On Exponentially Perfect Numbers Relatively Prime to 15

by

Joseph F. Kolenick

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Joseph F. Kolenick

Signature:

Joseph F. Kolenick, Student Date

Approvals:

Jacek Fabrykowski, Thesis Advisor Date

Thomas Smotzer, Committee Member Date

Eric Wingler, Committee Member Date

Peter J.Kasvinsky, Dean of School of Graduate Studies & Research Date

ABSTRACT

If the natural number n has the canonical form $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, then we say that an exponential divisor of n has the form $d = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$, where $b_i|a_i$ for $i = 1, 2, \ldots r$. We denote the sum of the exponential divisors of n by $\sigma^{(e)}(n)$. A natural number n is said to be exponentially perfect (or e-perfect) if $\sigma^{(e)}(n) = 2n$.

The purpose of this thesis is to investigate the existence of e-perfect numbers relatively prime to 15. In particular, if such numbers exist, are they bounded below? How many distinct prime divisors must they have? Several lemmas are utilized throughout the paper on route to answering these questions. Also, computer programs written in Maple are used for numerical estimates.

1 Introduction and History

The sum of positive divisors of a positive integer n is denoted by $\sigma(n)$. A positive integer n is said to be perfect if $\sigma(n) = 2n$. The Grolier Multimedia Encyclopedia says that perfect numbers are "another example of Greek progress in number theory," and credits the Pythagoreans for coining the term "perfect". Euclid noticed that numbers of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime number, are perfect numbers [4]. Later, L. Euler proved that every even perfect number is of the above stated form.

For $2^n - 1$ to be prime, it is necessary but not sufficient that n should be prime. Prime numbers of the form $2^n - 1$ are known as Mersenne primes. Consequently, since all even perfect numbers are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime number, we know only as many perfect numbers as we do Mersenne primes.

Computers have played an important role in discovering perfect numbers. So far, 44 such numbers are known, the largest being $2^{32,582,656}(2^{32,582,657} - 1)$ with 19,616,714 digits [7].

Since any even perfect number is of the form $2^{n-1}(2^n - 1)$, it is a triangular number, which is a number that is equal to the sum of all the natural numbers up to a certain point. Also, any even perfect number, with exception to the first, is the sum of the first $2^{(n-1)/2}$ odd cubes.

Although there are many results regarding odd perfect numbers, it is not known if any exist. For instance, if there exists an odd perfect number, then it is greater than 10^{300}, has at least 75 prime factors and at least 9 distinct prime factors [6], and its largest prime factor must be greater than 10^8 [5].

We will now move forward to the problem at hand involving exponential divisors. If an integer

$n > 1$ has the canonical form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \tag{1.1}$$

we then say a divisor d of n is an exponential divisor of n if it is of the form

$$d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \quad \text{where} \quad b_i | a_i, \quad i = 1, 2, \ldots, r.$$

These divisors were first introduced and studied by Subbarao [2], and later by Straus and Subbarao [3] as well as others. Let $\sigma^{(e)}(n)$ denote the sum of exponential divisors of n. Conventionally, 1 is considered an exponential divisor of itself, so that $\sigma^{(e)}(1) = 1$. Also, note that $\sigma^{(e)}(n)$ is a multiplicative arithmetic function, that is, $\sigma^{(e)}(mn) = \sigma^{(e)}(m)\sigma^{(e)}(n)$, whenever $\gcd(m, n) = 1$. It follows that for $n > 1$ given by (1.1) we have

$$\sigma^{(e)}(n) = \prod_{i=1}^{r} \left(\sum_{b_i | a_i} p_i^{b_i} \right)$$

A positive integer n is said to be exponentially perfect (or e-perfect) if $\sigma^{(e)}(n) = 2n$. If m is a square free, then $\sigma^{(e)}(m) = m$. Therefore, if n is an e-perfect number with $\gcd(m, n) = 1$, then mn is also an e-perfect number. Consequently, we will only look at e-perfect numbers that contain no primes to the first power, which are referred to as powerful e-perfect numbers.

Not much is known about e-perfect numbers. Straus and Subbarao [3] were able to show that all e-perfect numbers are even and that for every r the set of powerful e-perfect numbers with r prime factors is finite.

The first ten powerful e-perfect numbers have been noted in [3] and are listed below as

$$2^2 3^2,\ 2^3 3^2 5^2,\ 2^2 3^3 5^2,\ 2^4 3^2 11^2,\ 2^4 3^3 5^2 11^2,\ 2^6 3^2 7^2 13^2,\ 2^7 3^2 5^2 7^2 13^2$$

$$2^6 3^2 5^2 7^2 13^2,\ 2^8 3^2 5^2 7^2 139^2,\ 2^{19} 3^3 5^2 7^2 11^2 13^2 19^2 37^2 79^2 109^2 157^2 313^2.$$

Notice that all of the numbers listed above are divisible by 3. What is not known, however, is if in fact all e-perfect numbers are divisible by 3. Fabrykowski and Subbarao [1] showed that if there is an e-perfect number not divisible by three, then it is divisible by 2^{117} and greater than 10^{664} and must have 118 distinct prime factors.

In light of this result, we will show in Section 2 of this paper that if there is an e-perfect number relatively prime to 15, then it is divisible by 2^{3152} and greater than 10^{29008} and must have at least 3153 distinct prime factors.

2 Main Result

Theorem. *If there is an e-perfect number relatively prime to 15, then it is divisible by* 2^{3152} *and greater than* 10^{29008}.

To prove the theorem, we will make use of several Lemmas and numerical results. Throughout the paper p, q will denote prime numbers.

Lemma 2.1. *For every prime* p *and integer* $a \geq 2$ *we have:*

$$\frac{\sigma^{(e)}(p^a)}{p^a} \leq 1 + \frac{1}{p^2} + \frac{1}{p^3} \qquad \text{for } a \geq 3$$
$$\frac{\sigma^{(e)}(p^a)}{p^a} = 1 + \frac{1}{p} \qquad \text{for } a = 2$$

Proof. For $a = 2, 3, 4, 5$ the result is trivial and is easily verified directly. For $a \geq 6$ we will consider possible divisors of a. The first, second, and third largest divisors of a are at most $a, a/2$, and $a/3$ respectively. The rest of the divisors of a are less than or equal to $(a/3) - 1, (a/3) -$

$2, (a/3)-3, \ldots$. Consequently, we have

$$\begin{aligned}
\frac{\sigma^{(e)}(p^a)}{p^a} &\leq \frac{1}{p^a}\left(p^a + p^{a/2} + p^{a/3} + p^{(a/3)-1} + p^{(a/3)-2} + \ldots\right) \\
&= 1 + \frac{1}{p^{a/2}} + \frac{p^{a/3}(1+p^{-1}+p^{-2}+\ldots)}{p^a} \\
&= 1 + \frac{1}{p^{a/2}} + \frac{1}{p^{2a/3}} \cdot \frac{p}{p-1} \\
&\leq 1 + \frac{1}{p^2} + \frac{1}{p^3}
\end{aligned}$$

$\square$

Definition. *The Riemann-Zeta Function $\zeta(s)$ is defined by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ , \qquad\qquad Re\, s > 1.$$

An equivalent form of the Riemann-Zeta Function is

$$\zeta(s) = \prod_{p \in \mathbf{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad\qquad Re\, s > 1.$$

where **P** is the set of all prime numbers.

Lemma 2.2.

$$\frac{\zeta(2)\zeta(3)}{\zeta(4)\zeta(6)} = \prod_{p \in \boldsymbol{P}} \left(1 + \frac{1}{p^2}\right)\left(1 + \frac{1}{p^3}\right) \tag{2.2}$$

Proof. From the definition of the Riemann-Zeta Function we get

$$\frac{\zeta(2)\zeta(3)}{\zeta(4)\zeta(6)} = \frac{\prod_{p \in \mathbf{P}} \left(1 - \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^3}\right)^{-1}}{\prod_{p \in \mathbf{P}} \left(1 - \frac{1}{p^4}\right)^{-1}\left(1 - \frac{1}{p^6}\right)^{-1}} \qquad = \frac{\prod_{p \in \mathbf{P}} \left(1 - \frac{1}{p^4}\right)\left(1 - \frac{1}{p^6}\right)}{\prod_{p \in \mathbf{P}} \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p^3}\right)}.$$

Factoring the numerator yields

$$\frac{\prod_{p\in\mathbf{P}}\left(1-\frac{1}{p^2}\right)\left(1+\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)\left(1+\frac{1}{p^3}\right)}{\prod_{p\in\mathbf{P}}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)}=\prod_{p\in\mathbf{P}}\left(1+\frac{1}{p^2}\right)\left(1+\frac{1}{p^3}\right)$$

which proves the lemma. □

Lemma 2.3. *If $q \equiv 1$, 2, or 3 (mod 5) and $q \equiv 1$ (mod 6), then $q \equiv 1$, 7, or 13 (mod 30).*

Proof. Suppose $q \equiv 2 \pmod 5$ and $q \equiv 1 \pmod 6$. Then this implies that

$$q = 5m + 2$$

$$q = 6n + 1$$

for some integers m and n. If we multiply the first equation by 6 and the second by 5 we get

$$6q = 30m + 12$$

$$5q = 30n + 5$$

and subtracting these two equations yields $q = 30(m-n) + 7$, which proves that $q \equiv 7 \pmod{30}$. A similar proof can be shown for the other two cases and is therefore omitted. Hence, $q \equiv 1$, 7, or 13 (mod 30). □

Lemma 2.4. *For $x \geq 7$, the function $f(x) = \left(1+\frac{1}{x}\right)\left(1+\frac{1}{x^2}+\frac{1}{x^3}\right)^{-1}$ is a decreasing function and has values greater than one.*

Proof. Let us write

$$f(x) = \left(1+\frac{1}{x}\right)\left(1+\frac{1}{x^2}+\frac{1}{x^3}\right)^{-1} = \frac{x^3+x^2}{x^3+x+1}$$

By the quotient rule we have

$$f'(x) = \frac{(3x^2+2x)(x^3+x+1)-(x^3+x^2)(3x^2+1)}{(x^3+x+1)^2}$$
$$= \frac{-x^4+2x^3+4x^2+2x}{(x^3+x+1)^2}$$

Assuming $x \geq 0$, then $f'(x) < 0$ is equivalent to $x^3 > 2(x+1)^2$, which certainly holds for all real $x \geq 4$. Therefore $f(x)$ is a decreasing function for $x \geq 7$.

Next we must show that $f(x) > 1$ for $x \geq 7$ or equivalently that

$$\frac{x^3+x^2}{x^3+x+1} > 1.$$

This inequality holds when $x^3+x^2 > x^3+x+1$ or when $x^2-x-1 > 0$, which clearly holds for $x \geq 7$. Therefore $f(x)$ has values greater than 1 when $x \geq 7$. □

Lemma 2.5. *Let q_j^* be the jth prime congruent to 1, 7, or 13 (mod 30). Then the smallest integer $h \geq 2$ for which the inequality*

$$1.891843\frac{2^h}{\sigma^{(e)}(2^h)} < \prod_{j=1}^{h}\left(1+\frac{1}{q_j^*}\right)\left(1+\frac{1}{q_j^{*2}}+\frac{1}{q_j^{*3}}\right)^{-1}$$

holds is $h = 3152$.

Proof. The proof follows from a computer program. See appendix. □

Lemma 2.6. *The smallest integer k for which the inequality*

$$1.891843\frac{2^{3152}}{\sigma^{(e)}(2^{3152})} < \prod_{j=1}^{k}\left(1+\frac{1}{q_j^*}\right)\left(1+\frac{1}{q_j^{*2}}+\frac{1}{q_j^{*3}}\right)^{-1}$$

holds is $k = 3152$. Here q_j^* has the same meaning as in Lemma 2.5.

Proof. This follows from a computer printout. See appendix. □

Proof of Theorem.

Proof. Let $N = 2^h M$, where $\gcd(M, 30) = 1$, be a powerful e-perfect number. Let us write $M = LK$, where for every prime q such that $q|L$ we have $q^2 \| L$ and $\gcd(L, K) = 1$. We will show every prime q that is a factor of L is in the arithmetic progression 1 (mod 30), 7 (mod 30), or 13 (mod 30). Since we assumed that $3 \nmid N$, it is obvious that $3 \nmid \sigma^{(e)}(N)$ since N is an e-perfect number (i.e., it satisfies the equation $\sigma^{(e)}(N) = 2N$). Therefore, since $3 \nmid \sigma^{(e)}(N)$ we have that $3 \nmid \sigma^{(e)}(L)$. Hence,

$$\sigma^{(e)}(L) = \prod_{q^2||L} \sigma^{(e)}(q^2) = \prod_{q^2||L} q(q+1) \not\equiv 0 \pmod{3}.$$

This implies that $q \not\equiv 0$ (mod 3) and $q + 1 \not\equiv 0$ (mod 3), which in turn shows that $q \equiv$1 (mod 6). Also, by assumption, $5 \nmid N$ implies $5 \nmid \sigma^{(e)}(N)$. Thus, $5 \nmid \sigma^{(e)}(L)$ and we have

$$\sigma^{(e)}(L) = \prod_{q^2||L} \sigma^{(e)}(q^2) = \prod_{q^2||L} q(q+1) \not\equiv 0 \pmod{5}.$$

Consequently $q \not\equiv 0$ (mod 5) and $q + 1 \not\equiv 0$ (mod 5) which implies $q \equiv 1$, 2, or 3 (mod 5). Now since $q \equiv 1$, 2, or 3 (mod 5) and $q \equiv 1$ (mod 6), by Lemma 2.3 we have that $q \equiv 1$, 7, or 13 (mod 30). Hence, if $q \mid L$, then it is in the arithmetic progression 1 (mod 30), 7 (mod 30), or 13 (mod 30). Further, since N is an e-perfect number, it satisfies the equation $\sigma^{(e)}(N) = 2N$ or equivalently

$$\sigma^{(e)}(2^h)\sigma^{(e)}(L)\sigma^{(e)}(K) = 2 \cdot 2^h LK. \tag{2.5}$$

Now the highest power of 2 that divides the right hand side of (2.5) is $h + 1$. If θ is the highest power of 2 dividing the left hand side, then we have $\theta \geq 1 + s$, where s is the number of prime divisors of L. Thus $h + 1 = \theta \geq s + 1$ and we have

$$h \geq s \tag{2.6}$$

Rearranging equation (2.5) gives

$$\frac{2\cdot 2^h}{\sigma^{(e)}(2^h)} = \frac{\sigma^{(e)}(L)}{L}\frac{\sigma^{(e)}(K)}{K}$$

and applying Lemma 2.1 yields

$$\begin{aligned}
\frac{\sigma^{(e)}(L)}{L}\frac{\sigma^{(e)}(K)}{K} &\leq \prod_{q|L}\left(1+\frac{1}{q}\right)\prod_{p|K}\left(1+\frac{1}{p^2}+\frac{1}{p^3}\right)\\
&= \prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\prod_{p|LK}\left(1+\frac{1}{p^2}+\frac{1}{p^3}\right)\\
&\leq \prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\prod_{p\neq 2,3,5}\left(1+\frac{1}{p^2}+\frac{1}{p^3}\right)\\
&\leq \prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\prod_{p\neq 2,3,5}\left(1+\frac{1}{p^2}\right)\left(1+\frac{1}{p^3}\right)
\end{aligned}$$

Next, applying Lemma 2.4 to this result gives

$$\begin{aligned}
&\prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\prod_{p\neq 2,3,5}\left(1+\frac{1}{p^2}\right)\left(1+\frac{1}{p^3}\right)\\
&= \prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\cdot\left(1+\frac{1}{2^2}\right)^{-1}\left(1+\frac{1}{2^3}\right)^{-1}\\
&\left(1+\frac{1}{3^2}\right)^{-1}\left(1+\frac{1}{3^3}\right)^{-1}\left(1+\frac{1}{5^2}\right)^{-1}\left(1+\frac{1}{5^3}\right)^{-1}\frac{\zeta(2)\zeta(3)}{\zeta(4)\zeta(6)}\\
&= \frac{3^5 5^5\zeta(3)}{7\cdot 13\cdot\pi^8}\prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}\\
&= 1.05716934481\ldots\prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}
\end{aligned}$$

on using values in [8]. Hence,

$$1.891843\frac{2^h}{\sigma^{(e)}(2^h)} < \prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}$$

Now from Lemma 2.4, we have that $\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1}$ is a decreasing function of q and has values greater than 1 for $q \geq 7$. Therefore, since h is not less than the number of prime divisors of L,

$$\prod_{q|L}\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}+\frac{1}{q^3}\right)^{-1} < \prod_{j=1}^{h}\left(1+\frac{1}{q_j^*}\right)\left(1+\frac{1}{q_j^{*2}}+\frac{1}{q_j^{*3}}\right)^{-1}$$

where the q_j^*'s have the same meaning as in Lemma 2.5. Therefore, we get

$$1.891843\frac{2^h}{\sigma^{(e)}(2^h)} < \prod_{j=1}^{h}\left(1+\frac{1}{q_j^*}\right)\left(1+\frac{1}{q_j^{*2}}+\frac{1}{q_j^{*3}}\right)^{-1}.$$

It follows from Lemma 2.5 that h is at least 3152. Also, Lemma 2.6 implies that N must have at least 3153 distinct prime factors. Therefore since N is a powerful e-perfect number we have

$$N \geq 2^{3152}\prod_{j=1}^{3152} q_j^{*2} > 10^{29008}$$

thus proving the theorem. □

3 Some Remarks

In view of the numerical result presented in this paper, it is reasonable to make the following:

Conjecture 3.1. *All e-perfect numbers are divisible either by 3 or by 5.*

Conjecture 3.2. *If $n > 1$ is a powerful e-perfect number of the form $n = 2^hK$, where K and h are integers, $\gcd(2, K) = 1$, and r is the number of prime divisors of K, then $h \geq r$.*

This conjecture is supported by the list of the first 10 powerful e-perfect numbers that is presented in this paper.

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Appendix

All of the numerical estimates requiring a computer program were solved using programs that were written in MAPLE, in particular, using the traditional worksheet of Maple 11. The following lists the function of each program as well as the programs themselves.

Prime Numbers. These functions produce prime numbers that are congruent to 1, 7, or 13 modulo 30.

```
> F1 := n -> if ithprime(n) mod 30 = 1 then ithprime(n) else NULL end if;

> F2 := n -> if ithprime(n) mod 30 = 7 then ithprime(n) else NULL end if;

> F3 := n -> if ithprime(n) mod 30 = 13 then ithprime(n) else NULL end if;
```

List of Primes. This code creates an ordered set P of prime numbers that are congruent to 1, 7, and 13 modulo 30. The entries are ordered from least to greatest and are generated via the previous functions.

```
> P := {seq(F1(n), n=1...10000), seq(F2(n), n=1...10000), seq(F3(n), n=1...10000)};
```

Functions. These functions are used for finding solutions to Lemmas 2.3 and 2.4.

```
> F := n -> product ( (1 + 1/P[k])*(1 + 1/P[k]^2 + 1/P[k]^3)^(-1), k=1...n );

> with(numtheory):

> G := n -> 1.891843*2^n/(sum(2^divisors(n)[k], k=1...tau(n)));
```

Lemma 2.5. This code yields the result for Lemma 2.5 which is n = 3152.

```
> for n from 1 to 3152 do
> if G(n) < F(n) then print(n); end if;
> NULL;
> end do;
```

Lemma 2.6. This code gives a result for Lemma 2.6 which is n = 3152.

```
> for n from 1 to 3152 do
> if G(3152) < F(n) then print(n); end if;
> NULL;
> end do;
```

Function. This function is used for the main theorem.

```
> K := n -> 2^n*product(P[k]^2, k=1...n);
```

Theorem. This code produces the lower bound for the main theorem which is 10^{29008}.

```
> floor(log10(K(3152)));
```