, and the proof is complete. Hence, f is continuous. ■

Proposition 2.1.1. If X is a locally compact space which is either Hausdorff or regular, then the family of closed compact neighborhoods of each point is a base for its neighborhood system.

Proof:

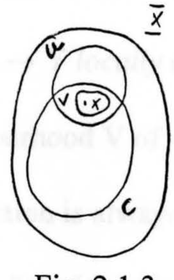


Fig. 2.1.3.

Let x be a point of X, C a compact neighborhood of x, and U an arbitrary neighborhood of x. If X is regular, then there is a closed neighborhood V of x which is a subset of the intersection of U and the interior of C, and evidently V is closed in C and hence, compact. But locally compact $T_2 \Rightarrow T_{3.5} \Rightarrow$ regular, and the above holds when X is locally compact Hausdorff space. ■

Proposition 2.1.2. Let the function $f: X \to Y$ have a closed graph. If K is a compact subset of Y, then $f^{-1}(K)$ is a closed subset of X.

Proof:

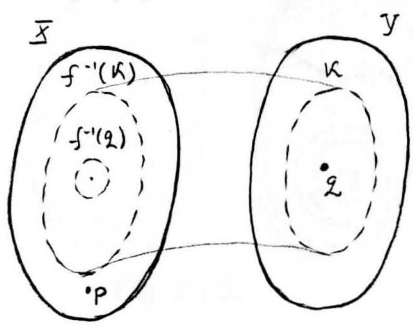


Fig. 2.1.4.

Let K be a compact subset of Y. Suppose $f^{-1}(K)$ is not closed. Then there is a $p \in X \setminus f^{-1}(K)$ and a net x_a in $f^{-1}(K)$ such that $x_a \to p$. Evidently $f(x_a)$ has a subnet $f(x_{N_b})$ which converges to some q in K. Thus, we have $(x_{N_b}, f(x_{N_b})) \to (p,q)$ so that

$p \in f^{-1}(q) \subset f^{-1}(K)$. But this contradicts the choice of p. ■

The characteristic function of the interval (0,1] mapping **R** onto {0,1} shows that a closed function does not always have a closed graph.

Let us call a function $f: X \to Y$ *locally closed* if for every neighborhood U, for each point p in X, there is a neighborhood V of p, such that $V \subset U$ and $f(V)$ is closed in Y. It is not clear that a closed function is always locally closed; but if the domain of a closed function is regular, then the function is locally closed.

A locally closed function need not be closed as the following example shows:

Example 2.1.1. Let X be the reals with the discrete topology, Y the reals with the usual topology, and $f: X \to Y$ be the identity function. Then f is locally closed and, in fact, continuous, but certainly not closed.

Theorem 2.1.3[12]. Let $f: X \to Y$ be any function where Y is locally compact Hausdorff space. If for each compact $K \subset Y$, $f^{-1}(K)$ is closed, then gr(f) is closed.

Proof:

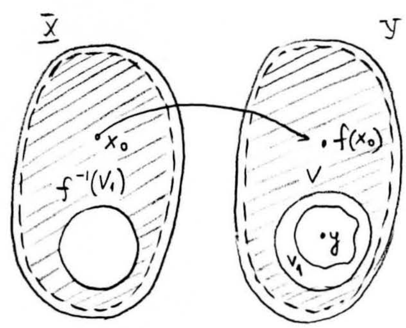


Fig. 2.1.5.

Let $x_0 \in X$ and $y \neq f(x_0)$. Since Y is T_2, there exists a neighborhood V containing y such that $f(x)$ is not an element of V. But we know by the Proposition 2.1.1. that the collection of the closed compact neighborhoods of y form a base for its neighborhood system. Thus, there exist a closed compact neighborhood $V_1 \subset V$ containing y, and, from the hypothesis, it follows that $f^{-1}(V_1)$ is closed in X. Consequently, $X \setminus f^{-1}(V_1) = U$ is open and $f(U) \cap V_1 = \varnothing$ which implies that gr(f) is closed. ■

2.2 Properties of Almost Continuous Functions

Definition 2.2.1. A function $f: X \rightarrow Y$ is **almost continuous** if for every x in X and for each open set V subset of Y containing $f(x)$, $Cl(f^{-1}(V))$ is neighborhood of x.

Here we will examine properties of composition, sum, and limits of almost continuous functions.

If f and g are two almost continuous function, then $f \circ g$ is not necessarily an almost continuous.

Define $f: [0,1] \rightarrow [0,1]$ by:

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ x, & \text{if } x \in (\mathbf{R} \backslash \mathbf{Q}) \cap [0,1] \\ 0, & \text{if } x \in \mathbf{Q} \cap [0,1) \end{cases}$$

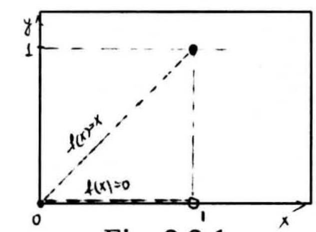


Fig. 2.2.1.

and $g: [0,1] \rightarrow [0,1]$ by:

$$g(x) = \begin{cases} 1, & \text{if } x = 1 \\ x, & \text{if } x \in \mathbf{Q} \cap [0,1) \\ 0, & \text{if } x \in (\mathbf{R} \backslash \mathbf{Q}) \cap [0,1] \end{cases}$$

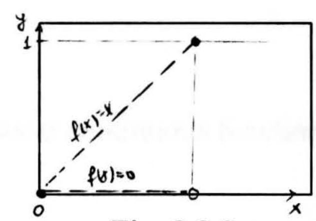


Fig. 2.2.2.

$$g(f(x)) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (\mathbf{R} \backslash \mathbf{Q}) \cap [0,1] \\ 0, & \text{if } x \in \mathbf{Q} \cap [0,1) \end{cases}$$

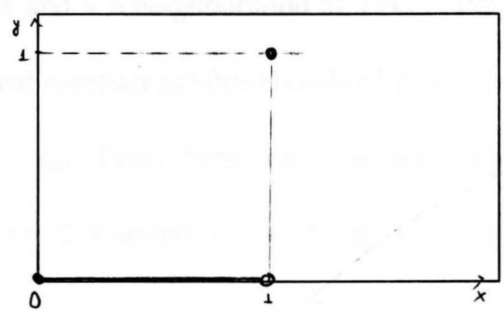


Fig. 2.2.3.

Let V be an open neighborhood of 1 not containing 0. So, consider $f^{-1}(V)$ is a neighborhood of 1. Then $Cl(f^{-1}(V))$ does not contain an open set containing 1 (in X), which implies that $f \circ g$ is not almost continuous.

A similar argument exists for the sum of two almost continuous functions.

$$(g+f)(x) = \begin{cases} 2, & \text{if } x = 1 \\ x, & \text{if } x \in (\mathbf{R} \backslash \mathbf{Q}) \cap [0,1] \\ x, & \text{if } x \in \mathbf{Q} \cap [0,1) \end{cases}$$

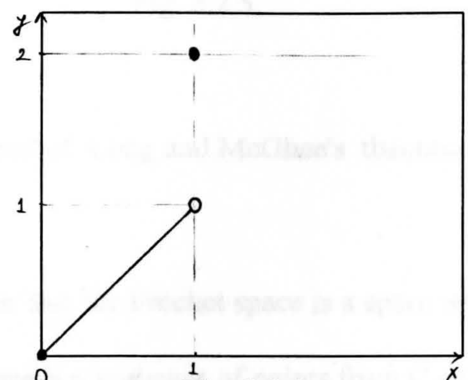


Fig. 2.2.4.

Also, limits of sequences of almost continuous functions do not have to be almost continuous.

There are various conditions that guarantee that an almost continuous function is continuous.

Theorem 2.2.2. Let $f: X \to Y$ be almost continuous where Y is locally compact. If Y is a Hausdorff space and $gr(f)$ is closed, then f is continuous.

Proof: Let $x_0 \in X$ and V a neighborhood of $f(x_0)$. Proposition 2.1.1 implies that the collection of closed compact neighborhoods of $f(x_0)$ is a base for the neighborhood system of $f(x_0)$. Thus, there exist a compact neighborhood W of $f(x_0)$ such that $W \subset V$. Since $gr(f)$ is closed, $f^{-1}(W)$ is closed by Proposition 2.1.2. so that

$Cl(f^{-1}(W)) = f^{-1}(W)$. But $Cl(f^{-1}(W))$ is a neighborhood of x_0 because f is almost continuous which implies $f^{-1}(V)$ is a neighborhood of x_0. Thus, f is continuous at x_0.

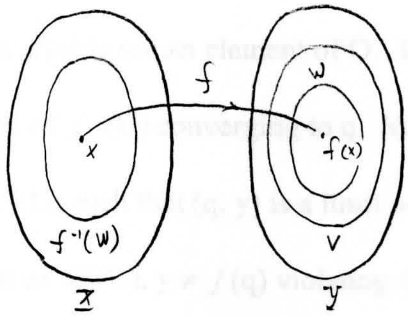
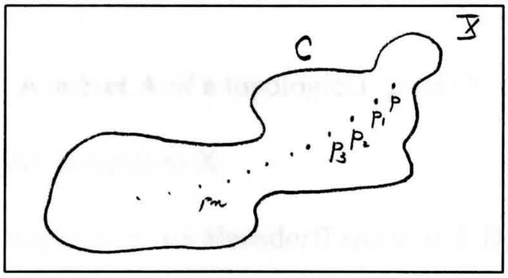


Fig. 2.2.5.

■

The hypothesis of Long and McGhee's theorem can be modified and that was done by Mahavier.

First consider that the Fréchet space is a space where if $p \in X$ is a limit point of a set $C \subset X$, then there is a sequence of points from C converging to p.



Fréchet space

Following is the Mahavier's theorem:

Theorem 2.2.3. Let $f: X \rightarrow Y$ be almost continuous where Y is locally countably compact and regular and X is a Fréchet space. If the graph of f is closed, then f is continuous.

Proof: Suppose f is not continuous at $p \in X$. Let $O \subset Y$ be an open set containing $f(p)$ such that $f^{-1}(O)$ is not a neighborhood of p. Let U be an open set containing f(p) such that Cl(U) $\subset$ O and Cl(U) is countably compact. By the almost continuity of f, there is an open set V containing p such that V $\subset$ Cl(f^{-1}(U)). There is a point q $\in$ V such that f(q) is not an element of O. There must then be a sequence {(q_i)}(i $\in$ N) of points of f^{-1}(U) converging to q. Since Cl(U) is countably compact, there is a point y $\in$ Cl(U) such that (q, y) is a limit point of {(q_i, f(q_i): i $\in$ N}. But, since f(q) is not an element of Cl(U), y $\neq$ f(q) violating the hypothesis that the graph of f is closed.

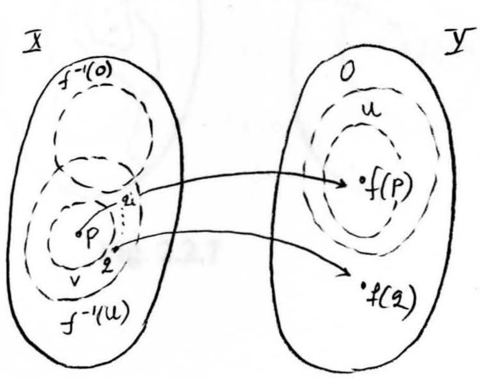


Fig 2.2.6.

■

Definition 2.2.2. A subset A of a topological space (X,τ) is **dense** in X provided that the closure of A, Cl(A), is equal to X.

Theorem 2.2.4. Suppose X is a Hausdorff space and D_1 and D_2 are disjoint dense subsets of X with $D_1 \cup D_2 = X$. Let Y be the topological sum of the subspaces D_1 and D_2 and let f: X $\rightarrow$ Y be the identity map. Then f is almost continuous; the graph of f is closed but f is nowhere continuous.

Proof: To see that f is almost continuous, let x $\in$ X and O be an open set in Y containing f(x). Suppose x $\in$ D_1. Then there is an open set V $\subset$ X containing x such that V $\cap$ D_1 $\subset$ O. Thus, Cl(f^{-1}(O)) $\supset$ Cl(f^{-1}(V $\cap$ D_1)) $\supset$ V, since D_1 is dense in X.

To see that the graph of f is closed, suppose (p,q) is the limit point of the graph of f where $q \in D_1$. If $p \neq q$, then there are disjoint open sets $U, V \subset X$ with $p \in U$ and $q \in V$. But, $U \times (V \cap D_1)$ misses the graph of f. So, p = q, i.e. $(p,q) \in gr(f)$. If $x \in D_1$, then $f^{-1}(D_1) = D_1$ is not a neighborhood of x since D_2 is dense in X. So, f is nowhere continuous.

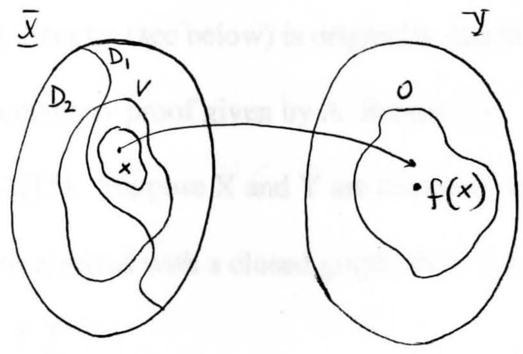


Fig. 2.2.7

■

2.3 The domain and the range space as complete metric spaces

We will start with the definition of "complete metric space" : A metric space on (X,d) is **complete** provided that every Cauchy sequence in X converges (to a point of X).

The following theorem (see below) is originally due to Pettis [15], however we shall provide more elementary proof given by A. Berner [2].

Theorem 2.3.1[15]. Suppose X and Y are complete metric spaces and if $f : X \rightarrow Y$ is almost continuous with a closed graph, then f is continuous.

Proof:

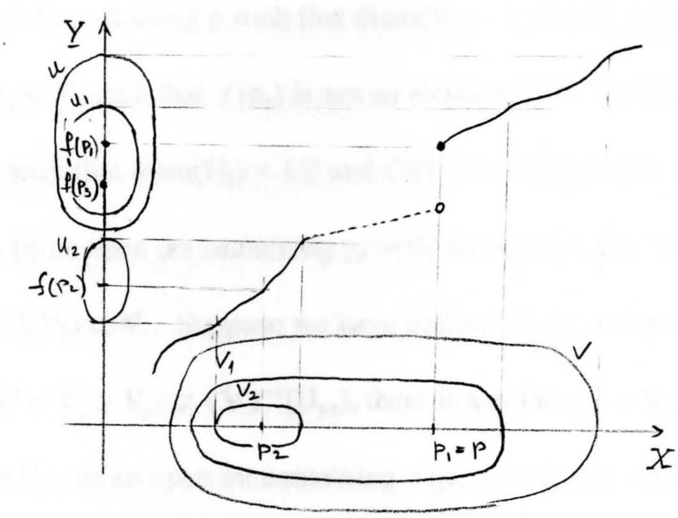


Fig. 2.3.1

Suppose f is not continuous at a point $p \in X$. We will inductively define the sequence $\{ p_i \}$ $(i \in N)$ of points of X, a sequence $\{ V_i \}$ $(i \in N)$ of open subsets of X, and a sequence $\{ U_i \}$ $(i \in N)$ of open subsets of Y satisfying the following conditions:

a. $p_i \in V_i$

b. $f(p_i) \in U_i$

c. $Cl(U_1) \cap Cl(U_2) = \varnothing$

d. If i and j are either both even or odd and $i < j$ then $Cl(U_j) \subset U_i$

e. If $i < j$ then $Cl(V_j) \subset V_i$

f. $diam\ (U_i) < 1/i$

g. $diam\ (V_i) < 1/i$

h. $V_i \subset Cl(f^{-1}(U_i))$

Let $p_1 = p$. There is an open set $U \subset Y$ containing $f(p)$ such that if V is a neighborhood of p, then $f(V)$ is not the subset of U. Let U_1 be an open set containing $f(p)$ such that $diam(U_1) < 1$ and $Cl(U_1) \subset U$. Because of the almost continuity of f, there is an open set $V_1 \subset X$ containing p such that $diam(V_1) < 1$ and $V_1 \subset Cl(f^{-1}(U_1))$. There must be a point $p_2 \in V_1$ such that $f(p_2)$ is not an element of U. Let U_2 be an open set containing $f(p_2)$ such that $diam(U_2) < 1/2$ and $Cl(U_2) \cap Cl(U_1) = \varnothing$. Again, using almost continuity, let V_2 be an open set containing p_2 with $diam(V_2) < 1/2$, $V_2 \subset Cl(f^{-1}(U_2)$ and $Cl(V_2) \subset V_1$. Suppose we have defined V_i, U_i and p_i satisfying a.- h. for all $i \leq j$. Since $\varnothing \neq V_j \subset V_{j-1} \subset Cl(f^{-1}(U_{j-1})$, there is a point $p_{j+1} \in V_j$ such that $f(p_{j+1}) \in U_{j-1.}$ Let U_{j+1} be an open set containing $f(p_{j+1})$ such that $Cl(U_{j+1}) \subset U_{j-1}$ and $diam(U_{j+1}) < 1/(j+1)$. By the almost continuity of f, we can choose an open set V_{j+1} containing p_{j+1} such that $V_{j+1} \subset Cl(f^{-1}(U_{j+1}))$, $diam(V_{j+1}) < 1/j + 1$ and $Cl(V_{j+1}) \subset V_j$. This completes the inductive definitions.

Since X is a complete metric space, there exists an x such that $\{p_i\}(i \in N)$ converges to x. Also, since Y is a complete metric space, there are points y and z such that $\{f(p_{2i})\}$ $(i \in N)$ converges to y and $\{f(p_{2i-1})\}(i \in N)$ converges to z. Since

$y \in Cl(U_1)$ and $z \in Cl(U_2)$, $y \neq z$. But, the points (x, y) and (x, z) are both the limit points of the graph of f, contradicting the fact that the graph f is closed. ∎

Definition 2.3.1. A space X is called **topologically complete** if it is homeomorphic to a complete metric space.

We can not use Theorem 2.2.4. and have both D_1 and D_2 be complete metric spaces.

The following theorem is J. D. Weston's theorem [22], however, we will provide Berner's [2] proof which is in correspondence to the notation of the Theorem 2.2.4.

Theorem 2.3.2. A Hausdorff space can not be decomposed into two disjoint dense subspaces that are topologically complete.

Proof: Suppose that X is Hausdorff and D_1, D_2 are disjoint dense subspaces each of which are topologically complete spaces, i.e. spaces which are homeomorphic to the complete metric spaces (there is no assumption about X being metric). Let d_1 and d_2 be complete metric spaces on D_1 and D_2 respectively. For $x \in D_i$, $B(x, \varepsilon) = \{y \in D_i : d_i(x, y) < \varepsilon\}$. Also, if $S \subset D_i$, the closure of S in D_i will be denoted as $Cl_i(S)$. Pick $p_1 \in D_1$ and let O_1 be an open subset of X such that $O_1 \cap D_1 = B(p_1, 1)$. Since D_2 is dense in X, we can choose $p_2 \in O_1 \cap D_2$, and let O_2 be an open subset of X such that $O_2 \subset O_1$, and $O_2 \cap D_2 \subset B(p_2, 1/2)$.

Suppose now that for each $i \leq 2n$ we have defined an open set O_i and a point $p_i \in O_i$ such that:

a.) if $i < j \leq 2n$, then $O_i \subset O_j$

b.) if i is odd, then $O_i \cap D_1 \subset B(p_i, 1/i)$

c.) if i is even, then $O_i \cap D_2 \subset B(p_i, 1/i)$

e.) if i and j are both odd and $i < j \leq 2n$, then $Cl_1(O_j \cap D_1) \subset O_i \cap D_1$

f.) if i and j are both even $i < j \leq 2n$, then $Cl_2(O_j \cap D_2) \subset O_i \cap D_2$.

Pick $p_{2n+1} \in O_{2n} \cap D_1$. Let O_{2n+1} be an open set such that $O_{2n+1} \subset O_i$, $O_{2n+1} \cap D_1 \subset B(p_{2n+1}, 1/(2n+1))$ and $Cl_1(O_{2n+1} \cap D_1) \subset O_{2n-1} \cap D_1$. Similarly pick p_{2n+2} and O_{2n+1}. Since D_1 and D_2 are complete metric spaces, there is a point $p \in D_1$ such that $\{p\} = \cap (O_i \cap D_1)$ (i is odd), and a point $q \in D_2$ such that $\{q\} = \cap(O_i \cap D_2)$ (i is even). Let U and V be disjoint open subsets of X containing p and q respectively. It is evident from the construction that $q \in O_i$ for every $i \in N$; thus, for each i, $O_i \cap V \cap D_1 \neq \varnothing$. Also, since $(O_i \cap V \cap D_1) \subset (O_i \cap D_1) \subset B(p_i, 1/i)$ (i is odd), $\cap Cl_1(O_i \cap V \cap D_i)$ cannot be empty. But (taking all intersections over odd values of i), $\cap Cl_1(O_i \cap V \cap D_1) \subset Cl_1(O_i \cap D_1) \setminus U = \cap (O_i \cap D_1 \setminus U) = \{p\} \setminus U = \varnothing$. So, D_1 and D_2 cannot both be complete metric spaces. ■

The space of irrationals is an example of a topologically complete space which cannot be decomposed into two disjoint dense subspaces that are topologically complete. The proof of this can be seen in the following example taken from L. A. Steen & J. A. Seebach Jr. [19].

Example 2.3.1. If $\{r_i\}$ is an enumeration of **Q**, we can define a new metric on **R** by $d(x,y) = |x - y| + \sum_{i=1}^{\infty} 1/2^i \inf(1, |\max_{j \leq i} 1/|x - r_j| - \max_{j \leq i} 1/|y - r_j|)$. The metric d adds to the Euclidean distance between x and y a contribution which measures the relative distance of x and y from the rationals **Q**. If $B(\mathbf{R}, \varepsilon)$ is Euclidean metric ball and $\Delta(\mathbf{R}, d)$ is a ball with regards to the metric d, it is clear that

$\Delta(\mathbf{R}, d) \subset B(\mathbf{R}, \varepsilon)$. The converse fails, since, if r is rational and ε is sufficiently small, $\Delta(\mathbf{R}, \varepsilon) = \{r\}$; hence, in the metric space $(\mathbf{R}, d)$, the rationals are open.

But if we restrict d to the irrationals $\mathbf{R}\backslash\mathbf{Q}$, we can always find for each ε, a δ so that $B(\mathbf{R}, \delta) \subset \Delta(\mathbf{R}, d)$. Thus, the metric space $(\mathbf{R}\backslash\mathbf{Q}, d)$ is homeomorphic to Euclidean irrationals.

But $(\mathbf{R}\backslash\mathbf{Q}, d)$ is complete, since no sequence $\{x_n\}$ which converges to Euclidean topology to a rational r_k can be Cauchy; for each x_n in such a sequence, there exists a term x_m (where m>n) such that $d(x_n, x_m) \geq | x_n - x_m | + 1/2^i$. Of course, those sequences which are Cauchy sequences converge to irrationals, so $\mathbf{R}\backslash\mathbf{Q}$ is topologically complete.

This example shows that the Theorem 2.2.4 would be unnecessarily restricted in its applications if we additionally require that both D_1, D_2 are topologically complete.

2.4 Points of Continuity

Definition 2.4.1. If $f: X \to Y$ is a function between topological spaces, then $C(f) = \{x \in X : f \text{ is continuous at } x\}$.

Definition 2.4.2. Given topological spaces X and Y we say that f is pointwise discontinuous if the set $C(f)$ of points of continuity is dense.

The following theorem was originally proven by Anna Neubrunnova in the 70's. However, here we provide Berner's [2] proof.

Theorem 2.4.1. Suppose Y is a regular space and $f: X \to Y$ is almost continuous. If f is pointwise discontinuous in X, then f is continuous (i.e. $C(f) = X$).

Proof:

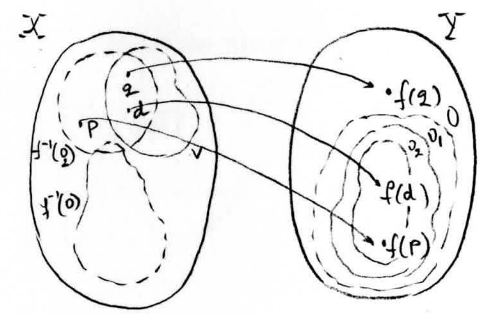


Fig. 2.4.1.

Suppose there is a point $p \in X \setminus C(f)$. Let O be an open subset of Y such that $f(p) \in O$ but $f^{-1}(O)$ is not a neighborhood of p. Let O_1 and O_2 be open subsets of Y such that $f(p) \in O_2$ and $Cl(O_2) \subset O_1$ and $Cl(O_1) \subset O$. Since f is almost continuous, there is an open set $U \subset X$ containing p such that $U \subset Cl(f^{-1}(O_2))$. There is a point q in U such that $f(q) \notin O$, since $U \not\subset f^{-1}(O)$. Then, again by the almost continuity of f, there is an open set V subset of X containing q such that $V \subset Cl(f^{-1}(Y \setminus Cl(O_1)))$. Since $U \cap V$ contains q, there is a point $d \in U \cap V \cap C(f)$. Since f is continuous at d and

$U \cap V \subset Cl(f^{-1}(Y \backslash Cl(O_1)))$, it follows that $f(d) \in Cl(O_2)$. But since $d \in U \cap V \subset V \subset Cl(f^{-1}(Y \backslash Cl(O_1)))$, it follows that $f(d) \in Cl(Y \backslash Cl(O_1)) \subset Y \backslash O_1$. This cannot be since $Cl(O_2) \subset O_1$. Therefore, there cannot be a point in $X \backslash C(f)$. Hence, f is continuous. ■

Example 2.4.1. The assumption that Y is regular cannot be dropped. Let X be the reals with the usual topology augmented to make each rational singleton open (this space is metrizable) and Y the reals with the topology generated by the sets: $(a,b) \cap (\mathbf{Q} \cup \{r\})$ where (a,b) is a usual open interval and $r \in \mathbf{R}$. The identity map from X to Y is almost continuous, and the set of points of continuity is **Q**, which is dense in X.

2.5. Closed Graph versus Separate Continuity

Definition 2.5.1. Suppose X, Y, Z are topological spaces and f maps X × Y into Z. Associate to each x ∈ X and to each y ∈ Y the mappings:

$$f_x: Y \to Z, \quad f^y: X \to Z$$

by defining

$$f_x(y) = f(x,y) = f^y(x).$$

If every f_x and f^y are continuous then f is said to be **separately continuous.**

The following condition is known as **Intermediate Value Property (IVP).**

Suppose f is a function that is continuous on the interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c between a and b with f(c) = k.

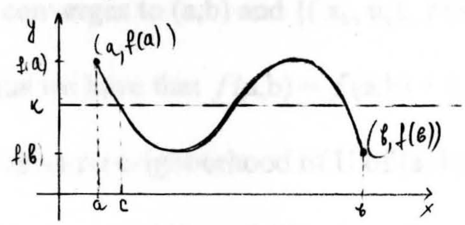


Fig. 2.5.1.

The inspiration for this section was the problem from the "YSU Problem Book" which was solved by a faculty member in the YSU Mathematics Department, Dr. Eric Wingler [18], and the problem was posed by my thesis advisor Dr. Zbigniew Piotrowski [18]. The problem states :

If $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a separately continuous function with a closed graph, is f continuous?

Theorem 2.5.1. Let $f: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ be a separately continuous function with a closed graph, then f is continuous.

Proof: Let $(a,b) \in \mathbf{R} \times \mathbf{R}$. First, it needs to be shown that the function f must be bounded in the neighborhood of (a,b). So, assume that f is not bounded above, then there exists a sequence $\{(x_n, y_n)\}$ converging to (a,b) such that $\{f(x_n, y_n)\}$ diverges to $+\infty$. Since f is separately continuous, there exists $\delta > 0$, such that $f(x, b) < f(a,b) + 1$, for all x such that $|x - a| < \delta$. Also, there is a number N such that for $k \geq N$ we have both $|x - a| < \delta$ and $f(x_k, y_k) > f(a,b) + 1$. Since, $f(x_k, y)$ is a continuos function of y, and for each $k \geq N$ (by IVP),

$$f(x_k,b) < f(a,b) + 1 < f(x_k, y_k),$$

there is a number u_k with $0 < | u_k - b| < | y_k - b|$ such that $f(x_k, u_k) = f(a,b) + 1$. It is clear that $\{(x_k, u_k)\}$ converges to (a,b) and $\{(x_k, u_k), f(x_k, u_k)\}$ converges to $((a, b), (f(a, b) + 1))$. Thus we have that $f(a,b) = f(a,b) + 1$, which is impossible. Therefore, f is bounded in some neighborhood of U of (a, b).

Now, if we consider the restriction $f\,|U$, we see that this mapping maps U into a compact subset K of **R**, and since $f\,|U$ has a closed graph it is continuous, and, therefore, f is continuous at (a, b). ■

To generalize this result we have to define necessary and sufficient conditions of spaces X, Y and Z that would guarantee that a separately continuous function $f: X \times Y \rightarrow Z$ with a closed graph is continuous.

Example 2.5.1. Let $X = Y = [0,1] \setminus (1/n : n \in N)$ with the usual topology.

Define $f: X \rightarrow Y$ by:

$$f(x) = \begin{cases} n, & \text{if } x,y \in (1/(n+1),1/n) \text{ for some } n \in N \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that f is separately continuous, and has a closed graph, but it is not continuous at (0,0).

If X or Y is locally connected, the function with the properties above will be continuous. We also may replace the co-domain **R** by any locally compact space Z, but what if Z is not locally compact?

As a reference, I supply the following example:

Example 2.5.2. Let I = [0,1] and let Z be a separable Hilbert space with orthonormal basis $\{ e_n \}^{\infty}_{n=1}$. Let Φ be defined by:

$$\Phi(x,y) = \begin{cases} 1- x^2-y^2, & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{if } x^2 + y^2 > 1 \end{cases}$$

and let

$$\Phi_n(x,y) = \Phi[2n(n+1)x - (2n+1), 2n(n+1)y - (2n+1)],$$

for each $n \in N$. Each function Φ_n is 1 at the center [(2n+1)/(2n(n+1)),(2n+1)/(2n(n+1))] of the circle inscribed in the square $[1/(n+1),1/n] \times [1/(n+1),1/n]$ and vanishes outside of this circle. Define $f: I \times I \to Z$ by $f(0,x) = \sum^{\infty}_{n=1} \Phi_n(x,y)\, e_n$ and outside this square, f vanishes. It is easy to see that at each $(x,y) \neq (0,0)$ f is continuous, and since $f(0,x) = f(x,0) = 0$ for each $x \in I$, f is separately continuous at (0,0). In addition to this, f has a closed graph. However, f is not continuous at (0,0) since

$$\| f\,[(2n+1)/(2n(n+1)), (2n+1)/(2n(n+1))] - f(0,0)\| = \| e_n \|=1$$

for every $n \in N$.

2.6. A topological version of the Closed Graph Theorem for multilinear mappings

Let us recall the classical Closed Graph Theorem in Banach spaces:

(1) If E and F are Banach spaces and u: E $\rightarrow$ F is a linear mapping with a closed graph, then u is continuous.

In her paper [4], Fernandez states that (1) is equivalent to the following statement:

(2) If E, F are Banach spaces and u: E $\rightarrow$ F is linear, surjective and continuous, then there is a constant A > 0 such that for any y $\in$ F with $\| y \| = 1$, there is an element x $\in$ E such that $u(x) = y$ and $\| x \| \leq A$.

Fernandez also notices that by the title of Cohen's [3] article we would expect the counter example for the bilinear version of (1) such as:

(1') If E_1, E_2, F are Banach spaces and f: $E_1 \times E_2 \rightarrow F$ is a bilinear mapping with a closed graph, then f is continuous.

However, the counter example to a bilinear version of (2) in Cohen's article pertains to the following version:

(2') If E_1, E_2, F are Banach spaces and f: $E_1 \times E_2 \rightarrow F$ is bilinear, surjective and continuous, then there is a constant A > 0 such that for any y $\in$ F with $\| y \| = 1$, there is an element $(x_1, x_2) \in E_1 \times E_2$, such that $f(x_1, x_2) = y$ and $\| x_1 \| \, \| x_2 \| \leq A$.

But (2') is not equivalent to (1').

As we noted in the previous chapter, when we replace linearity by almost continuity, we would expect the following topological interpretation of (1'):

(3) Let X, Y and Z be a complete metric spaces and let $f : X \times Y \to Z$ be a separately almost continuous function having a closed graph, then f is continuous.

We shall prove that (3) is false, see the Theorem 2.6.2. below.

It is very important to note that neither separate almost continuity implies almost continuity, nor vice versa, see examples 2.6.1. and 2.6.2. below. And this happens even in the case of $X = Y = \mathbf{R}$.

Theorem 2.6.1. Let X, Y, Z be complete metric spaces, and let $f: X \times Y \to Z$ be a closed graph function, then f is separately almost continuous if and only if f is separately continuous.

Proof: Clearly, separate continuity implies separate almost continuity.

The converse of the above theorem follows from the definition of product topology that the closed set $\{(x, y, f(x,y))\}$ is closed in every section $\{x\} \times Y$ and $X \times \{y\}$ for all $x \in X$ and $y \in Y$. Now by Pettis' theorem (Theorem 2.3.1) all x-sections of f_x and all y-sections f^y are now continuous, each being a closed graph almost continuous function from a complete metric space Y to Z, and X to Z, respectively.

So, the above means that f is separately continuous. ■

Now, we can show that (3) does not hold.

Theorem 2.6.2. There exists a complete metric space X, Y and Z and a closed graph separately almost continuous function f: $X \times Y \to Z$, such that f is *not* continuous.

Proof: In view of a previous theorem, we shall exhibit an example of such an f being (even) separately continuous. But, such a construction is presented in a previous section 2.5., Example 2.5.1.

Theorem 2.6.3. Assume X × Y has a countable base, and let card(X) > $\aleph_0$, and card(Y) > $\aleph_0$. Then there exists a real-valued function $f: X \times Y \to \mathbf{R}$, such that f is almost continuous, but not separately almost continuous.

Proof:

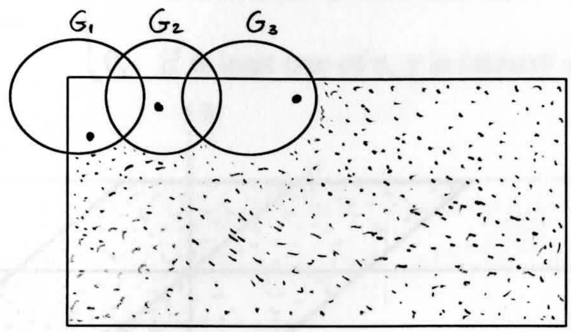


Fig. 2.6.1.

Pick (x_1, y_1) from G_1. Then pick (x_2, y_2) from G_2, such that $x_2 \neq x_1$, and $y_2 \neq y_1$. Pick (x_3, y_3) from G_3, such that x_3 is neither one of the previously chosen x_i, I<3 and y_3 is neither one of the previously chosen y_i, $i < 3$.

Since we have a countable base and uncountably many points in X and Y we can find (x_i, y_i) as indicated above. Since $\{ \boldsymbol{B}_i \}^{\infty}_{i=1}$ is a base, every open set contains an element of $\boldsymbol{B}$. So, D = $\{(x_i, y_i): i = 1,2,3,...\}$ is dense, and its complement is as well.

Now apply the characteristic function:

$$\chi_D = \begin{cases} 1, & \text{if (x,y) in D} \\ 0, & \text{otherwise} \end{cases}$$

χ_D is almost continuous.

Now, there are countably many sections, both x-sections and y-sections, which are not almost continuous.

Consider yet another example of almost continuous function which is not separately almost continuous that was given by Neunbrunn [13]:

Example 2.6.1. On the interval [-1,1] × [-1,1] in $\mathbf{R}^2$ define a real function f as:

$$f(x,y) = \begin{cases} 1, & \text{if both x and y are irrational or } (x,y) = (0,0) \\ 0, & \text{if at least one of x, y is rational and } (x,y) \neq (0,0). \end{cases}$$

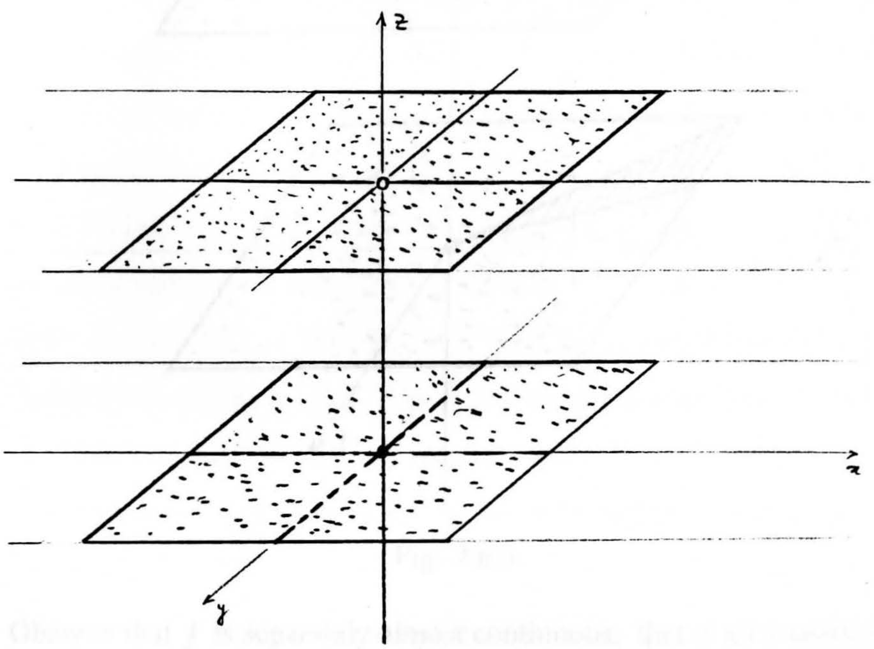


Fig. 2.6.2.

f is almost continuous at each point (x,y), but the sections f_{x0}, f^{y0} are not almost continuous when $(x_0, y_0) = (0,0)$, because none of them is continuous at the point 0.

Now, we shall provide an example of a separately almost continuous function which is *not* almost continuous.

Example 2.6.2. (Neunbrunn [13]) On the interval [-1, 1] × [-1, 1] consider the set F = {(x, y): 0 ≤x ≤ 1, (½) x ≤y≤ x}.

Define: $f: [-1, 1] \times [-1, 1] \to \mathbf{R}$ as:

$$f(x, y) = \begin{cases} 0, & \text{if } (x, y) \in F \setminus \{(0, 0)\} \\ 0, & \text{if both x, y are simultaneously rational or irrational and } (x, y) \notin F \\ 1, & \text{if x is rational, y irrational or conversely and } (x, y) \notin F \end{cases}$$

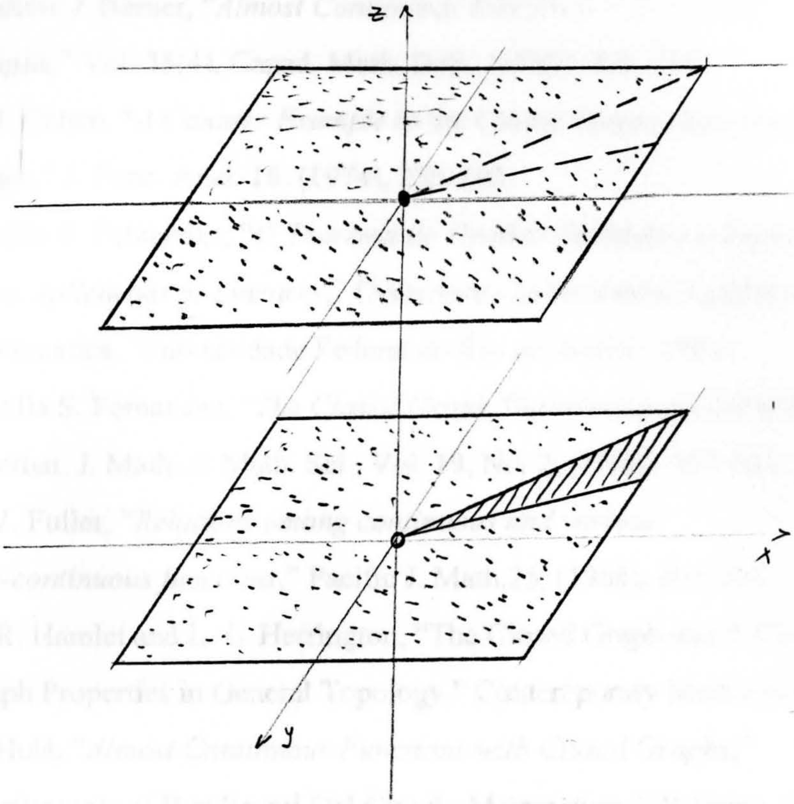


Fig. 2.6.3.

Observe that f is separately almost continuous, that is all x-sections and y-sections are almost continuous, but f is not almost continuous at (0,0) since $f(0, 0) = 1$.

References

[1] S. Banach, "Théorie des opérations linéares", Warszawa, 1932.

[2] Andrew J. Berner, "*Almost Continuous Functions with Closed Graphs,*" Vol. **25**(4), Canad. Math. Bull., (1982), 428-434.

[3] P. J. Cohen, "*A Counter Example to the Closed Graph Theorem for Bilinear Maps,*" J. Func. Anal. **16**, (1974), 235-240.

[4] Cecilia S. Fernandez, "*O Teorema do Grafico Fechado e o Teorema de Goldstine para Aplicacoes m-Lineares,*" Dissertacao de Mestrado, Instituto de Matematica, Universidade Federal do Rio de Janeiro, 1993.

[5] Cecilia S. Fernandez, "*The Closed Graph Theorem for Multilinear Mappings,*" Internat. J. Math. & Math. Sci., Vol. **19**, No. 2, (1996), 407-408.

[6] R.V. Fuller, "*Relations among continuous and various non-continuous functions,*" Pacific J. Math.25, (1968), 495-509.

[7] T. R. Hamlet and L. L. Herrington, "The Closed Graph and P-Closed Graph Properties in General Topology," Contemporary Mathematics, AMS, 1981

[8] L. Holá, "*Almost Continuous Functions with Closed Graphs,*" Supplemento ai Rendiconti Del Circolo Matematico di Palermo, Serie II, numero **14**, (1987), 229-333.

[9] T. Husain, "*Almost Continuous Mappings,*" Prace. Math. **10**, MR36 #3322, 1966

[10] John L. Kelley, "General Topology", D. Van Nostrand Company Inc, 1955.

[11] I. I. Kolodner, "*The Compact Graph Theorem,*" Amer. Math. Mounthly 75, 167-168.

[12] P. L. Long and E. E. McGehee Jr., "*Properties of Almost Continuous Functions,*" Proc. Amer. Math. Soc. **24**, (1970), 175-180.

[13] Tibor Neubrunn, "*Generalized Continuity and Separate Continuity*", Math. Slovaca **27**, (1977), No.3, 307-314.

[14] Walter Rudin, "Functional Analysis", McGraw-Hill Inc., 1991.

[15] B. J. Pettis, "*On Continuity and Opennes of Homeomorphisms in Topological*

Groups," Annals of Mathematics, Vol. **32**, No. 2, (1950), 293-307.

[16] Z. Piotrowski, "*A Topological Version of the Closed Graph Theorem for Multilinear Mappings*," Internat. J. Math. & Math. Sci., (submitted).

[17] Z. Piotrowski and A. Szymanski, "*Closed Graph Theorem : Topological approach*," Supplemento ai Rendiconti Del Circolo Matematico di Palermo, Serie II, Tomo XXXVII, (1988), 88-99.

[18] Z. Piotrowski and E. Wingler, "*A Note on Continuity Points of Functions*", Real Analysis Exchange, vol. **16**, (1990-1991), 408-414.

[19] Lynn A. Steen and J. Arthur Seebach Jr., "Counter Examples in Topology" , Holt, Rinehart and Winston Inc., 1970.

[20] G. F. Simmons, "Topology and Modern Analysis", McGraw-Hill Inc., 1963.

[21] Gilbert Strong, "Calculus", Wellesley-Cambridge Press, 1991.

[22] J. D. Weston, "*Incomplete Subspaces of a Hausdorff Space*," Archiv der Math. **10**, (1959), 40-41.

[23] Albert Wilansky, "Topics in Functional Analysis" , Lehigh University, Bethlehem, Pennsylvania, 1967.