THE STURM-LIOUVILLE MATHEMATICAL SYSTEM

by

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ABSTRACT

THE STURM-LIOUVILLE MATHEMATICAL SYSTEM

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The Sturm-Liouville Mathematical System consists of a mathematical framework of special linear and homogeneous boundary value problems. That is, the system contains a linear and homogeneous ordinary differential equation together with homogeneous boundary conditions.

In addition to historical notes, the qualitative theory of differential equations is highlighted here, for such Sturm-Liouville Systems, featuring theorems on 'separation', 'comparison', and 'oscillation' of solutions. Such theories lead to a generalization of the standard 'eigenvalue problem'.

Illustrations are provided to delineate these qualitative features of said Sturm-Liouville type problems, showing that the System is of steadily increasing importance today in both pure and applied mathematics.

ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION

The Sturm-Liouville Mathematical System

1-1 Objective

This thesis is an expository report on the Sturm-Liouville Mathematical System. This system has been of steadily increasing significance in both pure mathematics and in mathematical physics. According to Bell [1], the Sturm-Liouville Theory of the 1830's was the first step towards an unified treatment of numerous boundary-value problems and their solutions that had been accumulating in applied mathematics since the early eighteenth century. Expressed in this presentation will be a historical perspective of the originators of the system, a listing of some theoretical properties and significant features, and stability criteria. Physical applications as well as present influences will also be presented.

1-2 Mathematical Biography of Charles Sturm and Joseph Liouville

Jacques Charles Francois Sturm (1803-1855) was a French mathematician of Swiss origin who spent most of his life in Paris. In 1823, as a tutor for the de Broglie family, he went to Paris, where he at last succeeded Poisson in the Chair of Mechanics at the Sorbonne. In 1836, he was elected to the French Academy of Sciences and later in 1838, was appointed to the staff of the École Polytechnique (see Manougian and Northcutt [11]).

Sturm's primary interests were realized in the fields of algebra, geometry, physics, and differential equations. Some of his published works, according to Bôcher [3], include:
An experimental memoir in collaboration with Colladon on the compressibility of liquids; several papers in geometrical optics; some papers, partly in collaboration with Liouville, on the real and imaginary roots of algebraic polynomial equations; and many minor geometrical papers.

Sturm was also recognized for the Three Great Memoires; explicitly,

1835. "Mémoire sur la résolution des equations numériques." Mémoires des savants éstangers;

1836. "Mémoire sur les équations différentielles linéaires du second ordre." Liouville's Journal; and

1836. "Mémoire sur une classe d'équations à différences partielles." Liouville's Journal.

Joseph Fourier's work had a major impact on Charles Sturm.

The primary subjects of Fourier's life work had been the theory of heat and the theory of the solutions of numerical differential equations. After Fourier's death in 1830, both subjects were carried forward by Sturm, the first in the two Memoires of 1836, the second in that of 1835. In one of his Memoires of 1836, he extended the results of a special difference equation to the following differential equation:

$$\frac{d}{dx}\left(K(x)\,\frac{dy}{dx}\right) - G(x)y = 0. \tag{1-1}$$

Sturm applied mathematics to other disciplines. With respect to physics, he considered problems in small vibrations and celestial mechanics. He made the first accurate determination of the velocity of sound in water and in 1827, Sturm won a prize for his essay on compressible fluids. He also was acknowledged for his work in differential equations by winning the Grand Prix des Science Mathematique [11].

Sturm pursued the study of real solutions of algebraic equations and also of ordinary and partial differential equations. It was here that Sturm's most important and suggestive work was done. His paper of July 27, 1829 was devoted to the analytic treatment of systems of linear homogeneous differential equations with constant coefficients. A result of this paper is a method of treating the algebraic characteristic equation of the system. This led to his outstanding work done which today is called the Sturm-Liouville Theory of Differential Equations, described in recent writings of Simmons [14].

One famous by-product of Sturm's research was his theorem on the separation of roots of an algebraic equation- the Sturm Separation Theorem. Another consists of a theorem that rules out the possibility of infinitely many oscillations (of a solution to a differential equation) on closed intervals- the Sturm Comparison Theorem.

Coupled with Sturm's name, in all this work in differential equations, one finds the name of his young friend, Joseph Liouville. Joseph Liouville (1809-1882) was a French mathematician who graduated from École Polytechnique in 1827. In 1833, he was appointed professor at the Sorbonne, and as a young man of thirty, he was elected to the French Academy of Sciences. In 1836, he founded and edited the Journal des Mathematique Pure et Appliquees, which was one of the foremost high grade mathematical periodicals that played a significant role in French mathematical life throughout the nineteenth century.

Among Liouville's notable achievements in analysis are the proofs of the existence of transcendental numbers and research in differential equations and boundary-value problems. He also produced outstanding work in the theory of numbers and differential geometry. Liouville was the first to solve a boundary-value problem by solving an equivalent integral equation. The scientific significance of integral equations, noted by Liouville in the 1830's, but first elaborated on in 1904 by Hilbert, is that in many important instances, one integral equation is equivalent analytically to a differential equation together with its boundary conditions. The solution of an integral equation inherently gives a solution to an associated boundary-value problem in such instances.

Recent interest in the study of fractional calculus by Oldham and Spanier [12], reveals the discovery that one of the first major efforts in this branch of Calculus was started by Liouville in 1832. His ingenious theory of fractional differentiation answered the long standing question of what reasonable meaning can be assigned to the symbol $\frac{d^u y}{dx^u}$, when u is not a positive integer. For example, he demonstrated that

$$\frac{d^{1/2}(x^n)}{dx^{1/2}} = \frac{\Gamma(n+1)x^{n-1/2}}{\Gamma(n+1/2)}, \quad \text{for } n > -1; \tag{1-2}$$

with $\Gamma(n)$ denoting the Gamma Function of argument n.

Fractional operators are useful in solving problems in mechanics and geometry. Liouville's brilliant solution came several decades too early, finding its proper place in analysis only now in the twentieth century.

As mentioned by Simmons, the most original of all his achievements was his theory of the integrals of elementary functions, for here he proved that such integrals as $\int e^{-x^2}dx$, $\int \frac{\sin x}{x}dx$, $\int \frac{e^x}{x}dx$, $\int \frac{dx}{\log x}$, as well as Jacobian elliptic integrals of the first, second, and third kinds, cannot be expressed in terms of a finite number of elementary functions.

The fascinating and difficult theory of transcendental numbers is an important branch in mathematics that originated in Liouville's work. In the eighteenth century, Lambert and Euler proved the irrationality of π and e. Liouville expanded on their discoveries and in 1844, he showed that e does not satisfy any polynomial equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0, \qquad (1\text{-}3)$$

with integer coefficients. This led him to conjecture that e is transcendental. He was unable to prove this, yet his ideas contributed to Hermite's success in 1873 and then to Lindemann's proof in 1882 that π is also transcendental. In 1844, Liouville invented a method for constructing any one of an extensive class of transcendental numbers and used it to produce examples that are provably transcendental. An example, given by Simmons [14] is:

$$\sum_{n=1}^{\infty} 10^{-n!} = [10^{-1} + 10^{-2} + 10^{-6} + \ldots \quad] = 0.11000100\ldots \qquad (1\text{-}4)$$

Another accomplishment of Liouville was his discovery of the fundamental result in complex analysis known as <u>Liouville's Theorem</u>- that a bounded entire function is necessarily constant- and used it as the basis for his own theory of elliptic functions. There is also a well-known Liouville theorem in Hamiltonian Mechanics, which states that volume integrals are time invariant in phase space.

Sturm's work was practically completed before Liouville's work began. Except for alternative proofs which

Liouville gave of some of Sturm's results and an extension to certain differential equations of higher order, Liouville's work dealt with a problem not treated by Sturm, namely the proof that the development of an arbitrary function which occurs in Sturm's paper is valid. The functions introduced by Sturm are called <u>oscillating</u>. In 1837, Liouville was led to a linear integral equation, which he solved by the method of successive substitutions. Liouville's problem was that of finding those solutions, if any, of a linear second order differential equation which assumes preassigned values of the independent variable. He was thus concerned with Sturm's oscillating functions, and it is customary to name the resulting theory after both men.

CHAPTER II

INTRODUCTION TO BOUNDARY VALUE PROBLEMS

2-1 Second Order Boundary Value Problems

A boundary-value problem (BVP) is a problem involving a differential equation and associated or supplementary condition(s).

A typical second order BVP has the form given by Equation (2-1):

$$\begin{cases} y'' = f(x, y, y'), & a \leqslant x \leqslant b \\ y(a) = A, & \\ y(b) = B; & \text{where } A, B \text{ are constants.} \end{cases} \tag{2-1}$$

Basic questions in the theory of differential equations concern the existence of a solution to a differential equation and the conditions that make the solution unique. The following theorem, taken from the text of Burden and Faires [6] in the subject of numerical analysis, gives general conditions that ensure that the solution to a second order BVP will exist and be unique.

Theorem 2.1

Suppose the function $y'' = f(x, y, y')$ of Equation (2-1) is continuous over set $D = \{(x, y, y') \mid a \leqslant x \leqslant b,\ -\infty < y < \infty,\ -\infty < y' < \infty\}$ and also that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are continuous on D. If

(1) $\frac{\partial f}{\partial y}(x, y, y') > 0$ for all (x, y, y') in D, and

(2) a constant M exists such that $\left|\frac{\partial}{\partial y'} f(x, y, y')\right| \leq M$ holds for all (x, y, y') in D, then the BVP has a unique solution.

A compact way of expressing differential equations is to use operator notation. With respect to the second order linear differential equations, operator L is defined as a rule that assigns to each twice-differentiable function , y, on some interval I, the function $L[y]$,

where $L[y]((x)) \equiv a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$; (2-2)

that is, $L[y] \equiv a_0y'' + a_1y' + a_2y$; where $a_0 \neq 0$ (see Brauer and Nohel, [5]). Therefore the second order linear non-homogeneous differential equation

$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = f(x)$ can be expressed as

$$L[y]((x)) = f(x). \tag{2-3}$$

Operator L is a member of a particular class of operators called <u>linear operators</u>. By definition, L is linear if

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2),$$

where c_1 and c_2 are arbitrary constants, y_1 and y_2 are any two functions in S, where S is the collection of twice-differentiable functions defined on the interval I. Linear operator theory can be extended to differential equations of higher order. For example,

$$L_n[y] = a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny; \tag{2-4}$$

where y is any function which is n times differentiable on some interval I, and a_0, a_1, ..., a_n are continuous functions on I, and $a_0 \neq 0$.

The second order linear non-homogeneous BVP

$$\begin{cases} a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = f(x); & a \leqslant x \leqslant b, \\ y(a) = A, \quad y(b) = B, \end{cases} \tag{2-5}$$

can be rewritten using previous operator notation as:

$$\begin{cases} L[y]\{(x)\} = f(x); & a \leqslant x \leqslant b \\ y(a) = A, \quad y(b) = B. \end{cases} \tag{2-6}$$

When $f(x, y, y')$ can be expressed in the form

$$f(x, y, y') = p(x)y' + q(x)y + r(x),$$

the differential equation $y'' = f(x, y, y')$ is called _linear_.

The existence and uniqueness of solutions is ensured by a simplification of Theorem 2.1 via the following corollary supplied by Burden and Faires [6].

Corollary 2.1

If the linear boundary-value problem

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x), & a \leqslant x \leqslant b, \\ y(a) = A, \quad y(b) = B, \end{cases} \tag{2-7}$$

satisfies the conditions that

(1) $p(x)$, $q(x)$, and $r(x)$ are continuous on $a \leqslant x \leqslant b$, and

(2) $q(x) > 0$ on $a \leqslant x \leqslant b$,

then the problem has a unique solution.

2-2 Green's Function and Adjoint Forms

Another approach to the solution of nonhomogeneous boundary value problems is by means of the construction of auxiliary functions called Green's functions. The knowledge of this function leads to solutions expressed as definite integrals rather than as infinite series.

In the non-homogeneous case, $L[y] = r(x)$. Given the BVP, from Corollary 2.1, we seek $y(x) = y_c(x) + y_p(x)$,

where $y_c(x)$ is the general solution of the associated homogeneous BVP; that is, the 'complementary function', and

$y_p(x)$ is a 'particular solution' of the nonhomogeneous BVP.

To find $y_p(x)$, it is useful to introduce <u>Green's function</u>, as detailed by Waltman [16]; that is, let

$$y_p(x) = \int_a^b G(x,z)f(z)dz, \tag{2-8}$$

where G is Green's function defined as follows:

$$G(x,z) = \begin{cases} \dfrac{y_1(x)y_2(z)}{W(y_1,y_2)(z)}, & a \leqslant x \leqslant z \\[2ex] \dfrac{y_1(z)y_2(x)}{W(y_1,y_2)(z)}, & z \leqslant x \leqslant b\,. \end{cases} \tag{2-9}$$

Herein, $y_1(x)$ and $y_2(x)$ are two independent solutions of $L[y] = 0$; and $W(y_1,y_2)(z) = [y_1(z)y'_2(z) - y_2(z)y'_1(z)]$ is the 'Wronskian' of y_1 and y_2. It is noted that $G(x,z)$ (for the BVP of Corollary 2.1) is continuous. At $x = z$ both parts are equivalent; however, there exists a discontinuity in $\dfrac{\partial G(x,z)}{\partial x}$, which has a jump of one "unit", at $x = z$.

Because this derivative is discontinuous, the second derivative (namely y'') does not exist.

The advantage of expressing solutions in integral form is that the Green's function is independent of the nonhomogeneous term in the differential equation. Therefore, once this function is determined, the solutions to all possible problems with different functions f are known, provided the integral $\int_a^b G(x,z)f(z)dz$ exists.

Any second order linear differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad (2\text{-}10)$$

can be written in the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0 \qquad (2\text{-}11)$$

after multiplication by a suitable factor. Differential Equation (2-10) is said to be in adjoint form (see Leighton [10]) if:

$$a_1(x) = a'_0(x). \qquad (2\text{-}12)$$

Suppose $a_0(x) > 0$ and $a_0(x)$, $a_1(x)$, and $a_2(x)$ are continuous on $a \leqslant x \leqslant b$. Then Equation (2-10) can be transformed into Equation (2-11) by the function μ, where

$$\mu = \frac{1}{a_0(x)}\exp\left\{\int\frac{a_1(x)dx}{a_0(x)}\right\},\ p(x) = a_0(x)\mu,\ \text{and}\ q(x) = \frac{a_2(x)}{a_0(x)}p(x). \qquad (2\text{-}13)$$

Adjoint problems are of particular interest physically because they occur frequently in applications. They are also of interest mathematically because their theory is especially well-developed and elegant, as detailed by Boyce and DiPrima [4]. The importance of the adjoint form(s) in the study of differential equations of second order can hardly be over-emphasized. They arise naturally in mechanics and have a central role in the calculus of variations. Many properties of the solutions of differential equations can be discovered by studying the equations themselves, without solving them in the traditional sense. In addition to transforming a second order linear differential equation into adjoint form,

$$y'' + P(x)y' + Q(x)y = 0 \tag{2-14}$$

can be written as

$$u'' + q(x)u = 0 \tag{2-15}$$

by a simple change of the dependent variable. Simmons [14] notes that Equation (2-14) is considered to be in <u>standard form</u>, while Equation (2-15) is referred to as <u>Liouville normal form</u>. To write Equation (2-14) in Liouville normal form, let

$$y(x) = u(x)v(x);$$

then

$$y'(x) = u(x)v'(x) + v(x)u'(x),$$

and

$$y''(x) = u(x)v''(x) + 2u'(x)v'(x) + u''(x)v(x).$$

By substituting this addition and product rule differentiation, Equation (2-14) reads:

$$\left[(uv'' + 2u'v' + u''v) + P(uv' + u'v) + Q(uv)\right] = 0, \qquad \text{or}$$

$$vu'' + (2v' + Pv)u' + (v'' + Pv' + Qv)u = 0. \tag{2-16}$$

Setting the coefficient of u' equal to zero, in Equation (2-16), and solving for v yields:

$$v = C\exp\left\{-\tfrac{1}{2}\int P(x)dx\right\}; \tag{2-17}$$

where C is an arbitrary non-zero constant.

Since v(x), given by Equation (2-17) is non-zero, the transformation of Equation (2-14) into Equation (2-15) has no effect whatever on the zeros of the solutions and therefore does not alter the <u>oscillation phenomena</u> in the physical interpretation of the system. (A solution $u = \Phi(x)$, of Equation (2-15) is said to be <u>oscillatory</u> if there exists a sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = \infty$, and such that $\Phi(x_n) = 0$, for $n = 1, 2, 3, \ldots$.)

With the coefficient of u' set equal to zero, in Equation (2-16), the equation has the form:

$$u'' + q(x)u = 0.$$

If $q(x) < 0$, then the solutions of Equation (2-15) do <u>not</u> oscillate at all. Thus we are led to Theorem 2.2.

<u>Theorem 2.2</u>

If $q(x) < 0$, and $u(x)$ is a non-trivial solution of $u'' + q(x)u = 0$, then $u(x) = 0$ has at most one zero.

In consideration of the oscillation of solutions, this leads us to confine our study of Equation (2-15) to the special case in which $q(x) > 0$. Even in this case, it is not necessarily true that all solutions will oscillate. The central feature of the behavior of solutions of Equation (2-15) is that they oscillate in such a manner that their zeros are distinct and occur alternately. In this direction, we are led to the <u>Sturm Separation Theorem</u>.

2-3 <u>The Sturm Separation Theorem</u>

<u>Theorem 2.3</u>

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then the zeros of these functions occur alternately– in the sense that $y_1(x)$ vanishes exactly once between any two zeros of $y_2(x)$, and conversely.

Proof:

Since y_1 and y_2 are linearly independent, their Wronskian does not vanish; that is,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = (y_1 y'_2 - y_2 y'_1) \neq 0. \qquad (2\text{-}18)$$

Since the Wronskian is continuous, it must have a constant sign. Further, it is noted that y_1 and y_2 cannot have a common zero, for if they do, then the Wronskian would vanish at that point, which is impossible.

Assume next that x_1 and x_2 are successive zeros of y_2. We want to show that y_1 vanishes between these two points. At x_1 and at x_2,

$$W(y_1, y_2) = y_1(x)y'_2(x) - y_2(x)y'_1(x)$$
$$\equiv y_1(x)y'_2(x) \neq 0.$$

This implies that $y_1(x) \neq 0$ and $y'_2(x) \neq 0$.

Furthermore, $y'_2(x_1)$ and $y'_2(x_2)$ must have opposite signs, because if y_2 is increasing at x_1, it must be decreasing at x_2, and vice versa.

Since $W(y_1, y_2)$ has a constant sign, $y_1(x_1)$ and $y_1(x_2)$ must also have opposite signs, and therefore, by continuity, $y_1(x)$ must vanish at some point(s) between x_1 and x_2. Function y_1 cannot vanish more than once between x_1 and x_2; for if it does, then the same argument shows that y_2 must vanish between these zeros of y_1, which contradicts the original assumption that x_1 and x_2 are successive zeros of y_2.

The Sturm Separation Theorem tells us that the zeros of any two (non-trivial) solutions of $y'' + P(x)y' + Q(x)y = 0$ either coincide or occur alternately, depending on whether these solutions are linearly *dependent or independent. Thus, all solutions of $y'' + P(x)y' + Q(x)y = 0$ oscillate with essentially the same rapidity, in the sense that on a given interval, the number of zeros of any solution cannot differ by more than one from the number of zeros of any other solution. Waltman [16] discusses the interlacing of these zeros in Corollary 2.2.*

Corollary 2.2

The zeros of linearly independent solutions of $u'' + q(x)u = 0$ interlace, as depicted in Figure 2-1.

A consequence of Corollary 2.2 is that under the hypothesis of the Sturm Separation Theorem, if one solution of $z'' + Q_2(x)z = 0$ is oscillatory, all solutions of $y'' + Q_1(x)y = 0$ are oscillatory, if $Q_1(x) \geq Q_2(x)$.

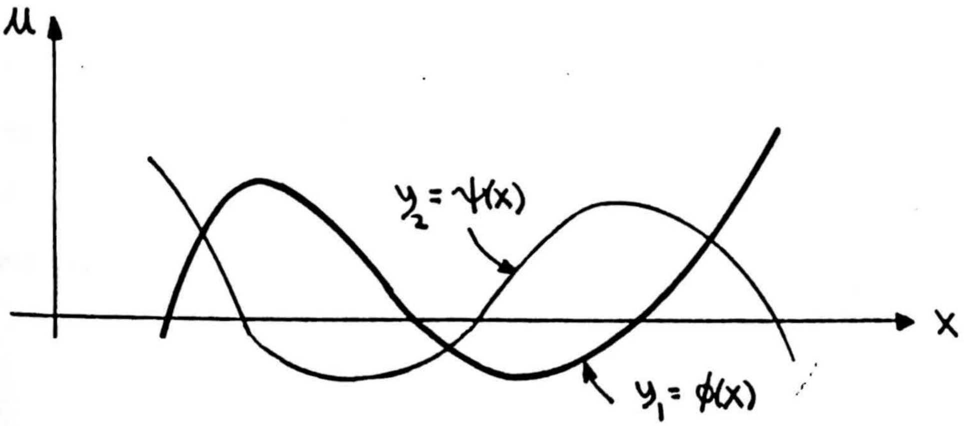

Fig. 2-1 The Interlacing (Intersecting) of Solutions

A key feature exhibited by Equation (2-15) is that, as $q(x)$ (the amplitude) gets larger, the solutions of this equation oscillate more rapidly. This feature is stressed in the Sturm Comparison Theorem.

2-4 The Sturm Comparison Theorem

Theorem 2.4

Let $y(x)$ and $z(x)$ represent non-trivial solutions of the following differential system:

$$\begin{cases} y'' + q(x)y = 0 \\ z'' + r(x)z = 0, \end{cases} \tag{2-19}$$

where $q(x)$ and $r(x)$ are both positive functions such that $q(x) > r(x)$, where $x_1 \leq x \leq x_2$. Then $y(x)$ vanishes at least once between any two zeros of $z(x) = 0$.

Proof:

Let x_1 and x_2 be successive zeros of $z(x)$; that is, $z(x_1) = z(x_2) = 0$ and assume $y(x)$ does not vanish on the open interval $x_1 < x < x_2$.
Proof of the theorem is achieved by contradiction.
Assume $y(x)$ and $z(x)$ are positive on $x_1 < x < x_2$.

With $$W(y, z) = yz' - zy',$$

then $$\frac{dW(y,z)}{dx} = yz'' + y'z' - z'y' - zy''$$

$$= yz'' - zy''. \tag{2-20}$$

From the system given by Equation (2-19), we re-write Equation (2-20) as:

$$\frac{dW(y,z)}{dx} = y(-rz) - z(-qy) = yz(q-r) > 0, \qquad (2\text{-}21)$$

on $x_1 < x < x_2$. Now by our assumption that y and z are both positive and $q > r$, then

$$\int_{x_1}^{x_2} \frac{dW(x)dx}{dx} = W(x_2) - W(x_1) > 0\ , \qquad (2\text{-}22)$$

or $\quad W(x_2) > W(x_1)$.

Now, at x_1 and x_2, $\quad W(y,z) = yz'$

$$\begin{aligned} \text{and} \quad & W(x_1) = y(x_1)z'(x_1) \geq 0, \\ \text{and} \quad & W(x_2) = y(x_2)z'(x_2) \leq 0. \end{aligned} \qquad (2\text{-}23)$$

Hence, Equations (2-22) and (2-23) lead to a contradiction.

Figure 2-2 illustrates The Sturm Comparison Theorem.

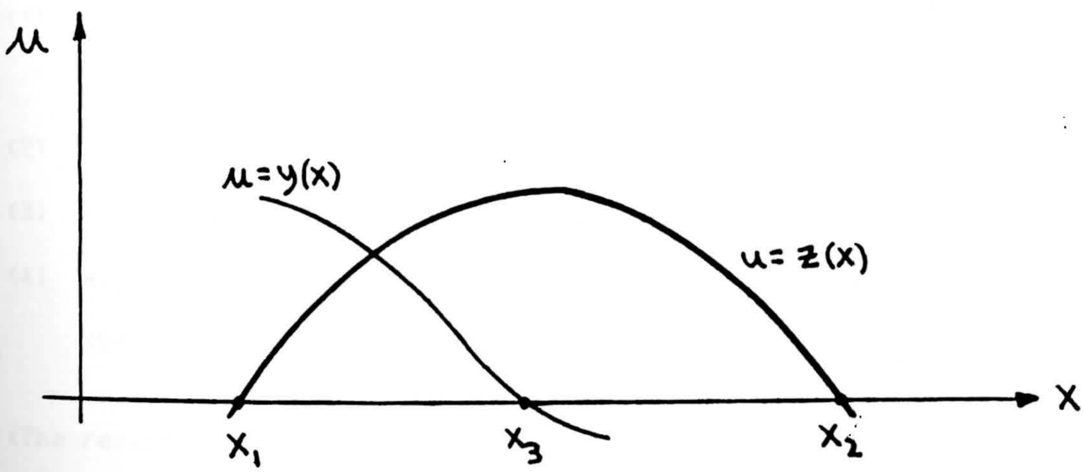

Fig. 2-2 Graphical Interpretation of the Sturm Comparison Theorem.

CHAPTER III

THE STURM-LIOUVILLE SYSTEM

3-1 The Regular Sturm-Liouville Mathematical System

The following boundary-value problem

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) - q(x)y + \lambda r(x)y = 0; \\ \\ a_1 y(a) + a_2 y'(a) = 0, \\ \\ b_1 y(b) + b_2 y'(b) = 0, \end{cases} \tag{3-1}$$

is called a regular Sturm-Liouville System (S-L system) over the interval $a \leqslant x \leqslant b$ when the following conditions are satisfied:

(1) $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are real-valued continuous functions for $a \leqslant x \leqslant b$,

(2) $p(x) > 0$ and $r(x) > 0$ for $a \leqslant x \leqslant b$,

(3) λ is a parameter independent of x, and

(4) a_1, a_2, b_1, and b_2 are real constants with $(a^2_1 + a^2_2) > 0$ and $(b^2_1 + b^2_2) > 0$.

(The restrictions imposed on p, q, and r are necessary in order to ensure the differential equation has solutions. Notice, if $p(x)$ vanishes at any points in $a \leqslant x \leqslant b$, or if $q(x)$ or $r(x)$ become unbounded in the interval, the solution of the differential equation may become unbounded.)

In the regular S-L system, we seek to find special values for the parameter λ that yield non-trivial solutions, to find those solutions, and to identify properties of such solutions.

The special values of λ for which there are non-trivial solutions of the S-L system are called eigenvalues and the corresponding solutions are called eigenfunctions. Note that Equation (3-1) describes a homogeneous, linear, second order ordinary differential equation. If $y(x)$ is an eigenfunction associated with this differential equation, then $Cy(x)$ is also an eigenfunction, where C is a constant. The set of all eigenvalues is called the spectrum of the system.

If we re-define the S-L system as follows:

$$y'' + \frac{p'y'}{p} - \frac{q\,y}{p} + \frac{\lambda r y}{p} = 0, \qquad (3\text{-}2)$$

and let

$$y_1 = y; \quad y_2 = y', \qquad (3\text{-}3)$$

then the following equivalent network results:

$$\left\{\begin{array}{l} y'_1 = y_2\,, \\ y'_2 = -\dfrac{p'y_2}{p} + \left(\dfrac{q}{p} - \dfrac{\lambda r}{p}\right)y_1\,, \\ \qquad a \leqslant x \leqslant b. \\ l_1:\quad a_1y_1(a) + a_2y_2(a) = 0\,, \\ l_2:\quad b_1y_1(b) + b_2y_2(b) = 0\,. \end{array}\right. \qquad (3\text{-}4)$$

We can now obtain a simple geometric view of the existence of eigenvalues for the S-L system as follows: in the $y_1 - y_2$ plane, the boundary conditions can be viewed as straight lines through the origin (see Figure 3-1). We seek a solution of the differential system given by Equation (3-4) that starts at t = a on line l_1, and terminates at t = b, which lies on line l_2. Of course, the solution may encircle the origin many times before terminating on l_2, at time t = b. If λ is not an eigenvalue, by definition, the only solution that starts on l_1 at t = a and terminates on l_2 at t = b is the trivial solution, represented by the origin.

The geometrical notion here relies on varying λ until all the elements of the set of solutions at t = a on l_1 terminate at t = b on l_2. This value of λ <u>will be</u> an eigenvalue. This view of the boundary value problem is a "shooting method", when one interprets the geometrical action in a numerical computation sense.

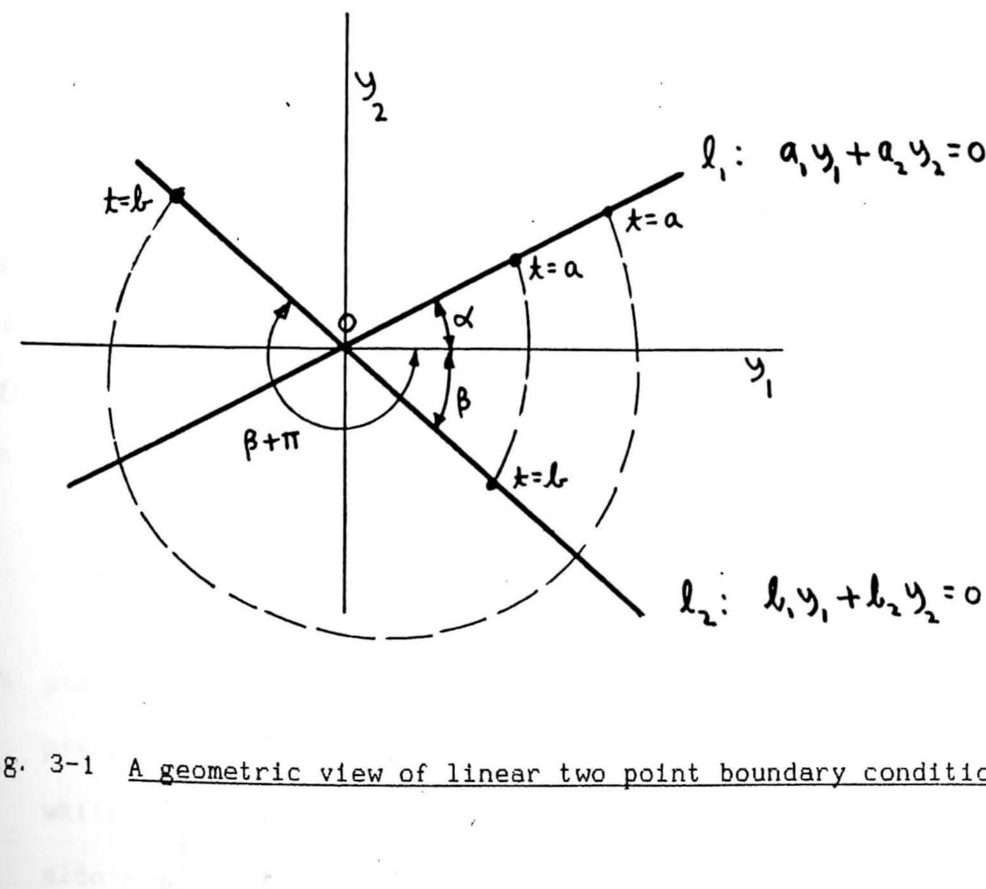

Fig. 3-1 <u>A geometric view of linear two point boundary conditions.</u>

3-2 Theoretic Properties of the S-L System

Clearly, Equation (3-1) is a self-adjoint differential equation. Recall the definition of the adjoint differential equation (Equation (2-10)) and use of operator $L[u]$ (as employed in Equation (2-2)), then the eigenvalue problem of Equation (3-1) is said to be self-adjoint if

$$\int_a^b (vL[u] - uL[v])dx = 0. \tag{3-5}$$

Since Equation (3-5) is satisfied for the regular S-L system given by Equation (3-1), this is a self-adjoint system. Two key properties associated with self-adjoint systems are that:

(1) the eigenvalues are real, and

(2) the eigenfunctions corresponding to different eigenvalues are orthogonal.

The proof of these properties follows in the manner described by Zauderen [18] in Theorems 3.1 and 3.2.

Theorem 3.1

All the eigenvalues of the regular Sturm-Liouville System are real.

Proof:

Given
$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \left(\lambda r(x) - q(x)\right)y = 0; \quad a \leqslant x \leqslant b,$$

and
$$\begin{aligned} a_1 y(a) + a_2 y'(a) &= 0, \\ b_1 y(b) + b_2 y'(b) &= 0. \end{aligned} \tag{3-6}$$

where $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are continuous on $a \leqslant x \leqslant b$;

$p(x)$, $q(x)$, $r(x)$, a_1, a_2, b_1, and b_2 are real-valued.

while λ and y may be complex;

also $p(x) > 0$ and $r(x) > 0$ for $a \leqslant x \leqslant b$.

Taking the complex conjugate of the original system yields Equation (3-7):

$$\frac{d}{dx}\left(p(x)\frac{d\bar{y}}{dx}\right) + \left(\bar{\lambda}r(x) - q(x)\right)\bar{y} = 0;$$

and

$$a_1\bar{y}(a) + a_2\bar{y}'(a) = 0, \tag{3-7}$$

$$b_1\bar{y}(b) + b_2\bar{y}'(b) = 0.$$

Combining the differential equations of Equations (3-6) and (3-7) gives:

$$\left[\bar{y}\left\{\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \left(\lambda r(x) - q(x)\right)y = 0\right\}\right.$$

$$\left. - y\left\{\frac{d}{dx}\left(p(x)\frac{d\bar{y}}{dx}\right) + \overline{\left(\lambda r(x) - q(x)\right)}y = 0\right\}\right]$$

$$\equiv \frac{d}{dx}\left\{ p(x)[\bar{y}y' - y\bar{y}'] \right\} + (\lambda - \bar{\lambda})r(x)\bar{y}y = 0. \tag{3-8}$$

Integrating Equation (3-8) from x = a to x = b, gives:

$$\int_a^b \frac{d}{dx}\{p(x)[\bar{y}y' - y\bar{y}']\}dx = (\bar{\lambda} - \lambda)\int_a^b r(x)\bar{y}ydx$$

$$p(x)\left[\bar{y}y' - y\bar{y}'\right]\Big|_a^b = (\bar{\lambda} - \lambda)\int_a^b r(x)|y|^2dx \tag{3-9}$$

From the boundary conditions stated in Equations (3-6) and (3-7), the left side of Equation (3-9) reduces to the following:

$$p(b)\left[\bar{y}(b)y'(b) - y(b)\bar{y}'(b)\right] - p(a)\left[\bar{y}(a)y'(a) - y(a)\bar{y}'(a)\right]$$

$$= p(b)\bar{y}(b)y(b)\left[-\frac{b_1}{b_2} + \frac{b_1}{b_2}\right] - p(a)\bar{y}(a)y(a)\left[\frac{a_1}{a_2} - \frac{a_1}{a_2}\right] = 0\ ; \tag{3-10}$$

where a_2 and $b_2 \neq 0$.

From the weight function $r(x) > 0$, defined over the open interval $a < x < b$, we have

$$\int_a^b r(x)|y|^2 dx > 0. \tag{3-11}$$

Therefore $(\bar{\lambda} - \lambda) = 0, \quad \Rightarrow \lambda = \bar{\lambda}$

ie., λ is real.

Theorem 3.2

If y_1 and y_2 are two eigenfunctions of the regular Sturm-Liouville System corresponding to eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then $\int_a^b r(x)y_1(x)y_2(x)dx = 0$; that is, y_1, y_2 are thusly orthogonal.

Proof:

Let y_1 and y_2 satisfy the operational statements

$$L[y_1] = \lambda_1 r(x) y_1 \quad ,$$
$$L[y_2] = \lambda_2 r(x) y_2 \quad . \tag{3-12}$$

By rearrangement, Equation (3-12) gives:

$$y_2 L[y_1] = \lambda_1 r(x) y_1 y_2 \quad ,$$

$$y_1 L[y_2] = \lambda_2 r(x) y_1 y_2$$, or by grouping

$$y_2 L[y_1] - y_1 L[y_2] = (\lambda_1 - \lambda_2) r(x) y_1 y_2, \text{ or} \tag{3-13}$$

$$\frac{d}{dx}\left\{ p(x)\left[y'_1 y_2 - y'_2 y_1\right]\right\} = (\lambda_1 - \lambda_2) r(x) y_1 y_2. \tag{3-14}$$

Integrating both sides of Equation (3-14), from $x = a$ to $x = b$, yields:

$$\int_a^b \frac{d}{dx}\{ p(x)[y'_1 y_2 - y'_2 y_1] \}dx = (\lambda_1 - \lambda_2)\int_a^b r(x) y_1 y_2 dx \ ,$$

or

$$p(x)\left[y'_1 y_2 - y'_2 y_1\right]\Big|_a^b = (\lambda_1 - \lambda_2)\int_a^b r(x) y_1 y_2 dx. \tag{3-15}$$

The left side of Equation (3-15) becomes

$$p(b)y_1(b)y_2(b)\left[-\frac{b_1}{b_2} + \frac{b_1}{b_2}\right] - p(a)y_1(a)y_2(a)\left[\frac{a_1}{a_2} - \frac{a_1}{a_2}\right] = 0\ ; \tag{3-16}$$

where $a_2,\ b_2 \neq 0$.

Therefore, we arrive at the statement:

$$0 = (\lambda_1 - \lambda_2)\int_a^b r(x)y_1y_2dx. \tag{3-17}$$

Since $\lambda_1 \neq \lambda_2$ by hypothesis, this implies that

$$\int_a^b r(x)y_1(x)y_2(x)dx = 0. \tag{3-18}$$

Equation (3-18) expresses the property of orthogonality of the eigenfunctions with respect to the weight function $r(x)$. Also, any orthogonal system $\{y_n\}$ with norm $||y_n|| = \sqrt{\int_a^b r(x)\left(y_n(x)\right)^2dx} > 0$ can be converted into an orthonormal system, given by

$$\{\Phi_n(x)\} \equiv \left\{\frac{y_n(x)}{||y_n(x)||}\right\}. \tag{3-19}$$

As stated by Danese [8], the orthogonal system is then said to be normalized.

Since the eigenfunctions corresponding to distinct eigenvalues of a S-L system are orthogonal with respect to the weight function, this suggests, the expansion of an arbitrary function $f(x)$, for which $\int_a^b r(x)\left(f(x)\right)^2dx$ is represented by a Fourier series in the terms of a finite number of eigenfunctions. That is, the eigenfunctions of the S-L system of Equation (3-1) are not only orthogonal, but also complete, as noted by Weinberger [17].

An orthonormal sequence of functions $\{\Phi_n\}$ in $a \leq x \leq b$ is complete with respect to a given class of functions if for every function f in this class,

$$\lim_{n\to\infty}\int_a^b \left[f(x) - \sum_{k=1}^{n} c_k\Phi_k(x)\right]^2 dx = 0, \tag{3-20}$$

at points of continuity of f, where

$$c_k = \int_a^b f(x)\Phi_k(x)\,dx. \tag{3-21}$$

The c_k's are called the <u>generalized Fourier coefficients</u> of f with respect to $\{\Phi_k\}$.

The class of functions which determines the completeness property of $\{\Phi_k(x)\}$ cannot be arbitrary. Now, let us consider some necessary restrictions on $\{\Phi_k\}$ in $a \leq x \leq b$. Since $\int_a^b (\Phi_k(x))^2 dx = 1$, for n = 1, 2, . . . , it follows that $\Phi_k^2(x)$ must be integrable in $a \leq x \leq b$. That is, Φ_k is <u>square-integrable</u> in $a \leq x \leq b$.

Obviously,

$$\int_a^b \left[f(x) - \sum_{k=1}^{n} c_k\Phi_k(x)\right]^2 dx \geq 0. \tag{3-22}$$

Upon expansion of (3-22), we have

$$\int_a^b f^2 dx - 2\sum_{k=1}^{n}\left(c_k\int_a^b f\Phi_k\,dx\right) + \sum_{k=1}^{n} c_k^2 \geq 0, \tag{3-23}$$

or

$$\int_a^b f^2 dx - 2\sum_{k=1}^{n} c_k^2 + \sum_{k=1}^{n} c_k^2 \geq 0, \tag{3-24}$$

which implies that

$$\sum_{k=1}^{n} c_k^2 \leq \int_a^b f^2 dx = ||f||^2. \tag{3-25}$$

Since the right hand side of Inequality (3-25) is independent of n, it follows that

$$\sum_{k=1}^{\infty} c_k^2 \leq ||f||^2. \tag{3-26}$$

This inequality, known as <u>Bessel's inequality</u>, is valid for every orthonormal system. It implies $\sum_{k=1}^{\infty} c_k^2$ is convergent and therefore

$\lim_{K \to \infty} c_k^2 = 0$ or that $\lim_{K \to \infty} c_k = 0$. For a complete orthonormal system of functions, Bessel's inequality becomes an equality for every function f. That is,

$$\sum_{K=1}^{\infty} c_k^2 = \int_a^b (f(x))^2 dx = ||f(x)||^2. \tag{3-27}$$

This relation is Parseval's equation and is also known as the "completeness relation" (see Courant and Hilbert [7]). In other words, the orthogonal sequence $\{\Phi_k\}$ in $a \leq x \leq b$ is complete with respect to the class of square-integrable functions if and only if for every function f in this class, Parseval's equation is true at points of continuity of f. It is important to note that if Equation (3-20) is satisfied, the functions $\sum_{K=1}^{\infty} c_k \Phi_x$ converges to the mean of the function.

In addition to Theorems 3.1 and 3.2, there are some other interesting basic properties of eigenvalues and eigenfunctions. For example, each eigenvalue is 'simple'. That is, since the S-L system is a second order system, there can be at most two linearly independent eigenfunctions for each distinct eigenvalue λ. However for the S-L system, there is only one linearly independent eigenfunction for each eigenvalue. Thus, each eigenvalue is termed 'simple'. It is also noted that there exists a set of eigenvalues having a limit point at infinity. The set of eigenvalues can be arranged as follows:

$$\lambda_1 < \lambda_2 < \lambda_3 \ldots \qquad \text{with} \quad \lambda_k \to \infty, \text{ as } k \to \infty.$$

The spectrum of the system is discrete and has a limit point at infinity. Another interesting property of the S-L system refers to the number of zeros of the eigenfunctions. In the case when $a_2 = b_2 = 0$, in Equation (3-1), Theorem 3.3 describes this feature.

Theorem 3.3

There exits an infinite sequence of characteristic numbers $\lambda_1, \lambda_2, \ldots$ of the simplified S-L system with the properties $\lambda_1 < \lambda_2 \ldots, \lambda_n \to \infty$, and a corresponding sequence of characteristic functions $y_1(x), y_2(x), \ldots$ defined on the interval $a \leq x \leq b$. The function $y_n(x)$ has precisely n zeros on the interval $a < x \leq b$.

The proof of this theorem is very detailed and can be found in texts as Birkhoff and Rota [2], Ince [9], and Leighton [10]

3-3 Green's Functions - Inversion of Differential Operators to Integral Operators

Now consider a regular S-L system in Liouville normal form, given as follows:

$$\begin{cases} y'' + [\lambda - q(x)]y = 0, \\ y(a) = y(b) = 0. \end{cases} \tag{3-28}$$

Rewriting Equation (3-28) yields:

$$y'' - q(x)y = -\lambda y, \tag{3-29}$$

where $(-\lambda y)$ can be viewed as a "forcing term" for the differential equation

$$y'' - q(x)y = 0. \tag{3-30}$$

Equation (3-29) is now viewed as a second order linear nonhomogeneous differential equation.

Let $G(x, z)$ be <u>Green's function</u> defined by

$$G(x, z) = \begin{cases} \dfrac{y_1(x)y_2(z)}{W(y_1, y_2)(z)}, & a \leqslant x \leqslant z \\ \\ \dfrac{y_1(z)y_2(x)}{W(y_1, y_2)(z)}, & z \leqslant x \leqslant b, \end{cases} \tag{3-31}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of Equation (3-30) with $y_1(a) = 0$ and $y_2(b) = 0$, and W is the Wronskian of $y_1(x)$ and $y_2(x)$. (In fact, in Equation (3-30), W will be constant, since the first derivative term does not appear).

A solution of the system, given by Equation (3-28), can be expressed by

$$y(x) = \lambda \int_a^{\ell} G(x, z)y(z)dz. \tag{3-32}$$

Equation (3-32) is an <u>integral equation</u> for an unknown function $y(x)$. This integral equation is the object under considerable mathematical study and perhaps, as indicated by Waltman [16], the most elegant way to present the Sturm-Liouville Theory.

CHAPTER IV

APPLICATIONS OF STURM-LIOUVILLE DIFFERENTIAL SYSTEMS

4-1 *Introduction*

Here three examples of physical problems are presented as illustrations of Sturm-Liouville systems. Explicitly, these examples describe (1) the 'harmonic oscillator'; (2) the elastic buckling problem with the differential equation and boundary condition(s) possessing the eigenvalue parameter; and (3) a detailed study of the one-dimensional heat conduction problem.

4-2 *The Harmonic Oscillator*

The 'harmonic oscillator' can be described by the following Sturm-Liouville mathematical system, for parameter $\lambda = -\hat{\lambda}$:

$$\begin{cases} y'' - \hat{\lambda} y = 0; \quad y = f\{x, \hat{\lambda}\}, \quad 0 \leq x \leq L \\ y(0) = 0 \\ y'(L) = 0. \end{cases} \tag{4-1}$$

If the parameter $\hat{\lambda}$ *is negative, then the general solution to the homogeneous, linear, second order, ordinary differential equation* describes "simple harmonic motion". Here, with $\hat{\lambda} < 0$, this 'harmonic oscillator' problem yields explicit solutions to system (4-1) called eigenfunctions:

$$y_n = A\sin\sqrt{-\hat{\lambda}_n}\,x \; ; \qquad (4-2)$$

with A, an arbitrary constant, and possessing eigenvalues,

$$\hat{\lambda}_n = -\left[\frac{\pi(2n + 1)}{2L}\right]^2 < 0; \qquad (4-3)$$

for $n = 0, 1, 2, 3, \ldots$.

<u>Note</u>: If $\hat{\lambda} \geq 0$, then $y \equiv 0$.

4-3 <u>Buckling of an Elastic Column</u>

In an investigation of the buckling of a uniform elastic column of length L by an axial load P, Timoshenko and Gere [15] lead to the development of the following fourth order ordinary differential equation:

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = q, \qquad (4-4)$$

where E is the modulus of elasticity,

I is the moment of inertia of the cross-section about an axis through the centroid perpendicular to the x-y plane,

P is the axial force in the beam-column,

q is the intensity of the uniformly distributed load, and

y is the deflection at location x.

In determining the critical buckling loads, the uniformly distributed load vanishes. (If P is less than the 'critical' load, the column is 'stable', implying that if the uniform loading is applied and then removed, the column returns to its initially straight position. If P exceeds the 'critical' load, the column becomes 'unstable', implying that a small uniformly applied load produces a deflection which does not disappear when the uniform loading is removed!). This implies that the differential equation for the column becomes,

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0; \tag{4-5}$$

Next, substituting $\lambda \equiv \frac{P}{EI}$, Equation (4-5) reduces to

$$y^{IV} + \lambda y'' = 0. \tag{4-6}$$

Assuming $P > 0$ and $\lambda > 0$, then the general solution of this equation is

$$y = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x + Cx + D. \tag{4-7}$$

The constants of this equation and the critical load(s) are found from the end conditions of the elastic column. The boundary conditions at $x = 0$ and $x = L$ depend on how the ends of the beam are supported.

In some buckling problems, the eigenvalue parameter appears in the boundary conditions as well as in the differential equation One particular case occurs when one end of the column is clamped and the other is free. In this case we are led to the following mathematical system

$$\left\{\begin{array}{ll} y^{IV} + \lambda y'' = 0; & \\ y(0) = 0 & (a) \\ y'(0) = 0 & (b) \\ y''(L) = 0 & (c) \\ [y'''(L) + \lambda y'(L)] = 0 & (d) \end{array}\right. \tag{4-8}$$

as shown in Figure 4-1.

Note: The end conditions (a) and (b) above represent clamped or fixed ends, and conditions (c) and (d) represent the free end at $x = L$.

Now, with $y'' = z$, the network in Equation (4-8) depicts a Sturm-Liouville system.

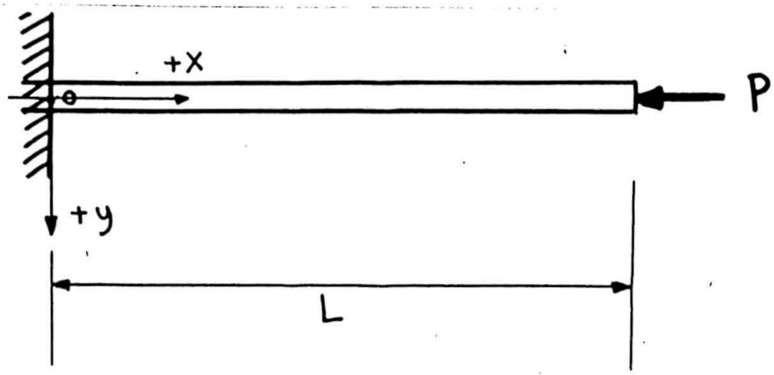

Fig. 4-1 Geometry and Loading Conditions for the Elastic Column

It is now desired to determine the eigenvalues and eigenfunctions of Equation (4-6) subject to the stated boundary conditions. In particular, the smallest eigenvalue gives the load at which the column buckles, or can assume a curved equilibrium position as shown in Figure 4-2. The eigenfunction corresponding to the buckling load then describes the configuration of this buckled column.

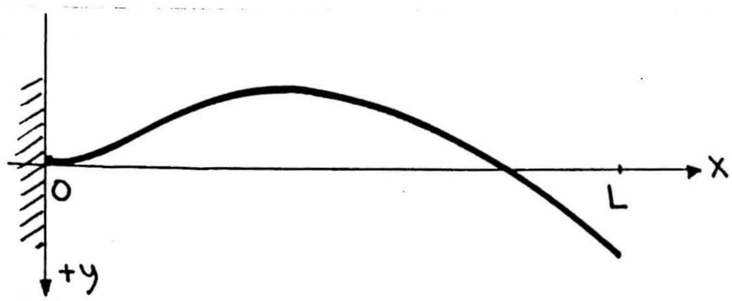

Fig. 4-2 Deflection Mode of the Elastic Column

As applied to Equation (4-7), Equations (a), (b), (c), and (d) give the following equations for determining the constants in the general solution:

$$\left\{\begin{aligned} B + D &= 0 \\ \sqrt{\lambda}A + C &= 0 \\ A\sin(\sqrt{\lambda}L) + B\cos(\sqrt{\lambda}L) &= 0 \\ \lambda C &= 0 \end{aligned}\right. \tag{4-9}$$

For a non-trivial solution of Equation (4-9)- a system of homogeneous, linear, algebraic equations- the determinant of the coefficients must equal zero; that is, the 'determinantal' equation

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ \sqrt{\lambda} & 0 & 1 & 0 \\ \sin\sqrt{\lambda}L & \cos\sqrt{\lambda}L & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{vmatrix} = 0 \tag{4-10}$$

must be satisfied. Expansion gives

$$\lambda\left(\sqrt{\lambda}\cos(\sqrt{\lambda}L)\right) = 0. \tag{4-11}$$

Since $\lambda > 0$, it follows that

$$\cos(\sqrt{\lambda}L) = 0. \tag{4-12}$$

Therefore

$$\sqrt{\lambda}L = \frac{(2n - 1)\pi}{2}, \quad \text{for } n = 1, 2, \ldots \tag{4-13}$$

so that

$$\sqrt{\lambda} = \frac{(2n - 1)\pi}{2L}, \quad \text{for } n = 1, 2, \ldots \tag{4-14}$$

The smallest eigenvalue corresponds to $n = 1$, therefore

$$\lambda_{min} = \frac{\pi^2}{4L^2} \tag{4-15}$$

and the equation for the corresponding deflection curve (eigenfunction) is

$$y = 1 - \cos\left(\frac{\pi x}{2L}\right) . \tag{4-16}$$

Also, $$P_{crit} = \lambda_{min} EI = \frac{\pi^2 EI}{4L^2} = \text{Euler Buckling Load}$$

4-4 <u>One-Dimensional Heat Conduction Problem</u>

An example of the One-Dimensional Heat Conduction Problem can be represented mathematically by the following system

$$\begin{cases} \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 & \text{(a)} \\ u(0, t) = u_x(0, t), \quad t > 0 & \text{(b)} \\ u(L. t) = -u_x(L, t), \quad t > 0 & \text{(c)} \\ u(x, 0) = 1, \quad 0 \leq x \leq L. & \text{(d)} \end{cases} \tag{4-17}$$

In this system, u represents the temperature (in a one dimensional bar as a function of position x and of time t); α is the 'conductivity' of the bar, Equations (b) and (c) represent the boundary data, and Equation (d) is the initial data. A visual representation of the system of Equation (4-17), is shown in Figure 4-3.

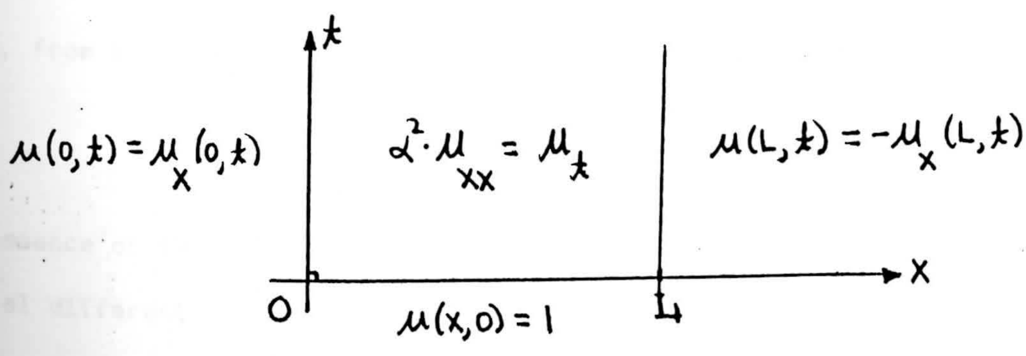

Fig. 4-3 <u>The Space-Time Frame Identifying the Boundary and Initial Conditions for the One-Dimensional Heat Conduction Problem</u>

The underlying problem of heat conduction is to find $u(x,t)$ which satisfies Equation (4-17). That is, this problem can be viewed as solving an *initial value problem* in the time variable t and a boundary value problem with respect to the spatial variable x.

To find a solution of the given system, we will employ the analytical technique- separation of variables. We seek a solution of Equation (a) of the form

$$u(x,t) = X(x)T(t). \tag{4-18}$$

Making the substitution of Equation (4-18) into Equation (a) yields:

$$\alpha^2 X''T = XT', \tag{4-19}$$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}. \tag{4-20}$$

In order for Equation (4-20) to be valid for $0 < x < L,\ t > 0$ it is essential that both sides of this equation be equal to the same constant. That is, Equation (4-20) becomes

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda, \tag{4-21}$$

where $-\lambda$ is referred to as the "separation constant".
(the choice of $-\lambda$ is for convenience).
Hence, from Equation (4-21) are two ordinary differential equations

$$X'' + \lambda X = 0 \tag{4-22}$$

and

$$T' + \alpha^2\lambda T = 0. \tag{4-23}$$

The essence of this method of separation of variables is that the partial differential equation has been replaced by two ordinary differential equations. The boundary conditions for Equation (4-22) are given as

$$\left(X(0) - X'(0)\right) = 0, \tag{4-24}$$

and

$$\left(X(L) + X'(L)\right) = 0. \tag{4-25}$$

Thus for the system comprised of Equations (4-22), (4-24), and (4-25) we have a <u>Sturm-Liouville System</u>.

To find the eigenvalues of Equation (4-22), we make use of the differential operator D as follows:

$$(D^2 + \lambda)\{x\} = 0 \tag{4-26}$$

which is satisfied for

$$D = \pm\sqrt{-\lambda} \tag{4-27}$$

For λ given, the general solution of Equation (4-22) is shown in the following three cases.

<u>Case I</u> For $\lambda = 0$; $X(x) = A + Bx$ (4-28)

<u>Case II</u> For $\lambda < 0$; $X(x) = C\exp\{\sqrt{-\lambda}x\} + D\exp\{-\sqrt{-\lambda}x\}$ (4-29)

<u>Case III</u> For $\lambda > 0$; $X(x) = E\sin\sqrt{\lambda}x + F\cos\sqrt{\lambda}x$ (4-30)

<u>Case I</u> If $\lambda = 0$; the boundary conditions, Equations (4-24) and (4-25) suggest that $A = B = 0$ and that $\lambda = 0$ is satisfied only for the trivial solution.

<u>Case II</u> If $\lambda < 0$; Equation (4-24) results in

$$C = D\frac{\left(1 + \sqrt{-\lambda}\right)}{\sqrt{-\lambda} - 1} \tag{4-31}$$

and Equation (4-25) gives

$$C\exp\{\sqrt{-\lambda}L\}\left(1 + \sqrt{-\lambda}\right) + D\exp\{-\sqrt{-\lambda}L\}\left(1 - \sqrt{-\lambda}\right) = 0 \tag{4-32}$$

Substituting Equation (4-31) into Equation (4-32) yields

$$D\left[\exp(\sqrt{-\lambda}L)\frac{(1+\sqrt{-\lambda})^2}{(\sqrt{-\lambda}-1)} + \exp(-\sqrt{-\lambda}L)(1-\sqrt{-\lambda})\right] = 0. \qquad (4\text{-}33)$$

The only nontrivial solution occurs for $\lambda = 0$, which contradicts our assumption that $\lambda < 0$. See Figure 4-4.

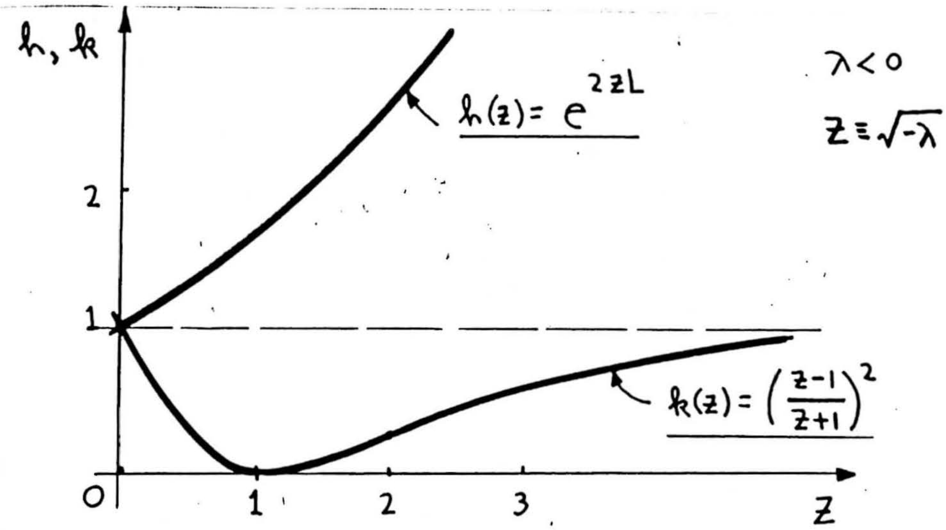

Fig. 4-4 Graphical Interpretation for the Case: $\lambda<0$

Case III If $\lambda > 0$; Equation (4-24) yields

$$F = \sqrt{\lambda}E \qquad (4\text{-}34)$$

and Equation (4-25) gives

$$\cos(\sqrt{\lambda}L)[F + E\sqrt{\lambda}] + \sin(\sqrt{\lambda}L)[E - F\sqrt{\lambda}] = 0. \qquad (4\text{-}35)$$

Substituting Equation (4-34) into Equation (4-35) results in

$$\tan(\sqrt{\lambda}L) = \frac{2\sqrt{\lambda}}{\lambda - 1} \qquad (4\text{-}36)$$

Non-trivial solutions exist where Equation (4-36) is satisfied.

To solve for λ graphically we make the substitution

$$\theta \equiv \sqrt{\lambda}L\ . \qquad (4\text{-}37)$$

Therefore, Equation (4-36) is now of the form

$$\tan\theta = \frac{2L\theta}{\theta^2 - L^2} \quad . \qquad (4\text{-}38)$$

Let $f(\theta) = \tan\theta$ and $g(\theta) = \dfrac{2L\theta}{\theta^2 - L^2}$.

The resulting graph of these two functions is shown in Figure 4-5. The approximate values of θ for which $f(\theta) = g(\theta)$ are identified in this figure, for the case $L = 1$:

$$\theta_1 \approx 1.3065424$$

$$\theta_2 \approx 3.6731944$$

$$\theta_3 \approx 6.5846200$$

$$\theta_4 \approx 9.6316846$$

Noting that

$$0 < \theta_1 < \frac{1}{2}\pi ,$$

$$\pi < \theta_2 < \frac{3}{2}\pi ,$$

$$2\pi < \theta_3 < \frac{5}{2}\pi ,$$

$$\vdots$$

$$(n-1)\pi < \theta_n < \frac{(2n - 1)}{2}\pi, \quad \text{for } n = 1, 2, \ldots .$$

Referring to said graphics, note that

$$\theta_n \approx (n - 1)\pi , \qquad (4\text{-}39)$$

for n sufficiently large.

To find the eigenvalues we recall the substitution for θ so that

$$\theta_n = \sqrt{\lambda_n}L \qquad (4\text{-}40)$$

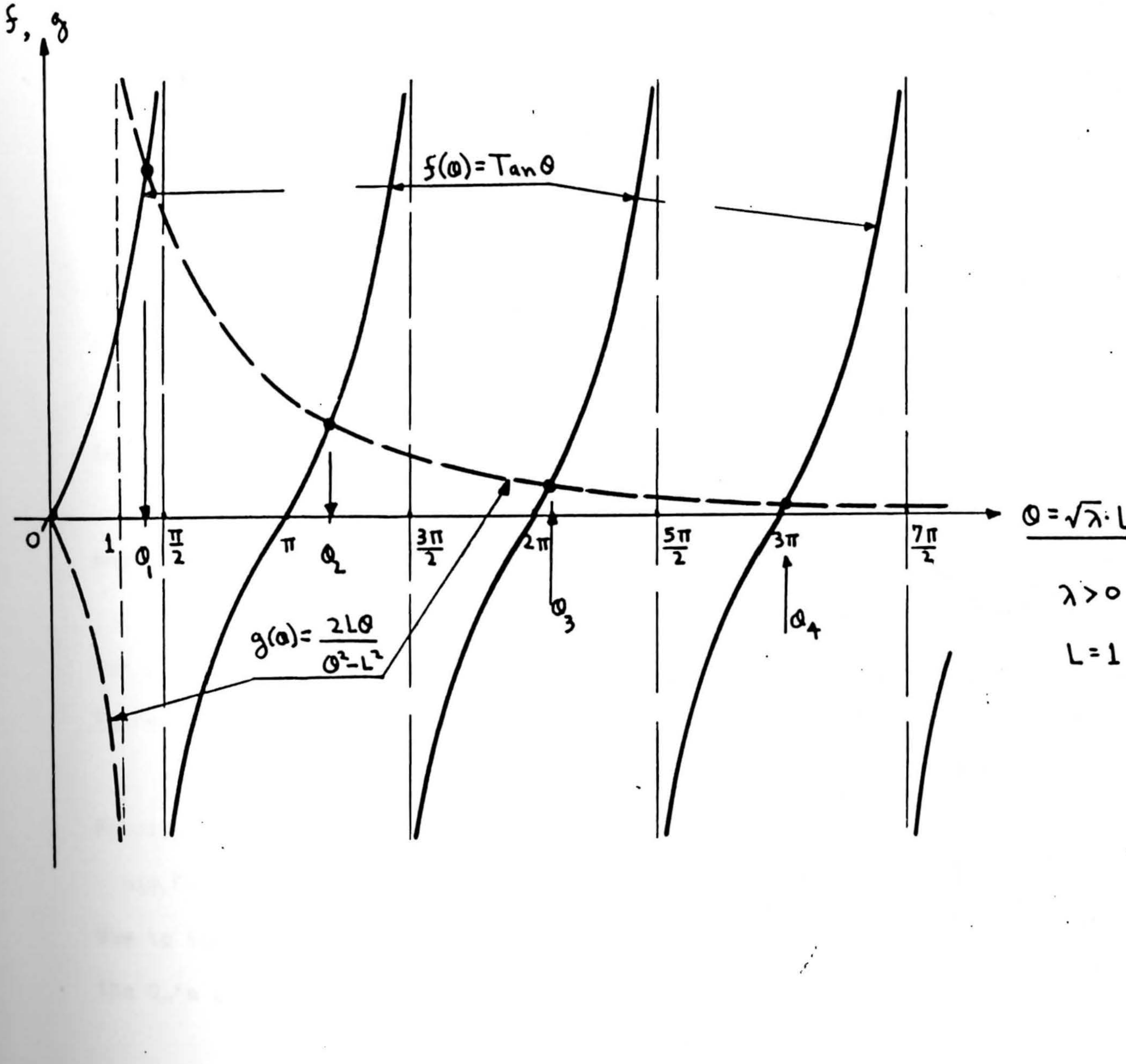

Fig 4-5 Graphical Interpretation for the Case: $\lambda > 0$

Upon solving Equation (4-40) for λ_n, we obtain

$$\lambda_n = \frac{\theta_n^2}{L^2}. \qquad (4\text{-}41)$$

An approximation for λ_n can be obtained by the substitution of Equation (4-39) into Equation (4-41) so that

$$\lambda_n \approx \frac{(n-1)^2\pi^2}{L^2}, \quad \text{for n suitably large.} \qquad (4\text{-}42)$$

Hence, the eigenfunction corresponding to λ_n is

$$X(x) = \Phi_n(x, \lambda_n) = \left(\sin(\sqrt{\lambda_n}x) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right), \text{ for } n \geq 1. \qquad (4\text{-}43)$$

Now upon considering Equation (4-43) where $\lambda > 0$, we find

$$T(t) = G\exp\{-\alpha^2\lambda t\}, \quad t > 0; \qquad (4\text{-}44)$$

where G is an arbitrary constant.

Consequently, the solutions of the heat conduction equation of the form of Equation (4-18) can be represented by

$$u(x, t) = G\exp\{-\alpha^2\lambda t\}\left(\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\cos(\sqrt{\lambda}x)\right) \qquad (4\text{-}45)$$

and the fundamental solutions of Equation (a) are given by

$$u_n(x, t) = G_n\exp\{-\alpha^2\lambda_n t\}\left[\sin(\sqrt{\lambda_n}x) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right], \qquad (4\text{-}46)$$

for $n = 1, 2, \ldots$.

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} G_n\exp\{-\alpha^2\lambda_n t\}\left[\sin(\sqrt{\lambda_n}x) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right]. \qquad (4\text{-}47)$$

Finally, it remains to impose the initial data

$$u(x, 0) = 1 = \sum_{n=1}^{\infty} G_n\left[\sin(\sqrt{\lambda_n}x) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right], \quad 0 < x < L. \qquad (4\text{-}48)$$

Due to the orthogonality feature of Sturm-Liouville problems, the G_n's can be determined as follows:

$$G_n = \frac{\int_0^L X(x)\,dx}{\int_0^L X^2(x)\,dx} \qquad (4\text{-}49)$$

Upon completion of the integration for the G_n coefficients, Equation (4-47) then reads:

$$u(x,t) = \sum_{n=1}^{\infty} G_n \exp\{-\alpha^2\lambda_n t\}[\sin(\sqrt{\lambda_n}x) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)]; \text{ where} \tag{4-50}$$

$$G_n = \frac{4[1 - \cos(\sqrt{\lambda_n}L) + \sqrt{\lambda_n}\sin(\sqrt{\lambda_n}L)]}{[2\sqrt{\lambda_n}(1+\lambda_n) + \sin(2\sqrt{\lambda_n}L)(\lambda_n-1) + 4\sqrt{\lambda_n}(\sin(\sqrt{\lambda_n}L))^2]} \tag{4-51}$$

4-4A Numerical Solutions of The One-Dimensional Heat Conduction Equation

(i) Computation By The Method of Finite Differences

Numerical methods complement analytical techniques in that they handle problems that appear non-tractable or too difficult to solve analytically. One fundamental and important technique for numerical solution of partial differential equations is the method of finite differences.

Here, we consider the one-dimensional heat conduction problem given by Equation (4-17) and employ finite differences, seeking the value $u(x,t)$ at discrete grid points (x_i, t_j), by defining a grid on the rectangle $u = \{(x,t): 0 \leqslant x \leqslant L, \; 0 \leqslant t \leqslant T\}$.

Let M and N be positive integers so that

$$h \equiv \Delta x = \frac{L}{M}, \quad \text{where } x_i = ih \quad \text{for } i = 0, 1, \ldots, M$$

and

$$k \equiv \Delta t = \frac{T}{N}, \quad \text{where } t_j = jk \quad \text{for } j = 0, 1, \ldots, N.$$

The resulting grid is shown in Figure 4-6.

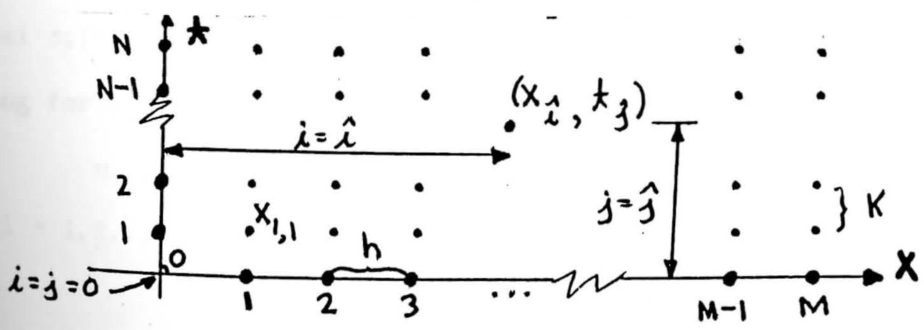

Fig. 4-6 Grid Arrangement for the Finite Difference Method of Computation

Next, we need to find a finite-difference approximation of the partial differential equation [Equation (a) of (4-17)] that will relate the values of u at various grid points. Namely,

$$\frac{\partial u(x_i, t_j)}{\partial t} \approx \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} \tag{4-52}$$

and $$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} \approx \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)}{h^2} \tag{4-53}$$

Replacing approximations (4-52) and (4-53) with

$$x_i + h = x_{i+1},$$

$$x_i - h = x_{i-1},$$

$$t_j + k = t_{j+1},$$

results in the following difference equation

$$\alpha^2\left[\frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}\right] = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} \tag{4-54}$$

for $i = 1, 2, \ldots, M-1$, and $j = 0, 1, 2, \ldots, N-1$.

or re-writing in subscript notation:

$$\alpha^2\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right) = \frac{u_{i,j+1} - u_{i,j}}{k} \tag{4-55}$$

for $i = 1, 2, \ldots, M-1$, and $j = 0, 1, 2, \ldots, N-1$.

Equation (4-55) then serves as the finite-difference approximation of partial differential equation in Equation (4-17).

Solving for $u_{i,j+1}$ yields

$$u_{i,j+1} = \left[ru_{i+1,j} + (1-2r)u_{i,j} + ru_{i-1,j}\right] \tag{4-56}$$

for $i = 1, 2, \ldots, M-1$, and $j = 0, 1, 2, \ldots, N-1$,

where

$$r = \frac{\alpha^2 k}{h^2}.$$

Now consider the given boundary conditions, (b) and (c) of Equation

(4-17). The central difference approximation to u_x at the grid point (x_i, t_j) is given as

$$\frac{\partial u(x_i, t_j)}{\partial x} \approx \frac{u(x_i + h, t_j) - u(x_i - h, t_j)}{2h} \qquad (4\text{-}57)$$

for $i = 1, 2, \ldots, M-1$, and $j = 0, 1, 2, \ldots, N$.

Using this approximation, we have

$$u_{0,j} = \frac{u_{1,j} - u_{-1,j}}{2h}, \qquad (4\text{-}58)$$

and

$$u_{M,j} = -\left[\frac{u_{M+1,j} - u_{M-1,j}}{2h}\right], \qquad (4\text{-}59)$$

for $j = 0, 1, 2, \ldots, N$.

We can eliminate $u_{-1,j}$ and $u_{M+1,j}$ from Equations (4-58) and (4-59) by setting $i = 0$ and $i = M$ in Equation (4-56). Thus

$$u_{0,j+1} = ru_{1,j} + (1-2r)u_{0,j} + ru_{-1,j} \qquad (4\text{-}60)$$

and

$$u_{M,j+1} = ru_{M+1,j} + (1-2r)u_{M,j} + ru_{M-1,j} \qquad (4\text{-}61)$$

Now by eliminating $u_{-1,j}$; from Equations (4-58) and (4-60), and eliminating $u_{M+1,j}$ from Equations (4-59) and (4-61) we obtain

$$u_{0,j+1} = 2ru_{1,j} + (1 - 2r - 2rh)u_{0,j} \qquad (4\text{-}62)$$

and

$$u_{M,j+1} = 2ru_{M-1,j} + (1 - 2r - 2rh)u_{M,j} \qquad (4\text{-}63)$$

where $j = 0, 1, 2, \ldots, N-1$.

Finally, the initial data [(d) in Equation (4-17)] is of the form

$$u_{i,0} = 1, \qquad \text{for } i = 0, 1, 2, \ldots, M. \qquad (4\text{-}64)$$

Therefore, the appropriate difference approximation to Equation (4-17) is given by Equations (4-56), (4-62), (4-63), and (4-64).

If we wish to approximate $u(0.1, 0.02)$

where

$$\alpha^2 = 0.10 \text{ cm/sec}^2$$

$$L = 1 \text{ cm.}$$

$$x = 0.1 \text{ cm.}$$

$$t = 0.02 \text{ sec.}$$

and choose $h = 0.05$ cm. and $k = 0.0025$ sec. so that $0 < r \leq \frac{1}{2}$ in order for this method to converge. (The development of stability conditions in the theory of finite differences can be found in the text of Burden and Faires [6]). The appropriate difference approximations for this problem are as follows:

$$u_{i,j} = 0.1u_{i+1,j} + 0.8u_{i,j} + 0.1u_{i-1,j} \tag{4-65}$$

for $i = 1, 2, \ldots, M-1$, and $j = 0, 1, 2, \ldots, N$.

$$u_{0,j+1} = 0.2u_{1,j} + 0.79u_{0,j} \tag{4-66}$$

for $j = 0, 1, 2, \ldots, N-1$.

$$u_{M,j+1} = 0.2u_{M-1,j} + 0.79u_{M,j} \tag{4-67}$$

for $j = 0, 1, 2, \ldots, N-1$.

$$u_{i,0} = 1.0 \tag{4-68}$$

for $i = 0, 1, 2, \ldots, M$.

The grid representation of $u(0.1, 0.02)$ is $u_{2,8}$. After successive iterations employing Equations (4-65), (4-66), (4-67), and (4-68),

$$u_{2,8} = 0.1(0.9988786) + 0.8(0.9899982) + 0.1(0.9889772)$$

$$\approx 0.9908.$$

(ii) <u>Computation of the (Fourier-Eigenfunction) Series Analytic Solution.</u>

In order to estimate the temperature of the bar at a specific position, $u(0.10, 0.02)$, we must evaluate Equation (4-50) with fixed α, L, t, and x and λ_n approximated.

In particular, $\alpha^2 = 0.10$ cm/sec^2

$L = 1$ cm.

$t = 0.02$ sec.

$x = 0.10$ cm.

Employing the Newton-Raphson iterative scheme, the eigenvalues λ_n are approximated to 6 decimal places:

$$\lambda_1 \approx 1.707053$$

$$\lambda_2 \approx 13.492357$$

$$\lambda_3 \approx 43.357221$$

$$\lambda_4 \approx 92.769349$$

$$\cdots \quad .$$

(The choice of four eigenvalues is due to the rapid convergence of this series summation).

Therefore Equation (4-50) gives

$$u(0.1, 0.02) = 0.9241226 - 1.1050899 \times 10^{-8} + 0.0699305$$
$$+ 4.7168876 \times 10^{-9}$$
$$\approx 0.9941.$$

4-5 Listing of Classical Problems in Mathematical Physics of Sturm-Liouville Type

In addition to the applications presented, many engineering and scientific problems are well described by Sturm-Liouville Mathematical Systems. An excellent compilation of several classical problems of S-L type is given by Segel [13]. The following table identifies these classical problems - defined by the S-L differential equation, Equation (3-1).

DE-Classical Name	p(x)	q(x)	r(x)	Interval
Bessel DE	x	$k^2 \frac{1}{x}$	x	$(0, 1]$
Fourier DE	1	0	1	$[-\pi, \pi]$
Hermite DE	e^{-x^2}	0	e^{-x^2}	$(-\infty, \infty)$
Laguerre DE	xe^{-x}	0	e^{-x}	$(0, \infty)$
Legendre DE	$1-x^2$	0	1	$(-1, 1)$

Table 1

CHAPTER V

SUMMARY

5-1 Thesis Review

Throughout this presentation, consideration has been given to the historical prominence of Sturm and Liouville as mathematicians of great influence in the nineteenth century. Both men have contributed key theorems of great significance in many branches of mathematics. It was Sturm who was influential in the origination of this "oscillating" function, while Liouville dealt with the validity of this function, hence, the *Sturm-Liouville Theory*. Chapter I delineates the productivity of investigators Sturm and Liouville.

Prior to the presentation of the formal theory, Chapter II was necessary to introduce several relevant concepts. Namely, the existence and uniqueness of solutions of the second order BVP; linear operators; Green's function; self-adjointness; the oscillating phenomena; and two theorems (separation and comparison) by Sturm which serve as the foundation in the development of the Sturm-Liouville Theory.

Chapter III defines what is meant by a regular S-L system.

(It is this particular system that was considered throughout this paper). The Sturm-Liouville Theory provides us with information about the eigenvalues and eigenfunctions. There are certain features common to <u>all</u> S-L problems. For instance, there are an infinite number of eigenvalues and eigenfunctions (the eigenvalues are discrete and can be ordered); the eigenvalues are all real; the eigenfunctions corresponding to distinct eigenvalues are orthogonal; the eigenvalues are *simple*; and most important, is that the infinite set of eigenfunctions constitutes an orthogonal basis and can be used to expand an essentially arbitrary function, f(x), defined on some interval such that the function can be represented in the following (Fourier-Eigenfunction) form:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \tag{5-1}$$

ie, a linear combination of the eigenfunctions.

Chapter IV contains results of extensive personal efforts of analysis and computation to delineate three prime examples of physical problems of the S-L type. These illustrations well portray the fact that the Sturm-Liouville system is a particular class of eigenvalue problems of frequent occurrence in applications.

Example 1, the 'Harmonic Oscillator', is obviously a *regular* S-L problem in that it satisfies Equation (3-1) with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a = 0$, $b = L$, $a_1 = 1$, $a_2 = 0$, $b_1 = 0$, and $b_2 = 1$. It is noted, that if $\hat{\lambda}$ is negative, then the general solution describes "simple harmonic motion".

Example 2, the Buckling of an Elastic Column, does not immediately fit the definition of a S-L system. However, if we make the substitution $y'' = z$, Equation (4-8) gives a regular S-L system. The calculations of Section 4-3 imply that the smallest eigenvalue is $\lambda_1 = \frac{\pi^2}{4L^2}$.

If we wish to determine the smallest critical load- the load at which the column buckles, this is given as

$$P_{crit} = \frac{\pi^2 EI}{4L^2} . \qquad (5-2)$$

In Example 3, a study is made of the one-dimensional heat conduction problem. By employing the method of *separation of variables*, the partial differential equation is replaced by two ordinary differential equations, of which the boundary value problem with respect to the spatial variable x gives a regular S-L system. This example also beautifully illustrates the orthogonality feature of S-L problems in the determination of the G_n coefficients. Hence, we obtain a lengthy analytical solution given by Equations (4-50) and (4-51).

In many instances, it may be more difficult to evaluate the analytical solution than to solve the original problem numerically. In Section 4-4A, the numerical solution to the one-dimensional heat conduction problem by the method of *finite differences* was illustrated in order to approximate the temperature at a specified position and time, in particular, $u(0.1, 0.02)$.

In comparing the results of these two techniques, we find the solutions differ in the third decimal place. However, the numerical approach would yield much more accurate results if we increased the number of grid points; that is, decrease increments h and k without violating the stability criteria for convergence.

Since all three examples presented are regular Sturm-Liouville systems, it is noted that the theoretical properties and significant features previously mentioned are satisfied.

Eigenvalue problems do arise which are <u>not</u> of Sturm-Liouville type. For example, problems involving higher-order differential equations, non-linear differential equations, problems in which the boundary conditions are periodic, etc. In many of those cases, there is no well-developed theory like the S-L Theory to direct us. Fortunately, there are a multitude of eigenvalue problems we encounter that are of the Sturm-Liouville type. Many of the functions important in physics satisfy differential equations of S-L type. However, in many equations of physical interest, some of the conditions that defined the regular S-L system are not satisfied. If p, q, and r satisfy those conditions on the open interval $a < x < b$, but fail to satisfy them at one or both of the boundary points this problem is referred to as the *singular* Sturm-Liouville system. Table 1 identifies some singular problems. For example, in Bessel's equation $p(0) = 0$ and $q(x)$ is discontinuous at $x = 0$, but the forementioned conditions are satisfied in $0 \leq x \leq 1$, except at $x = 0$. The general theory of singular boundary value problems is very difficult and the properties of their eigenvalues and eigenfunctions are often established individually for each equation. Therefore, the singular S-L system may or may not have the theoretical properties previously established for the regular system. Consequently, each of these problems must be treated independently.

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