THE TRANSFER HOMOMORPHISM AND SPLITTING THEOREMS

by

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1 Thesis Abstract

Let G be a group and $H \leq G$. Then we say G splits over H if there exists a subgroup $K \leq G$ such that $G = HK$ and $H \cap K = 1$. If it so happens that in addition $K \trianglelefteq G$ then we say G splits normally over H.

If the structure of the subgroup H or K is particularly nice, say H is cyclic or maybe abelian, then one can expect that the structure of the whole group G will be nice or at least influenced in some way by the structure of H or K. For instance, it's well known that if G splits normally over H and both H and K are solvable then G is also solvable. A more fundamental example is if G splits normally over H, G/K is abelian, and the commutator subgroup $G' \leq H$, then G is abelian.

It's this influence on the structure of G that makes it important to determine when a group will split over one of its subgroups. But what conditions placed on G or on H are sufficient in order to ensure that G splits over H? Is it possible to actually characterize whether or not G will split over a subgroup H in terms of group theoretic properties of H or G?

In the early 1900's mathematicians such as W. Burnside [1], F.G. Frobenius [2], P. Hall [3], and Schur and H. Zassenhaus [6] made efforts to finding the answers to this question for various subgroups H of various groups G. In this expository paper we chronical the development of their work and offer proofs of the theorems they were able to prove. We do this mainly through the use of special homomorphism called the transfer homomorphism.

2 Definitions

Definition A nonempty set G equipped with an operation $*$ on it is said to form a **group** under that operation if it obeys the following laws, called the **group axioms**:

(1) **Closure**: For any $a, b \in G$, we have $a * b \in G$.

(2) **Associativity**: For any $a, b, c \in G$, we have $a * (b * c) = (a * b) * c$.

(3) **Identity**: There exists $e \in G$ such that $a * e = e * a = a$ for all $a \in G$.

(4) **Inverse**: For each $a \in G$ there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. Such an element $a^{-1} \in G$ is called an **inverse** of $a \in G$. $\bigcirc$

We suppress the $*$ notation and write ab for $a * b$ and 1 for e.

Definition A group G with operation $*$ is said to be **abelian** if the operation $*$ on G obeys the **commutative** law or, in other words, if for every $a, b \in G$ we have $a * b = b * a$. $\bigcirc$

Definition Let G be any group. Then the **centre** of G, denoted $Z(G)$, consists of the elements of G that commute with every element of G. $\bigcirc$

In other words, $Z(G)=\{x \in G \mid xy = yx \text{ for all } y \in G\}$. Note that $1y = y = y1$ for all $y \in G$, so $1 \in Z(G)$, and the **centre** is a nonempty subset of G. In fact, $Z(G) \leq G$.

Definition Let G be a group and H a subgroup of G. If for all $g \in G$ we $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, then we say H is a **normal** subgroup of G and write $H \trianglelefteq G$. $\bigcirc$

Lemma 2.1 *Let G be a group, $g \in G$, and $n \in \mathbf{Z}^+$ such that $g^n = 1$. Then $|g| \mid n$.*

Proof The Division Algorithm implies there exists $q, r \in \mathbf{Z}$ such that $n = |g|q + r$

and $0 \leq r < |g|$. Then

$$\begin{aligned} 1 = g^n = g^{|g|q+r} &= g^{|g|q} g^r \\ &= (g^{|g|})^q g^r \\ &= 1^q g^r = g^r \end{aligned}$$

Therefore $1 = g^r$. But $r < |g|$, so $r = 0$ by the minimality of $|g|$. Hence, $n = |g|q$ and $|g| \mid n$. $\square$

Lemma 2.2 *Let G be a group, $g \in G$. Then $g^{|G|} = 1$.*

Proof Now $\langle g \rangle = \{1, g, g^2, g^3, ..., g^{|g|-1}\} \leq G$. Then by Lagrange $|\langle g \rangle| \mid |G|$, so $|g| \mid |G|$ .Therefore, there exists $k \in \mathbf{Z}$ such that $|g|k = |G|$. Hence,

$$g^{|G|} = g^{|g|k} = (g^{|g|})^k = 1^k = 1$$

$\square$

Lemma 2.3 *Let G be a group and $a, b \in G$ such that $ab \in Z(G)$. Then $ab = ba$.*

Proof Since $ab \in Z(G)$, $ab(b) = (b)ab$. This implies that $abbb^{-1} = babb^{-1}$. Hence $ab1 = ba1$ and we get $ab = ba$. $\square$

Theorem 2.4 First Isomorphism Theorem: *Let G_1 and G_2 be groups and ϕ: $G_1 \to G_2$ be a homomorphism. Then $G_1/\mathrm{Kern}\phi \cong G_1)\phi$.*

Theorem 2.5 The Second Isomorphism Theorem: *Let G be a group, $N \trianglelefteq G$, and $H \leq G$. Then $HN/N \cong H/H \cap N$.*

Proof Define

$$\Phi : H \to HN/N$$

by $(h)\Phi = hN$ for all $h \in H$. Notice $N \trianglelefteq G$ means $HN \leq G$ and $N \leq HN$, so $N \trianglelefteq HN$. Therefore, HN/N is a group. Also, $N \trianglelefteq G$ means $N \cap H \trianglelefteq H$, so $H/H \cap N$ is a group. Let $h_1, h_2 \in H$. Then $(h_1h_2)\Phi = h_1h_2N = h_1Nh_2N = (h_1)\Phi(h_2)\Phi$, so Φ is a homomorphism.

Let $hnN \in HN/N$. Then $(h)\Phi = hN = hnN$ since $(hn)^{-1}h = n^{-1} \in N$, so Φ is onto.

Finally, $h \in \text{Kern}\Phi$ if and only if $(h)\Phi = 1N$ if and only if $hN = 1N$ if and only if $1^{-1}h \in N$ if and only if $h \in N$ if and only if $h \in H \cap N$. Therefore, $\text{Kern}\Phi = H \cap N$. Now by the First Isomorphism Theorem $H/\text{Kern}\Phi \cong (H)\Phi$, which implies that

$$H/H \cap N \cong HN/N.$$

□

Definition Let G be a group and $H \leq G$. Then we say G **splits over** H if there exists a subgroup $K \leq G$ such that $G = HK$ and $H \cap K = 1$. ○

Example 1 In S_4, if

$$H = \{1, (1234), (13)(24), (1432), (14)(23), (12)(34), (13), (24)\},$$

and $K = \langle(123)\rangle$ then, $S_4 = HK$ and $H \cap K = 1$. Therefore, S_4 splits normally over H.

Example 2 In D_4, if $H = \{1, (24), (13), (13)(24)\}$ and $K = \langle(12)(34)\rangle$ then $D_4 = HK$ and $H \cap K = 1$.

Definition Let G be a group and $H \leq G$. Then we say G **splits normally over** H

if there exists a subgroup $K \trianglelefteq G$ such that $G = HK$ and $H \cap K = 1$. ○

Example 3 In S_3, if $H = \langle(123)\rangle$ and $K = \langle(12)\rangle$ then $H \trianglelefteq S_3$, $S_3 = HK$, and $H \cap K = 1$. Thus, S_3 splits normally over H.

Definition Let G be a group, $a, b \in G$, $H \trianglelefteq G$ and $K \leq G$. Then

(1) $[a, b] = a^{-1}b^{-1}ab$ is called the **commutator** of a and b.

(2) $[H, K] = \langle\{[h, k] \mid h \in H \text{ and } k \in K\}\rangle$.

(3) $G' = \langle\{[a, b] \mid a, b \in G\}\rangle$ is called the **commutator subgroup**. ○

Lemma 2.6 *Let G be a group, $a, b \in G$, $H \trianglelefteq G$, and $K \leq G$. Then*

(1) $[a, b] = 1$ if and only if $ab = ba$.

(2) $G' \trianglelefteq G$.

(3) G/G' is **abelian**.

(4) If G/H is abelian then $G' \leq H$.

(5) If $G' \leq K$ then K is **normal** *that is $K \trianglelefteq G$.*

Proof For (1) $[a, b] = 1$ if and only if $a^{-1}b^{-1}ab = 1$ if and only if $ab = ba$.

For (2) we know $G' \leq G$ by definition. Now let $g \in G$ and $x \in G'$.

Then $x = \prod_{i=1}^{m}[a_i, b_i]^{k_i}$ where $a_i, b_i \in G$, for all $1 \leq i \leq m$, and $k_i \in \mathbf{Z}^+$, for all $1 \leq i \leq m$.

Notice:

$$\begin{aligned} g^{-1}[a, b]g &= g^{-1}a^{-1}b^{-1}abg \\ &= g^{-1}a^{-1}gg^{-1}b^{-1}gg^{-1}agg^{-1}bg \\ &= (g^{-1}ag)^{-1}(g^{-1}bg)^{-1}(g^{-1}ag)(g^{-1}bg) \\ &= [g^{-1}ag, g^{-1}bg]. \end{aligned}$$

and

$$\begin{aligned} g^{-1}[a,b][a,b]g &= g^{-1}[a,b]gg^{-1}[a,b]g \\ &= [g^{-1}ag, g^{-1}bg][g^{-1}ag, g^{-1}bg] \\ &= [g^{-1}ag, g^{-1}bg]^2. \end{aligned}$$

Now,

$$\begin{aligned} g^{-1}xg &= g^{-1}\prod_{i=1}^{m}[a_i, b_i]^{k_i} g \\ &= \prod_{i=1}^{m} g^{-1}[a_i, b_i]^{k_i} g \\ &= \prod_{i=1}^{m}(g^{-1}[a_i, b_i]g)^{k_i} \\ &= \prod_{i=1}^{m}[g^{-1}a_i g, g^{-1}b_i g]^{k_i} \in G'. \end{aligned}$$

Thus, $G' \trianglelefteq G$.

For (3) let $G'a, G'b \in G/G'$. Then

$$\begin{aligned} [G'a, G'b] &= (G'a)^{-1}(G'b)^{-1}(G'a)(G'b) \\ &= G'a^{-1}G'b^{-1}G'aG'b \\ &= G'a^{-1}b^{-1}ab \\ &= G'[a,b] \\ &= G'1 \end{aligned}$$

since $[a,b]1^{-1} = [a,b] \in G'$. Hence $[G'a, G'b] = G'1$ or $G'aG'b = G'bG'a$ by (1), so G/G' is abelian.

For (4) let $a, b \in G$. Then $Ha^{-1}, Hb^{-1} \in G/H$, so $Ha^{-1}Hb^{-1} = Hb^{-1}Ha^{-1}$, since G/H is abelian, or $Ha^{-1}b^{-1} = Hb^{-1}a^{-1}$. Hence

$a^{-1}b^{-1}(b^{-1}a^{-1})^{-1} = a^{-1}b^{-1}ab \in H$. So $[a, b] \in H$. Now since $H \leq G$ we get $G' \leq H$ for all $[a, b] \in H$ for $a, b \in G$.

For (5) let $k \in K$ and $g \in G$. We show that $g^{-1}kg \in K$.

$k^{-1}g^{-1}kg = [k, g] \in G' \leq K$, so $k^{-1}g^{-1}kg \in K$. Thus, there exists $k_1 \in K$ such that $k^{-1}g^{-1}kg = k_1$. Hence $g^{-1}kg = kk_1 \in K$. Therefore, $K \trianglelefteq G$. $\square$

Definition Let G be a group and $a, b \in G$. Then a and b are **conjugates** if there exists $g \in G$ such that $a = g^{-1}bg$, written as b^g. $\bigcirc$

Definition Let G be a group and S a nonempty set. Then G **acts** on S if there is a homomorphism

$$\phi : G \to Sym(S),$$

where $Sym(S) = \{\phi : S \to S \mid \phi \text{ is one to one and onto}\}$. $\bigcirc$

Notation: Let G be a group and S be a set such that G acts on S via ϕ. If $g \in G$, $a \in S$ then $\phi(g)(a) = ga$.

Definition Let G be a group and S a set such that G acts on S and $a \in S$. The **stabilizer** in G of a, G_a, is defined by $G_a = \{g \in G \mid ga = a\}$. $\bigcirc$

Definition Let G be a group, S a set, and $a \in S$ such that G acts on S. The **orbit** of G on S containing a is

$$Ga = \{ga \mid g \in G\}$$

$\bigcirc$

THE SYLOW THEOREMS

Definition Let G be a group, p be a prime, and $n \in Z^+ \cup \{0\}$ such that $p^n \mid |G|$ but p^{n+1} does not $|G|$. Then

(1) $|G|_p = p^n$ (called the p^{th} part of G)

(2) If $P \leq G$ and $|P| = p^n$, we call P a **Sylow p-subgroup** of G.

(3) $Syl_p(G)$ is the set of **Sylow** p-subgroups of G. ○

Theorem 2.7 First Sylow Theorem *Let G be a group and p be a prime. Then $Syl_p(G) \neq \emptyset$.*

Theorem 2.8 Second Sylow Theorem *Let G be a group, p be a prime, and $H \leq G$ be a p-subgroup. Then there exists $P \in Syl_p(G)$ such that $H \leq P$. Moreover, G acts transitively on $Syl_p(G)$ by conjugation.*

3 The Transfer Homomorphism

Definition Let G be a group and $H \leq G$. A subset $\{t_i\}_{i=1}^n$ is called a **transversal** of H in G if

$$G = \bigcup_{i=1}^{n} Ht_i,$$

where $n = |G|/|H|$.

Theorem 3.1 *Let G be a group, and $H \leq G$ and $\Im$ is the set of transversals of H in G. Then*

(1) G acts on $\Im$ by $\{t_{i=1}^n\}g = \{t_i g\}_{i=1}^n$.

(2) H acts on $\Im$ by $h\{t_i\}_{i=1}^n = \{ht_i\}_{i=1}^n$.

Proof For (1), if $Ht_i g = Ht_j g$ for some i and j then $Ht_i = Ht_j$, so $i = j$. Hence $|\{Ht_i g\}_{i=1}^n| = |\{Ht_i\}_{i=1}^n|$. Therefore, $G = \bigcup_{i=1}^n Ht_i g$ and $\{t_i g\}_{i=1}^n \in \Im$.For (2), let $T = t_i\}_{i=1}^n \in \Im$, $h \in H$ then $G = \bigcup_{i=1}^n Ht_i$. We show that $|\{Hht_i\}_{i=1}^n| = |\{Ht_i\}_{i=1}^n|$. $\{Hht_i\}_{i=1}^n = \{Ht_i\}_{i=1}^n$ since $(ht_i)t_i^{-1} = h \in H$ for all $1 \leq i \leq n$. Hence $\bigcup_{i=1}^n Hht_i = \bigcup_{i=1}^n Ht_i = G$. So $\{ht_i\}_{i=1}^n \in \Im$. □

Definition Let G be a group, $J \trianglelefteq H \leq G$, such that H/J is abelian. If $T, U \in \Im$ where $T = \{t_i\}_{i=1}^n$ and $U = \{u_i\}_{i=1}^n$ and $Ht_i = Hu_i$ for all $1 \leq i \leq n$ define

$$T/U = \Pi_{i=1}^n Jt_i u_i^{-1} \in H/J.$$

○

Lemma 3.2 *Let G be a group and $J \trianglelefteq H \leq G$ such that H/J is abelian. If $T, U, V \in \Im$ then*

(1) $T/T = J$

(2) $T/U = (U/T)^{-1}$

(3) $T/U = (T/V)(V/U)$

Proof For (1)

$$\begin{aligned} T/T &= \Pi_{i=1}^n J t_i t_i^{-1} \\ &= \Pi_{i=1}^n J1 = J1J1\ldots J1 \\ &= J11\ldots 1 \\ &= J \end{aligned}$$

For (2) since H/J is abelian,

$$\begin{aligned} T/U &= \Pi_{i=1}^n J t_i u_i^{-1} \\ &= \Pi_{i=1}^n J (u_i t_i^{-1})^{-1} \\ &= (\Pi_{i=1}^n J u_i t_i^{-1})^{-1} \\ &= (U/T)^{-1} \end{aligned}$$

For (3) since H/J is abelian,

$$\begin{aligned} T/U &= \Pi_{i=1}^n J t_i u_i^{-1} \\ &= \Pi_{i=1}^n J t_i v_i^{-1} v_i u_i^{-1} \\ &= \Pi_{i=1}^n J t_i v_i^{-1} J v_i u_i^{-1} \\ &= \Pi_{i=1}^n J t_i v_i^{-1} \Pi_{i=1}^n J v_i u_i^{-1} \\ &= (T/V)(V/U). \end{aligned}$$

□

Lemma 3.3 *Let G be a group and $J \trianglelefteq H \leq G$ such that H/J is abelian and $T \in \Im$. Define the* **transfer homomorphism**

$$\tau : G \to H/J$$

by $(g)\tau = Tg/T$ for all $g \in G$. Then

(1) If $U \in \Im$, $g \in G$ and $h \in H$, then $Tg/Ug = T/U$ and $hT/hU = T/U$.

(2) τ is independent of T (i.e. $Tg/T = Ug/U$ for all $T, U \in \Im$)

(3) τ is a homomorphism.

Proof For (1)

$$\begin{aligned} Tg/Ug &= \Pi_{i=1}^{n} Jt_i g(u_i g)^{-1} \\ &= \Pi_{i=1}^{n} Jt_i g g^{-1} u_i^{-1} \\ &= \Pi_{i=1}^{n} Jt_i u_i^{-1} \\ &= T/U \end{aligned}$$

and since H/J is abelian,

$$\begin{aligned} hT/hU &= \Pi_{i=1}^{n} Jht_i(hu_i)^(-1) \\ &= \Pi_{i=1}^{n} Jht_i u_i^{-1} h^{-1} \\ &= \Pi_{i=1}^{n} JhJt_i u_i^{-1} Jh^{-1} \\ &= \Pi_{i=1}^{n} JhJt_i u_i (Jh)^{-1} \\ &= \Pi i = 1^n Jt_i u_i^{-1}(Jh)(Jh)^{-1} = J1Jt_i u_i^(- 1) \\ &= \Pi_{i=1}^{n} Jt_i u_i^{-1} \\ &= T/U. \end{aligned}$$

For (2) let $U \in \Im$. Then by (1)

$$\begin{aligned} Tg/T &= (Tg/Ug)(Ug/U)(U/T) \\ &= (T/U)(Ug/U)(U/T) \\ &= (T/U)(U/T)(Ug/U) \\ &= (T/U)(T/U)^{-1}(Ug/U) \\ &= JUg/U \\ &= Ug/U \end{aligned}$$

since H/J is abelian.

For (3) let $g_1, g_2 \in G$. Then

$$\begin{aligned} (g_1g_2)\tau &= Tg_1g_2/T \\ &= (Tg_1g_2/Tg_2)(Tg_2/T) \\ &= (Tg_1/T)(Tg_2/T) \\ &= (g_1)\tau(g_2)\tau. \end{aligned}$$

$\square$

Definition Let G be a group and $S \subseteq G$. Define the subgroup generated by S, $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H$. Then clearly, $\langle S \rangle \leq G$, and $S \subseteq \langle S \rangle$. $\bigcirc$

Lemma 3.4 *Let G be a group. Then $\langle S \rangle = s_1^{n_1} s_2^{n_2} \dots s_k^{n_k}$, where $s_i \in S$, $n_i \in \mathbf{Z}$, and $k \in \mathbf{Z}^+$.*

Proof Let $T = \{s_1^{n_1} s_2^{n_2} \dots s_k^{n_k} \mid s_i \in S, n_i \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^+\}$. Realize that T is closed, T contains all of its inverses, and $1 = s^0 \in T$. Therefore, $T \leq G$. Also, $S \subseteq T$ since $s = s^1 \in T$ for all $s \in S$. Hence $\langle S \rangle \leq T$.

Let $s_1^{n_1} s_2^{n_2} \dots s_k^{n_k} \in T$ and $S \subseteq H \leq G$. Then $s_i \in H$ for all $1 \leq i \leq k$, so $s_i^{n_i} \in H$ since $H \leq G$. So $s_1^{n_1} s_2^{n_2} \dots s_k^{n_k} \in H$ since $H \leq G$. Therefore, $T \leq H$, so $T \leq \bigcap_{S \subseteq H \leq G} H = \langle S \rangle$. □

Lemma 3.5 *Let G be a group, $J \trianglelefteq H \leq G$, H/J be abelian and suppose $gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$. Then $G' \bigcap Z(G) \bigcap H \leq J$.*

Proof Let $h \in G' \bigcap Z(G) \bigcap H$. Then by the First Isomorphism Theorem $G/\mathrm{Kern}\mathcal{T} \cong (G)\mathcal{T} \leq H/J$, so $G/\mathrm{Kern}\mathcal{T}$ is abelian. Hence, $G' \leq \mathrm{Kern}\mathcal{T}$, so $h \in \mathrm{Kern}\mathcal{T}$. Therefore, since $h \in Z(G)$,

$$\begin{aligned} J = (h)\mathcal{T} &= Th/T = \prod_{i=1}^{n} Jt_i h t_i^{-1} \\ &= \prod_{i=1}^{n} Jt_i h t_i^{-1} = \prod_{i=1}^{n} Jh = Jhh \dots h \\ &= Jh^n \end{aligned}$$

where $n = |G|/|H|$. Hence, $J = Jh^n$, so $h^n \in J$.

Now working H/J with Jh. Then $(Jh)^n = Jh^n = J$, so $|Jh| \mid \frac{|G|}{|H|}$ by a previous lemma. Also, $(Jh)^{\frac{|H|}{|J|}} = J$. Hence $|Jh| \mid \frac{|H|}{|J|}$. Therefore, $|Jh| = 1$ since $\gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$. Hence $Jh = J$, so $h \in J$. Therefore $G' \bigcap Z(G) \bigcap H \leq J$. □

Definition Let G be a group , $J \trianglelefteq H \leq G$ such that H/J is abelian. Define a relation on $\Im$ by

$$T \sim U \text{ if } T/U = J$$

Then $\sim$ is an equivalence relation on $\Im$ by lemma 3.5.

This follows from lemma 3.5 $T \sim T$. $T \sim T$ implies

$$\begin{aligned} T/T &= \prod_{n=i}^{n} Jt_i t_i^{-1} \\ &= \prod_{n=i}^{n} J1 \\ &= J1J1 \dots J1 \\ &= J1 \dots 1 \\ &= J1 = J. \end{aligned}$$

Hence $\sim$ is reflexive.

Also if $u \in U, T \sim U$ implies $U \sim T$. $T \sim U$ implies $T/U = J$. $U \sim T$ implies $U/T = (T/U)^{-1} = J^{-1} = J$. Hence, $U \sim T$.

Next, $T \sim U$ implies $T/U = \prod_{n=i}^{n} Jt_i u_i^{-1}$. $U \sim V$ implies $U/V = \prod_{n=i}^{n} Jt_i u_i^{-1}$. $T \sim U \sim V$ implies $T/U \cdot U/V = J \cdot J = J$. Hence, $T \sim U$. □

Lemma 3.6 *Let G be a group, $J \trianglelefteq H \leq G$, H/J is abelian, and $\sim$ be as above. For $T \in U$ let $[T] = \{U \in \Im \mid U \sim T\}$ (Equivalence Class) and $\Omega = \{[T] \mid T \in \Im\}$. Then G and H acts on Ω by $[T]g = [Tg]$ and $[T]h = [hT]$ for all $g \in G$ and for all $h \in H$.*

Proof It's enough to show this actions are well defined. Well, from a previous lemma if $T \sim U$ then $J = T/U = Tg/Ug$, so $Tg \sim Ug$. Hence, $[Tg] = [Ug]$. Therefore, $[T]g = [U]g$.

If $T \sim U$ then $J = T/U = hT/hU$. Hence, $hT \sim hU$ so $[hT] = [hU]$. Thus, $[T]h = [U]h$.

We show that if $[T] \in \Omega$ then $[Tg] \in \Omega$ and $[hT] \in \Omega$. If $[T] \in \Omega$ then $T \in \Im$ implies $Tg \in \Im$, $hT \in \Im$. Hence $[Tg], [hT] \in \Omega$. $\square$

Lemma 3.7 *Let G be a group, $J \trianglelefteq H \leq G$, H/J be abelian, $gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$ and $\sim, \tau, \Omega$ be as above. Then*

(1) H acts transitively on Ω.

(2) $H_{[T]} = J$ for all $[T] \in \Omega$.

Proof For (1) let $[T], [U] \in \Omega$ and let $n = \frac{|G|}{|H|}, m = \frac{|H|}{|J|}$. We want to find $h \in H$ such that $[T]h = [U]$. Now $[T]h = [U]$ if and only if $[hT] = [U]$ if and only if $hT \sim U$ if and only if $hT/U = J$. By Lemma 3.5

$$
\begin{aligned}
hT/U &= (hT/T)(T/U) \\
&= \prod_{i=1}^{n} Jht_it_i^{-1}(T/U) \\
&= \prod_{i=1}^{n} Jh(T/U) \\
&= \prod_{i=1}^{n} h^n(T/U)
\end{aligned}
$$

since $\gcd(n, m) = 1$ there exists $r, s \in \mathbf{Z}$ such that $rn + sm = -1$. Let $h \in H$ such that $Jh = (T/U)^r$ then

$$
\begin{aligned}
Jh^n(T/U) &= ((T/U)^r)^n(T/U) \\
&= (T/U)^{rn+1} \\
&= (T/U)^{-sm} \\
&= ((T/U)^m)^{-s}
\end{aligned}
$$

$$
\begin{aligned}
&= J^{-s} \\
&= J.
\end{aligned}
$$

Therefore, $[T]h = [U]$.

For (2), let $j \in J$. Then

$$
\begin{aligned}
jT/T &= \prod_{i=1}^{n} Jjt_it_i^{-1} \\
&= \prod_{i=1}^{n} Jj \\
&= \prod_{i=1}^{n} J \\
&= J.
\end{aligned}
$$

Therefore, $jT \sim T$ which implies $[jT] = [T]$. Hence, $[T]j = [T]$, so $J \leq H_{[T]}$.

Let $h \in H_{[T]}$ then $[T]h = [T]$ implies $[hT] = [T]$, so $hT \sim T$. Therefore,

$$
\begin{aligned}
J &= hT/T \\
&= \prod_{i=1}^{n} Jht_it_i^{-1} \\
&= \prod_{i=1}^{n} Jh \\
&= \prod_{i=1}^{n} Jhh\dots h \\
&= Jh^n.
\end{aligned}
$$

So $h^n \in J$. Now working H/J with Jh, $(Jh)^n = Jh^n = J$, so $|Jh| \mid n$ by lemma 2.1. Also, $(Jh)^{\frac{|H|}{|J|}} = J$. Hence, $|Jh| \mid \frac{|H|}{|J|} = m$, so $|Jh| = 1$, since $gcd(m, n) = 1$. Hence, $Jh = J$, so $h \in J$. Thus, $H_{[T]} \leq J$, $H_{[T]} = J$. $\square$

Let G be a group, $J \trianglelefteq H \leq G$ such that H/J is abelian and $g \in G$. Since τ is independent of the choice of the transversal used, we are going to find a transversal to use to make τ easier to compute.

Let $\langle g \rangle$ act on $S = \{Hx \mid x \in G\}$ by right multiplication. Then

$$S = \bigcup_{i=1}^{n} \Theta_i$$

where n is the number of orbits of $\langle g \rangle$ on S and

$$\Theta_i = \{Hx_i, Hx_i g, Hx_i g^2, \ldots, Hx_i g^{n_i - 1}\}$$

where $Hx_i g^{n_i} = Hx_i$ for all $1 \leq i \leq n$.

Let $T = \{x_i g^r \mid 0 \leq r \leq n_i - 1, 1 \leq i \leq n\}$. Then

$$Tg = \{x_i g^r \mid 0 \leq r \leq n_i, 1 \leq i \leq n\}$$

Hence, $(g)\tau = Tg/T = \prod_{i=1}^{n} Jx_i g^{n_i} x_i^{-1}$ where $x_i g^{n_i} x_i^{-1} \in H$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} n_i = n$, number of orbits of $\langle g \rangle$ on S where $n_n \neq n$.

4 The Splitting Theorems

Theorem 4.1 *Let G be a group, $J \trianglelefteq H \leq G$, H/J be abelian and $gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$. Then the following are equivalent:*

(1) G splits normally over H/J.

(2) If $h_1, h_2 \in H$ are conjugate in G then $Jh_1 = Jh_2$.

(3) $(h)\tau = Jh^n$ where $n = \frac{|G|}{|H|}$ for all $h \in H$.

(4) $Th \sim hT$ for all $h \in H, T \in \Im$.

Proof For (1) implies (2), by (1) there exists $K \trianglelefteq G$ such that $G = HK$ and $H \cap K = J$. Let $h^g \in H$ where $h \in H$. Now $g = h_1 k$ where $h_1 \in H$, $k \in K$, so $h^g = h^{h_1 k} = h_2^k$, where $h_2 = h^{h_1}$. Now $[h_2^{-1}, k] = (h_2^{-1})^{-1} k^{-1} h_2^{-1} k \in K$ since $K \trianglelefteq G$ and $[h_2^{-1}, k] = h_2^{-1}(h_2^{-1})^k \in H$ since $h_2^k \in H$. Therefore, $h_2^{-1}(h_2^{-1}) \in H \cap K = J$, so $Jh_2 = Jh_2^k$. Thus, $h_2(h_2^k)^{-1} \in J$. Then $Jh_2 = Jh_2^k$, so $Jh^{h_1} = Jh^g$. Hence, $Jh^{Jh_1} = Jh^g$, so $Jh = Jh^g$ since H/J is abelian. Therefore, (1) implies (2).

For (2) implies (3), let $h \in H$. Now $(h^{n_i})^{x_i^{-1}}, h^{n_i} \in H$ for all $1 \leq i \leq n$ and are conjugate since

$$(h^{n_i x_i^{-1}})^{x_i} = h^{n_i}$$

Hence, by (2)

$$\begin{aligned}
(h)\tau &= \prod_{i=1}^{n} Jx_i h^{n_i} x_i^{-1} \\
&= \prod_{i=1}^{n} J(h^{n_i})^{x_i} \\
&= \prod_{i=1}^{n} Jh^{n_i} \\
&= Jh^{\sum_{i=1}^{n} n_i} = Jh^n.
\end{aligned}$$

Therefore, $(h)\tau = Jh^n$.

For (3) implies (4), Let $h \in H$. Then $Th/hT = (Th/T)(T/hT) = ((h)\tau)(T/hT) = Jh^n \prod_{i=1}^{n} Jh^{-1} = Jh^n Jh^{-n} = Jh^n h^{-n} = Jh^0 = J1 = J$. Therefore, $Th \sim hT$.

For (4) implies (1), since $Th \sim hT$ for all $h \in H$ we have $[Th] = [hT]$ for all $h \in H$. Hence, since H acts transitively on Ω on the left, we know H act transitively on Ω on the right.

Let $g \in G$ and $T \in \Im$. Claim: $G = G_{[T]}H$.

Now $[Tg] \in \Omega$ and $[T] \in \Omega$. Since H acts transitively on Ω on the right there exists $h \in H$ such that $[Tg]h = [T]$, so $[T]gh = [T]$. Therefore, $gh \in G_{[T]}$, so $g \in G_{[T]}H$. Hence, $G = G_{[T]}H$. Also, $H \cap G_{[T]} = H_{[T]} = J$ by Theorem 3.10. Therefore, G splits over H/J.

Next we show that $G_{[T]} \trianglelefteq G$. We claim that $G_{[T]} = \text{Kern}\tau$.

$$\begin{aligned}
g \in G_{[T]} &= [T]g = [T] \Leftrightarrow \\
&= [Tg] = [T] \Leftrightarrow \\
&= Tg \sim T \Leftrightarrow \\
&= Tg/T = J \Leftrightarrow \\
&= (g)\tau = J \Leftrightarrow \\
&= g \in \text{Kern}\tau.
\end{aligned}$$

Hence, $G_{[T]} = \text{Kern}\tau \trianglelefteq G$. $\square$

Definition: Let G be a group and π be a set of primes.

(1) $\pi(G)$=$\{p \mid p$ is prime and $p \mid |G|\}$.

(2) π'=$\{p \mid p$ is prime and $p \notin \pi\}$.

(3) G is a π-group if $\pi(G) \subseteq \pi$.

(4) A subgroup $H \leq G$ is a Hall π-subgroup if H is a π-group and $\pi(G/H) \subseteq \pi'$.

(5) $\text{Hall}_\pi(G) = \{H \leq G \mid H$ is a Hall π-subgroup$\}$. ○

Example 1 Let $|G| = 2^4 \times 3^2 \times 5^6$. Then $\pi(G) = \{2,3,5\}$ and G is a $\{2,3,5\}$ - group. Also, if $H \in \text{Hall}_{\{3,5\}}(G)$ then $|H| = 3^2 \times 5^6$.

Example 2 Consider A_5. $|A_5| = 5!/2 = 2^2 \times 3 \times 5$

We know that $(A_5)_1 \cong A_4$, so $|(A_5)_1| = |A_4| = 2^2 \times 3$. Thus, $(A_5)_1$ is a $\{2,3\}$ - subgroup. Also,

$$\frac{|A_5|}{|(A_5)_1|} = \frac{2^2 \times 3 \times 5}{2^2 \times 3} = 5 \in \{2,3\}'.$$

$\pi(A_5/(A_5)_1) \subseteq \{2,3\}'$. Thus, $(A_5)_1 \in \textbf{Hall}_{\{2,3\}}(A_5)$

Example 3 We show that $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$. **Solution**: Let $H \in \text{Hall}_{\{2,5\}}(A_5)$. Then $|A_5| = 5!/2 = 5 \times 4 \times 3 \times 2 \times 1 \div 2 = 2^2 \times 3 \times 5$, so $|H| = 2^2 \times 5$ and $|A_5|/|H| = 3$. Let A_5 act on $S = \{Hx \mid x \in A_5\}$ by right multiplication via $\Phi : A_5 \to Sym(S) \cong S_3$ defined by $\text{Kern}\Phi \trianglelefteq A_5$. Since A_5 is simple, $\text{Kern}\Phi = 1$ or A_5. If $\text{Kern}\Phi = 1$, by First Isomorphism Theorem

$$A_5/\text{Kern}\Phi \cong (A_5)\Phi \leq S_3.$$

But then

$$|A_5| \mid |S_3| = A_5/1 \cong A_5 \cong (A_5)\Phi \leq S_3.$$

Now, $60 = |A_5| \mid |S_3| = 6$, a contradiction. If

$\text{Kern}\Phi = A_5$ then $A_5 = \text{Kern}\Phi = \bigcap_{g \in G} H^g \leq H$. Thus, $A_5 = H$, a contradiction. Hence, $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$. □

Theorem 4.2 *Let G be a group , $H \in Hall_\pi(G)$, and H be abelian. Then G splits normally over H if and only if whenever for all $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$ then $h_1 = h_2$.*

Proof Now, $1 \trianglelefteq H \leq G$ and $H/1 \cong H$ is abelian since $H \in Hall_\pi(G)$ we know $\gcd(\frac{|G|}{|H|}, \frac{|H|}{1}) = 1$. Also, G splits normally over H if and only if G splits normally over $H/1$. By Theorem 3.11 G splits normally over $H/1$ if and only if $h_1 \sim_G h_2$ implies $\{1\}h_1 = \{1\}h_2$, if and only if $h_1 \sim_G h_2$ implies $h_1 h_2^{-1} \in \{1\}$, which is true if and only if $h_1 \sim_G h_2$ implies $h_1 = h_2$. □

Theorem 4.3 *Let G be a group, $P \in Syl_p(G)$, and $x, y \in Z(P)$ such that $x \sim_G y$. Then $x \sim_{N_G(P)} y$.*

Proof Since $x \sim_G y$ there exists $g \in G$ such that $x = y^g$. Since $x, y \in Z(P)$, $P \leq C_G(x) \cap C_G(y)$, so $P \in Syl_p((C_G(x))$. Also, $P \leq C_G(y)$ implies $P^g \leq C_G(y)^g = C_G(y^g) = C_G(x)$, so $P, P^g \in Syl_p((C_G(x))$. Hence, there exists $g_0 \in C_G(x)$ such that $P^{gg_0} = P$. Thus, $gg_0 \in N_G(P)$, and $y^{gg_0} = x^{g_0} = x$. Therefore, $x \sim_{N_G(P)} y$. □

Theorem 4.4 Burnside *Let G be a group and $P \in Syl_p(G)$ such that $P \leq Z(N_G(P))$. Then G splits normally over P.*

Proof Since $P \leq Z(N_G(P))$, P is abelian. Also, $P \in Hall_\pi(G)$ for $\pi = \{p\}$. Let $x, y \in P$ such that $x \sim_G y$. Then there exists $g \in G$ such that $x = y^g$ since P is abelian we know $x, y \in Z(P)$, so by Theorem 3.13 $x \sim_{N_G(P)} y$. Hence, there exists $n \in N_G(P)$ such that $x = y^n = y$ since $P \leq Z(N_G(P))$ and $y \in P$. Therefore, by Theorem 3.12 G splits normally over P. □

Theorem 4.5 *Let G be a group, $H \in Hall_\pi(G)$ such that $H \trianglelefteq G$ and H is abelian. Then G splits over H and G acts transitively on the complements of H in G by conjugation.*

Proof Now, $1 \trianglelefteq H \leq G$ and $H/1 \cong H$ is abelian and $\gcd(\frac{|G|}{|H|}, \frac{|H|}{1}) = 1$ since $H \in Hall_\pi(G)$. Hence, by the previous theorem H acts transitively on Ω on the left and $H_{[T]} = 1$.

Claim: $G = G_{[T]}H$. We use a Frattini Argument. Let $g \in G$. Since H acts transitively on Ω, $\Omega = [T]H$ and $[T]g \in \Omega$. Therefore, there exists $h \in H$ such that $[T]g = [T]h$. Hence, $[T]gh^{-1} = [T]$. Thus, $gh^{-1} \in G_{[T]}$, so $g \in G_{[T]}H$. Hence, $G = G_{[T]}H$, and

$$H \cap G_{[T]} = H_{[T]} = 1.$$

Therefore, G splits over H.

Next, let L be another complement of H in G. Then $G = HL$ and $H \cap L = 1$. Also,

$$\begin{aligned} |L| &= \frac{|L|}{1} \\ &= \frac{|L|}{|H \cap L|} \\ &= \frac{|LH|}{|H|} \\ &= \frac{|G|}{|H|}. \end{aligned}$$

Claim: $L \in \Im$. If $l_1, l_2 \in L$ such that $Hl_1 = Hl_2$ then $l_1, l_2^{-1} \in H \cap L = 1$, so l_1, l_2. Hence, $L \in \Im$ and $[L] \in \Omega$. For $l \in L, lL = L$ then $[L]l = [lL] = [L]$, so $L \leq G_{[L]}$. Now there exists $h \in H$ such that $[L]h = [T]$. But then $L^h \leq G_{[L]h} = G_{[L]}h = G_{[T]}$.

Now

$$|L^h| = |L| = \frac{|G|}{|H|} = \frac{|G_{[T]}H|}{|H|} = \frac{|G_{[T]}|}{|G_{[T]} \cap H|} = \frac{|G_{[T]}|}{1} = |G_{[T]}|.$$

Hence, $|L^h| = |G_{[T]}|$ and $L^h \leq G_{[T]}$ implies $L^h = G_{[T]}$. Therefore, G acts transitively on complements of H in G. □

Lemma 4.6 *Let G be a group, $N \trianglelefteq G, H \in Hall_\pi(G)$. Then*

(1) $HN/N \in Hall_\pi(G/N)$.

(2) $H \cap N \in Hall_\pi(N)$.

Proof For (1),

$$|\frac{HN}{N}| = \frac{|HN|}{|N|} = \frac{|H||N|}{|N||H \cap N|} = \frac{|H|}{|H \cap N|}$$

, so $\pi(\frac{HN}{N}) \subseteq \pi$ since $H \in Hall_\pi(G)$. Hence, $\frac{HN}{N}$ is a π-group.

Also,

$$\frac{|G/N|}{|HN/N|} = \frac{|G|/|N|}{|HN|/|N|} = \frac{|G|}{|HN|}$$

and

$$\frac{|G|}{|HN|}\frac{|HN|}{|H|} = \frac{|G|}{|H|}.$$

Hence, $|G|/|HN| \mid |G|/|H|$ and $\pi(G/H) \subseteq \pi'$. Therefore,$\pi(G/HN) \subseteq \pi'$. Thus, $\pi(\frac{G/N}{HN/N}) \subseteq \pi'$, so $HN/N \in Hall_\pi(G/N)$

For (2), Since H is a π-group, $H \cap N$ is a π-group. Also,

$$|G/N|_\pi = \frac{|G|_\pi}{|N|_\pi} = \frac{|H|}{|N|_\pi}.$$

By (1)

$$|G/N|_\pi = |\frac{HN}{N}| = \frac{|HN|}{|N|}_\pi = \frac{|H|}{|H \cap N|}$$

therefore $|N|_\pi = |H \cap N|$. Hence, $H \cap N \in Hall_\pi(N)$. □

Lemma 4.7 *Let G be a group, and $H \trianglelefteq G$. Then $Z(H) \trianglelefteq G$.*

Proof Let $h \in H, g \in G$, and $h_1 \in Z(H)$. Then

$$\begin{aligned}(g^{-1}h_1g)h &= g^{-1}h_1gg^{-1}(ghg^{-1})g \\ &= g^{-1}(ghg^{-1})h_1g \\ &= hg^{-1}h_1g\end{aligned}$$

since $h_1 \in Z(H)$ and $ghg^{-1} \in H$. Therefore, $g^{-1}h_1g \in Z(H)$, so $Z(H) \trianglelefteq G$. □

Theorem 4.8 Schur and H. Zassenhaus *Let G be a group, $H \in Hall_\pi(G)$ such that $H \trianglelefteq G$. Then G splits over H.*

Proof Use induction on $|G|$. Let $P \in Syl_p(H)$. Then $G = N_G(P)H$ by Frattini Argument. First, suppose $|N_G(P)| < |G|$. Now, $H \cap N_G(P) \trianglelefteq N_G(P)$ and $H \cap N_G(P)$ is a π-group, so

$$\frac{|N_G(P)|}{|H \cap N_G(P)|} = \frac{|N_G(P)H|}{|H|} = \frac{|G|}{|H|} \subseteq \pi'$$

as $H \in Hall_\pi(G)$. Hence, $H \cap N_G(P) \in Hall_\pi(N_G(P))$, so by induction $N_G(P)$ splits over $H \cap N_G(P)$. Hence, there exists $K \leq N_G(P)$ such that $N_G(P) = K(H \cap N_G(P))$ and $K \cap (H \cap N_G(P)) = 1$.Therefore, $G = N_G(P)H = K(H \cap N_G(P))H = KH$. Also, $K \cap H \leq K \cap N_G(P) \cap H = 1$. Thus, G splits over H.

Secondly, if $|N_G(P)| = |G|$ then $G = N_G(P)$, so $P \trianglelefteq G$.

But then $Z(P) \trianglelefteq G$. By Lemma 4.7 $HZ(P)/Z(P) \in \mathrm{Hall}_\pi(G/Z(P))$. Also, $HZ(P)/Z(P) \trianglelefteq G/Z(P)$ since $H\,Z(P) \trianglelefteq G$. Now, $|G/Z(P)| = |G|/|Z(P)| < |G|$ since $Z(P) \neq 1$ as P is a p-group. Hence, by induction $G/\,Z(P)$ splits over $H\,Z(P)/\,Z(P)$, so there exists $K/Z(P) \leq G/Z(P)$ such that $G/Z(P) = (HZ(P)/Z(P))(K/Z(P))$

and $HZ(P)/Z(P) \bigcap K/Z(P) = Z(P)$. Hence, $G = HZ(P)K = HK$ and $H \cap K \leq Z(P)$. Since $H\ Z(P)/Z(P) \in Hall_\pi(G/Z(P))$ we know $K/Z(P)$ is a π'-group. Also, since $Z(P)$ is a π-group we also know that $Z(P) \in \mathrm{Hall}_\pi(K)$. Moreover, $Z(P) \trianglelefteq K$ and $Z(P)$ are abelian, so by Theorem 3.16 K splits over $Z(P)$. Therefore, there exists $L \leq K$ such that $K = Z(P)L$ and $Z(P) \cap L = 1$. Hence, $G = HK = H\ Z(P)L = HL$ and $H \cap L \leq H \cap L \cap K \leq L \cap\ Z(P) = 1$. Thus, G splits over H. □

Definition Let G be a group and $H \leq G$. Then there is **no fusion** of H in G if whenever $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$ then $h_1 \sim_H h_2$. ○

Example: There is **fusion** of A_3 in S_3. $(123) \sim_{S_3(132)}$ since $(123)^{(32)} = (132)$ and $(23) \notin A_3$.

Definition Let G be a group and $H \leq G$. Then the **focal subgroup** of H in G is $Foc_G(H) = \langle \{[h,g] | h \in H, g \in G, [h,g] \in H\} \rangle$. ○

Notice $H' \leq Foc_G(H) \leq G' \cap H$, so, by previous theorem $Foc_G(H) \trianglelefteq H$ and $H/Foc_G(H)$ is abelian.

Theorem 4.9 *Let G be a group and $H \leq G$ such that $gcd(\frac{|G|}{|H|}, \frac{|H|}{|H'|}) = 1$. Then G splits normally over $H/Foc_G(H)$ and $Foc_G(H) = H \cap G'$.*

Proof Notice that $H/Foc_G(H)$ is abelian and

$$\frac{|H|}{|H'|} = \frac{|H|}{|Foc_G(H)|} \frac{|Foc_G(H)|}{|H'|}$$

so

$$\frac{|H|}{|Foc_G(H)|} \mid \frac{|H|}{|H'|}.$$

Therefore, the $\gcd(\frac{|G|}{|H|}, \frac{|H|}{|Foc_G(H)|} = 1$; so we can use our previous theorem. Let $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$. Then there is $g \in G$ such that $h_1 = h_2^g$. Then $h_1h_2^{-1} = h_2^gh_2^{-1} = [g, h_2^{-1}] \in H$, so $[g, h_2^{-1}]^{-1} \in H$ or $[g, h_2^{-1}]^{-1} = [h_2, g] \in H$. Therefore, $(h_1h_2^{-1})^{-1} = [g, h_2^{-1}]^{-1} = [h_2, g] \in Foc_G(H)$. Thus, $h_1h_2^{-1} \in Foc_G(H)$ since $Foc_G(H) \leq G$. Hence, $Foc_G(H)h_1 = Foc_G(H)h_2$.and then by our theorem G splits normally over $H/Foc_G(H)$. Thus, there exists $K \trianglelefteq G$ such that $G = HK$ and $H \cap K = Foc_G(H)$. Now

$$G/K = HK/K \cong H/H \cap K = H/Foc_G(H),$$

so G/K is abelian and then $G\prime \leq K$. Hence, $Foc_G(H) \leq H \cap G' \leq H \cap K = Foc_G(H)$. It follows that $Foc_G(H) = H \cap G'$. □

Theorem 4.10 Focal Subgroup Theorem *Let G be a group and $P \in Syl_p(G)$. Then $Foc_G(P) = P \cap G'$.*

Proof Since $P \in Syl_p(G)$ we know $gcd(\frac{|G|}{|P|}\frac{|P|}{|P'|}) = 1$, so $FOC_G(P) = P \cap G'$ by the previous theorem and G splits normally over $P/P \cap G'$. □

Theorem 4.11 *Let G be a group, $J \trianglelefteq H \leq G$, H/J is a p-subgroup such that $gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$.Then the following are equivalent:*

(1) G splits normally over H/J.

(2) Whenever h_1, $h_2 \in H$ such that $h_1 \sim_G h_2$ then $Jh_1 \sim_{H/J} Jh_2$.

Proof For (1) implies (2) is the same as used before and shows that $Jh_1 = Jh_2$, so $Jh_1 \sim_{H/J} Jh_2$ by $J1$.

For (2)implies (1) use induction on $|H|/|J|$. Let $J_1/J = Z(H/J)$. Since H/J is a p-group $Z(H/J) \neq J$. Also, if H/J is abelian then we are done by Theorem 3.12.

Thus, H/J is not abelian, so $(H/J) \neq H/J$. Since $J_1/J \trianglelefteq H/J$ we get $J \leq J_1 \trianglelefteq H \leq G$. Then $|H|/|J_1| < |H|/|J|$, and $|H|/|J_1 \cdot |J_1|/|J| = |H|/|J|$, so $|H|/|J_1| \mid |H|/|J|$, so H/J_1 is a p-group. Also, $\gcd(\frac{|G|}{|H|}, \frac{|H|}{|J_1|}) = 1$. Let $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$. Then $Jh_1 \sim_{H/J} Jh_2$ by $J1$, so $J_1h_1 \sim_{H/J_1} J_1h_2$ by "the same element". Therefore, by induction (2)implies (1) here and then G splits normally over H/J_1. Thus, there exists $k_1 \trianglelefteq G = HK_1$ and $H \cap K_1 = J_1$.

Now $J \trianglelefteq J_1 \leq K_1$ and $J_1/J \leq H/J$, J_1/J is a p-group. Also. $\frac{|J_1|}{|J|} \mid \frac{|H|}{|J|}$ and $\frac{|K_1|}{|J_1|} = \frac{|G|}{|H|}$, so $gcd(\frac{|J_1|}{|J|}, \frac{|K_1|}{|J_1|}) = 1$. Let $x_1, x_2 \in J_1$ such that $x_1 \sim_{K_1} x_2$. Then since $x_1, x_2 \in H$ we have $Jx_1 \sim_{H/J} Jx_2$. But $Jx_1, Jx_2 \in J_1/J = Z(H/J)$, so $Jx_1 = Jx_2$. Therefore, $Jx_1 \sim_{J_1/J} Jx_2$ by the element $J1$.

Therefore, by induction, so K_1 splits normally over J_1/J. Hence, there exists $K \trianglelefteq K_1$ such that $K_1 = KJ_1$ and $K \cap J_1 = J$. Now $G = HK_1 = HKJ_1 = HK$. Also, $H \cap K = H \cap K_1 \cap K = J_1 \cap K = J$.

Let $h \in H$. Then $|K^hK/K| = |K^h|/|K^h \cap K|$. Now

$J^h = K^h \cap J_1^h = K^h \cap J_1 \leq K^h \cap K \leq K^h$ and J implies $J^h = J \leq K$, so $J^h \leq K \cap K^h$.

Hence

$$\frac{|K^h|}{|K^h \cap K|} \mid \frac{|K^h|}{|J^h|} = \frac{|K|}{|J|} = \frac{|G|}{|H|}$$

so

$$|\frac{K^hK}{K}| \mid \frac{|K_1|}{|K|} = \frac{|J_1|}{|J|} \mid \frac{|H|}{|J|}.$$

Therefore, $|K^h|/|K^h \cap K| = 1$, so $K^h = K^h \cap K$ which implies that $K^h \leq K$. But, since $|K^h| = |K|$, we get $K^h = K$, so $K \trianglelefteq G$. □

Lemma 4.12 *Let G be a group, $H \leq K \leq G$, $L \leq G$. Then $K \cap HL = H(K \cap L)$.*

Proof Let $hx \in H(K \cap L)$. Then $h \in H \leq K$ and $x \in K \cap L \leq K$. Thus, $hx \in K$. Also $h \in H$ and $x \in K \cap L \leq L$ implies that $hx \in HL$. Hence, $x \in K \cap HL$, so $K \cap HL \supseteq H(K \cap L)$. Let $hl \in K \cap HL$ where $h \in H$ and $l \in L$. Now $hl \in K$, so there exists $k \in K$ such that $hl = k$. Thus, $l = h^{-1}k \in HK = K$. Hence, $l \in K \cap L$ which implies that $hl \in H(L \cap K)$. Hence, $K \cap HLH(K \cap L)$. Therefore, $K \cap HL = H(K \cap L)$. □

Theorem 4.13 *Let G be a group, $J \trianglelefteq H \leq L \leq G$, H/J is a p-group, and $gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$. Further assume that whenever $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$ then $h_1 \sim_L h_2$. Then G splits normally over H/J if and only if L splits normally over H/J.*

Proof If G splits normally over H/J then there exists $K \trianglelefteq G$ such that $G = HK, H \cap K = J$. Then $L = L \cap G = L \cap HK = H(L \cap K)$ by Lemma 3.24. Since $K \trianglelefteq G$ then $K \cap L \trianglelefteq L$. Also, $H \cap L \cap K = J \cap L = J$, so L splits normally over H/J.

Suppose L splits normally over H/J. Let $h_1, h_2 \in H$ such that $h_1 \sim_G h_2$. Then $h_1 \sim_L h_2$ by assumption. Now $J \trianglelefteq H \leq L$, H/J is a p-group and $|G|/|H| = |G|/|L||L||H|$, so $|L|/|H| \mid |G|/|H|$. Therefore, $\gcd(\frac{|G|}{|H|}, \frac{|H|}{|J|}) = 1$ which implies that $Jh_1 \sim_{H/J} Jh_2$ by Theorem 3.2.23. Therefore, G splits normally over H/J by Theorem 3.2.23. □

Lemma 4.14 *Let G be a group, $P \in Syl_p(G)$, and $H \trianglelefteq G$ such that G/H is a p-subgroup. Then $G = PH$.*

Proof By a previous lemma $PH/H \in Syl_p(G/H)$ and since G/H is a p-group, $G/H = PH/H$. Let $g \in G$. Then $gH \in G/H = PH/H$, so there exists $p \in P, h \in H$

such that $gH = phH$. Thus, $(ph)^{-1}g \in H$ so there exists $h_1 \in H$ such that $(ph)^{-1}g = h_1$ which implies that $g = phh_1 \in PH$. Hence, $G = PH$. $\square$

Theorem 4.15 *Let G be a group, $P \in Syl_p(G)$ such that $N_G(Q)/C_G(Q)$ is a p-group for all $Q \leq P$. Then for all $P^* \in Syl_p(G)$ and for all $x \in P \cap P^*$, $P \sim_{C_G(x)} P^*$*

Proof Use induction on $|P|/|P \cap P^*|$. Let $Q = P \cap P^*$ and $x \in Q$. If $P = P^*$ then $P^1 = P^*$ and $1 \in C_G(x)$ and we are done.

Without loss of generality we may assume $P \neq P^*$. Then $Q < P$, so $Q < N_P(Q) \leq N_G(Q)$. By Sylow's Second Theorem there exists $Q_1 \in Syl_p(N_G(Q))$ such that $N_P(Q) \leq Q_1$. Again by Sylow's Second Theorem there exists $P_1 \in Syl_p(G)$ such that $Q_1 \leq P_1$. Hence, $Q < N_P(Q) \leq Q_1 \leq P_1$ then $Q < N_P(Q) \leq P \cap Q_1 \leq P \cap P_1 \leq P$, so $|P|/|P \cap P_1| < |P|/|Q|$. Also, $x \in Q \leq P \cap Q_1 \leq P \cap P_1$. By induction there exists $y_1 \in C_G(x)$ such that $P_1 = P^{y_1}$. Hence, $P \sim_{C_G(x)} P_1$.

Also, $Q < P^*$, so $Q < N_{P^*}(Q) \leq N_G(Q)$. By Sylow's Second Theorem there exists $w \in N_G(Q)$ such that $N_{P^*}(Q) \leq Q_1^w$. Since $N_G(Q)/C_G(Q)$ is a p-group by a previous lemma $N_G(Q) = Q_1 C_G(Q)$, so $w = qc$ where $q \in Q$ and $c \in C_G(Q)$. Now

$$Q < N_{P^*}(Q) \leq Q_1^w = Q_1^{qc} = Q_1^c \leq P_1^c = P^{y_1 c}.$$

Hence, $Q < N_{P^*}(Q) \leq P^* \cap P^{y_1 c} \leq P^{y_1 c}$. Also,

$$\frac{|P^{y_1 c}|}{|P \cap P^{y_1 c}|} < \frac{|P^{y_1 c}|}{|Q|} = \frac{|P|}{|Q|}$$

and $N_G(H)/C_G(H)$ is a p-group for all $H \leq P^{y_1 c}$. Now $x = x^{y_1 c} \in P^* \cap P^{y_1 c}$ so by induction there exists $y_2 \in C_G(x)$ such that $P^* = P^{y_1 c y_2}$ and $y_1 c y_2 \in C_G(x)$. Hence, $P \sim_{C_G(x)} P^*$. $\square$

Theorem 4.16 Frobenius *Let G be a group and $P \in Syl_p(G)$. Then G splits normally over P if and only if $N_G(Q)/C_G(Q)$ is a p-group for all $Q \leq P$.*

Proof Suppose G splits normally over P. Then there exists $K \trianglelefteq G$ such that $G = PK$ and $P \cap K = 1$. Let $Q \leq P$. Then $K \cap N_G(Q) \trianglelefteq N_G(Q)$ and $Q \trianglelefteq N_G(Q)$ so $[Q, K \cap N_G(Q)] \leq Q \cap K \cap N_G(Q) = Q \cap K \leq P \cap K = 1$. Therefore, $K \cap N_G(Q) \leq C_G(Q) \leq N_G(Q)$. Now by the second isomorphism theorem

$$\frac{N_G(Q)}{K \cap N_G(Q)} \cong \frac{N_G(Q)K}{K} \leq G/K = PK/K \cong P/P \cap K$$

so $N_G(Q)/K \cap N_G(Q)$ is a p-group!. Then

$$|N_G(Q)/C_G(Q)| = \frac{|N_G(Q)|}{|C_G(Q)}| = \frac{|N_G(Q)|/|K \cap N_G(Q)|}{|C_G(Q)|/|K \cap N_G(Q)|}$$

is a power of p. Therefore, $N_G(Q)/C_G(Q)$ is a p-group. Now suppose $N_G(Q)/C_G(Q)$ is a p-group for all $Q \leq P$. Then $P \trianglelefteq N_G(P)$ and $P \in Syl_p(N_G(P))$, so by a previous theorem we know that $N_G(P)$ splits over P. Therefore, there exists $K \leq N_G(P)$ such that $N_G(P) = PK$ and $P \cap K = 1$. We claim K is a p'-group(p prime group). Let $P_0 \in Syl_p(K)$. Then since $P \trianglelefteq N_G(P)$ and $K \leq N_G(P)$ we know $PP_0 \leq G$. Also, $P \leq PP_0$ and PP_0 is a p-group. Hence, $P = PP_0$ since $P \in Syl_p(G)$, so $P_0 \leq P \cap K = 1$ which implies $P_0 = 1$. Therefore, K is a p'-group. Now $KC_G(P)/C_G(P) \leq N_G(P)/C_G(P)$ and $N_G(P)/C_G(P)$ is a p-group, so $KC_G(P)/C_G(P)$ is a p-group. But

$$|KC_G(P)/C_G(P)| = \frac{|KC_G(P)|}{|C_G(P)|} = \frac{|K||C_G(P)|}{|K \cap C_G(P)|}\frac{1}{|C_G(P)|} = \frac{|K|}{|K \cap C_G(P)|}$$

and

$$\frac{|K|}{|K \cap C_G(P)|} \mid |K|.$$

Hence, since K is a p'-group, we get $|KC_G(P)/C_G(P)| = p^0 = 1$. Hence, $KC_G(P)/C_G(P) = 1C_G(P)$ or $C_G(P) = KC_G(P)$ or $K \leq C_G(P)$. Therefore, P and K commute. Hence, $K \trianglelefteq PK = N_G(P)$ so $1 \trianglelefteq P \leq N_G(P) \leq G$, $\gcd(\frac{|G|}{|P|}, \frac{|P|}{1}) = 1$, $P/1$ is a p-group and $N_G(P)$ splits normally over P. Let $x, y \in P$ such that $x \sim_G y$. Then there exists $g \in G$ such that $x = y^g$. Therefore, $x \in P \cap P^g$, so by Theorem 3.27 there exists $y_1 \in C_G(x)$ such that $P^{y_1} = P^g$. Therefore, $P = P^{gy_1^{-1}}$, so $gy_1^{-1} \in N_G(P)$. Also, $y^g = x$ implies $y^{gy_1^{-1}} = x^{y_1^{-1}} = x$ since $y_1^{-1} \in C_G(x)$. Therefore, $x \sim_{N_G(P)} y$, so G splits normally over P. $\square$

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