# AN EXPLORATION ON THE HAMILTONICITY OF CAYLEY DIGRAPHS

by

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An Exploration on Hamiltonicity of Cayley Digraphs

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## ABSTRACT

In 1969, Lászlo Lovász launched a conjecture that remains open to this day. Throughout the years, variations of the conjecture have surfaced; the version we used for this study is: “Every finite connected Cayley graph is Hamiltonian”. Several studies have determined and proved Hamiltonicity for the Cayley graphs of specific types of groups with a minimal generating set. However, there are few results on the Hamiltonicity of the directed Cayley graphs. In this thesis, we look at some of the cases for which the Hamiltonicity on Cayley digraphs has been determined and we prove that the Cayley digraph of group $\mathcal{G}$ such that $\mathcal{G} = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ is non-Hamiltonian with a standard generating set.

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# Contents

# 1 Introduction

Determining Hamiltonicity on any graph can have important and useful applications. To determine if a graph is Hamiltonian, we attempt to find a path that starts and ends at the same vertex and visits every vertex of the graph exactly once. It can be difficult to find such a path. It can be even more difficult to prove that there does not exist such a path. Examples of Hamiltonian graphs, the applications of Hamiltonian graphs, and more information on this type of problem can be found in Chapter 2.1.

Cayley digraphs generally have some form of structure; the group and generating set we choose dictate the structure of the Cayley digraph. Some of the groups whose Cayley digraphs are Hamiltonian (with minimal or standard generating sets) are abelian and/or normal. Chapter 2.2 will provide the necessary background information on these types of groups.

Examples of Cayley digraphs, both Hamiltonian and non-Hamiltonian, will be found in Chapter 3. In the same chapter we will show a few of the known results for determining Hamiltonicity on Cayley graphs and digraphs.

Lastly, in Chapter 4, we will share the results that sparked our curiosity and inspired our work. We will prove that the Cayley digraph of a group $\mathcal{G} = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, where $p$ and $q$ are distinct primes and $q \neq 2$, is non-Hamiltonian. The lemma we prove in this chapter is a new result. The proof for our main theorem is a new approach to a known result.

# 2 Preliminaries

In the following subsections we will provide the background information and definitions for our study. Unless cited otherwise, the definitions and notation in Chapters 2.1 and 3 were taken from the textbook *Graphs and Digraphs* [1], and the definitions and theorems in Chapter 2.2 were taken from the textbook *Algebra: Pure and Applied* [2].

## 2.1 Graph Theory

In 1736, Loenhard Euler (1707-1783) proved the answer to the question we call the Königsberg Bridge Problem [1]. The city formerly known as Königsberg in what is now Russia was made up of 4 landmasses that were divided by a river. The city had 7 bridges to grant access to each landmass. The following question was posed.

> *Starting from any of the four land areas, is it possible to cross each of the seven bridges exactly once and come back to the starting point without swimming across the river*

We can represent the city as a **graph**, a mathematical structure that models relations between objects using vertices and edges, where the vertices represent the landmasses and edges represent the bridges. The number of vertices of this graph, or **order**, is 4 and the number of edges of this graph, or **size**, is 7. One landmass is connected to two other landmasses by two bridges each. The remaining landmass is connected to the other three landmasses by one bridge. We get the following graph.

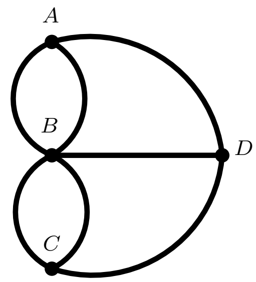


Figure 1: Graph Representing Königsberg Bridge Problem

Euler determined that the walk described above was not possible. Euler's proof of this result is considered to be the first theorem of graph theory. This being said, graph theory was studied only recreationally for several decades. It was not widely considered theoretical mathematics until the late 1900s. While it is a form of pure mathematics, there are various applications of graph theory. Like in the Königsberg Bridge Problem, concepts of graph theory provide answers to many questions that occur in everyday life.

Some of the applications of graph theory include finding communities in networks, ranking hyperlinks, and studying algorithms in computer science. Graphs are used to represent immense networks of communications [6]. Google uses PageRank, a link analysis algorithm that was developed using several topics from the field. This means each Google search requires the use of graph theory. Another common and practical application of the theory is found in every GPS search. For this, and many other applications, directed graphs are used. A **digraph**, or directed graph, is a graph in which each edge is given an orientation from one end to the other.

Much like a car on a road going from one location to another, a **directed path** is a sequence of edges which joins a sequence of distinct vertices, where all edges are directed in the same direction. Now, consider a **n-cycle**, or $C_n$, meaning a trail of order $n$ where the only repeating vertices are the first and last vertices. Then a directed $n$-cycle is a $n$-cycle where all the edges are oriented in the same direction. We can think of this as a car starting at one location, driving to $n-1$ locations, and finally arriving back to the original location.

A significant type of graph, of which several theorems and conjectures have been made, are Hamiltonian graphs. These are named after the mathematician and physicist Sir William Rowan Hamilton (1805-1865)[1]. One of Hamilton's best known contributions was to abstract algebra; he introduced an extension of the complex numbers which he called quaternions. Hamilton also laid the groundwork for a concept that became a popular area of study in graph theory, the Hamiltoncity of graphs and digraphs. A cycle in a graph $G$ that contains every vertex of $G$ is called a **Hamiltonian cycle** of $G$. And so, a graph that contains a Hamiltonian cycle is itself called **Hamiltonian**. See the figures below for an example of a Hamiltonian graph and a non-Hamiltonian graph.

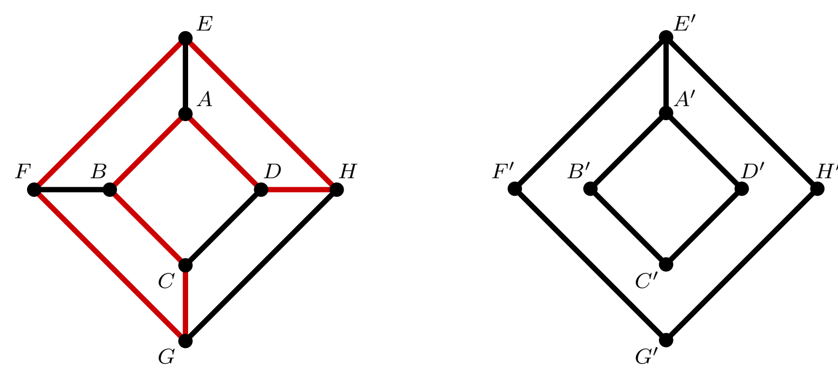


Figure 2: Hamiltonian vs. non-Hamiltonian

The graph on the left is Hamiltonian. The cycle in red is the Hamiltonian cycle. Starting on any vertex, we can follow the red path to visit every vertex and return to the vertex we picked. This confirms that this graph is Hamiltonian. However, there exists no such cycle in the graph on the right. Notice that the vertex $A'$ on this graph is a **cut vertex**, a vertex that when deleted from the graph, along with its incident edges, will yield a disconnected graph. The following theorem shows why the graph on the right is not Hamiltonian.

**Theorem 2.1.** *If $G$ contains a cut vertex, then $G$ is not a Hamiltonian graph.*

Notice, in order for the graph in Figure 2 to be Hamiltonian, a Hamiltonian cycle would enter and exit vertex $A'$ but would not be able to come back through it order to reach the starting vertex, or remaining vertices. Notice $A'$ is the cut vertex.

It has been shown that there are necessary conditions for a graph to be Hamiltonian[1]. Such as Theorem 2.1 Additionally, there are several known sufficient conditions for a graph to be Hamiltonian. However, there are no known necessary and sufficient conditions that make a graph Hamiltonian. This fact makes determining Hamiltonicity of graphs a particularly difficult problem.

Determining whether a graph is Hamiltonian has been described as “difficult to solve, easy to verify" [1]. Suppose we are asked to determine Hamiltonicity of a random graph of large order and size. We might first look for properties of the graph that allow us to use known theorems to determine that the graph is non-Hamiltonian. Alternatively, we could first attempt to find a Hamiltonian cycle. If the graph is indeed Hamiltonian, all we must do to prove this is to identify the Hamiltonian cycle. If the graph is non-Hamiltonian, it can be a rigourous and lengthy process to prove why there does not exist a Hamiltonian cycle. There exists a class of mathematical problems that cannot be solved, that we know of, by a polynomial-time algorithm. The set of problems containing such problems is called **NP**, which stands for nondeterministic polynomial-time. Furthermore, a problem is called **NP**-complete if its solution with a polynomial-time algorithm would result in a solution for all other problems in **NP**. There are thousands of NP-complete problems known to this day, and determining the Hamiltonicity of graphs is one of them.

Research on graph theory has gained popularity over the past several decades. Naturally, research on the relationship between topics in graph theory and those of other branches of mathematics has developed. For this study, the other field we focus on is group theory.

## 2.2 Group Theory

Group theory is a branch of mathematics that, while containing concepts that can be traced back to earlier mathematicians, was trailblazed by Éveriste Galois (1811-1831). He is considered the father of group theory[2]. In his short life, Galois proved under which conditions an equation can be solved, a problem which was the goal of several contemporary mathematicians to solve. He brilliantly used the limited knowledge on groups at the time to achieve his results. Group theory is also considered a theoretical branch of mathematics; however, we can find applications of group theory in physics, chemistry and cryptography.

A **group**, $\mathcal{G}$, is a nonempty set with a binary operation $*$ and satisfies the following axioms:

1. (closure) For any $a, b \in \mathcal{G}$, we have $a * b \in G$.

2. (associativity) For any $a, b, c \in \mathcal{G}$, we have $a * (b * c) = (a * b) * c$.

3. (identity) There exists an element $e \in \mathcal{G}$ such that for all $a \in \mathcal{G}$ we have $a * e = e * a = a$. Such an element $e \in \mathcal{G}$ is called an **identity** in $\mathcal{G}$.

4. (inverse) For each $a \in \mathcal{G}$ there exists an element $a^{-1} \in \mathcal{G}$ such that $a * a^{-1} = a^{-1} * a = e$. Such an element $a^{-1} \in \mathcal{G}$ is called an **inverse** of $a$ in $\mathcal{G}$.

An **isomorphism**, or a bijective homomorphism, maps a group to another. We say two groups are **isomorphic** if there exists an isomorphism mapping one group to the other group. If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic, we denote this $\mathcal{G} \cong \mathcal{H}$. An **automorphism** is an isomorphism from one group to itself. Consider the group $\mathbb{Z}_5$. Let the map $\phi : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_5$ be defined by $\phi(a) = a$. We call a map defined like this the **identity map**. We see that $\phi$ is bijective, a homomorphism, and maps $\mathbb{Z}_5$ to itself; it is an automorphism.

A group $\mathcal{G}$ is called **cyclic** if there exists an element $a \in \mathcal{G}$ such that $\mathcal{G} = \{a^n | n \in \mathbb{Z}\} = \langle a \rangle$. The element $a \in \mathcal{G}$ such that $\mathcal{G} = \langle a \rangle$ is called a **generator** of $\mathcal{G}$. Consider $1 \in \mathbb{Z}_4$. We see that $1^1 = 1$, $1^2 = 1 + 1 = 2$, $1^3 = 1 + 1 + 1 = 3$, and $1^4 = 1 + 1 + 1 + 1 = 0$. Then $\{1^n | 0 \leq n \leq 3\} = \mathbb{Z}_4$, making 1 a generator of $\mathbb{Z}_4$. Similarly, $1 \in \mathbb{Z}_n$ is a generator of $\mathbb{Z}_n$ for all $n \in \mathbb{Z}^+$. There are groups that cannot be generated by a single element but can be generated by a set of elements. This set is a **generating set** of $\mathcal{G}$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be groups. The group $\mathcal{G}_1 \times \mathcal{G}_2$ is a **direct product** where $\mathcal{G}_1 \times \mathcal{G}_2 = \{(a, b) | a \in \mathcal{G}, b \in \mathcal{G}\}$. The componentwise operation, $+$, on $\mathcal{G}_1 \times \mathcal{G}_2$ is defined by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. Consider the direct product $\mathbb{Z}_2 \times \mathbb{Z}_4$. While $1 \in \mathbb{Z}_2$ generates $\mathbb{Z}_2$ and $1 \in \mathbb{Z}$ generates $\mathbb{Z}_4$, we get that $\mathbb{Z}_2 \times \mathbb{Z}_4 \neq \langle (1, 1) \rangle = \{(1, 1), (0, 2), (1, 3), (0, 0)\}$. The set $\{(0, 1), (1, 0)\}$ is a generating set of $\mathbb{Z}_2 \times \mathbb{Z}_4$. However, in the direct product $\mathbb{Z}_4 \times \mathbb{Z}_5$, we see that $(1, 1)$ is the generator of $\mathbb{Z}_4 \times \mathbb{Z}_5$. The following theorem help us understand why some direct products are cyclic.

**Theorem 2.2.** *The group $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to the cyclic group $Z_{mn}$ if and only if $n$ and $m$ are relatively prime.*

A special type of group comes up quite often when dealing with cyclic groups called an abelian group. A group $\mathcal{G}$ is **abelian** if $ab = ba \ \forall a, b \in \mathcal{G}$. There are a large number of studies and theorems over abelian groups. One of these states that every subgroup of an abelian group is also abelian. Another important result is that all cyclic groups are abelian. The converse is not true. Another important type of group is normal subgroups. To define a normal subgroup we must first define cosets. The set $aH = \{ah | \forall h \in H\}$ is the **left coset** of $H$ in $\mathcal{G}$. Similarly the set $Ha = \{ha | h \in H\}$ is the **right coset** of $H$ in $\mathcal{G}$. We say the subgroup $H$ of $\mathcal{G}$ is **normal** if $\forall g \in \mathcal{G}$ we have that $gH = Hg$, denoted $H \trianglelefteq \mathcal{G}$. Then by definition, every subgroup of the abelian group $\mathcal{G}$ is a normal subgroup of $\mathcal{G}$. For example, consider $\mathcal{G} = \mathbb{Z}_8$. We know this group is cyclic; it can be generated by $1 \in \mathbb{Z}_8$. Now let $H$ be the group generated by 2. Then $H = \langle 2 \rangle = \{0, 2, 4, 6\}$. Notice $H \leq \mathcal{G}$ and moreover, $H \trianglelefteq \mathcal{G}$. The next theorem is powerful. It is used in several results shown in the coming chapter.

**Theorem 2.3** (Fundamental Theorem of Finite Abelian Groups)**.** *Let $\mathcal{G}$ be an Abelian group of finite order. Then $\mathcal{G} = \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_s^{r_s}}$ where $p_i$ are prime numbers not necessarily distinct.*

Let us look at groups as graphs. The elements of a group are the vertices of graph $\mathcal{G}$. An edge exists between elements $a$ and $b$ if $a+b = 0$. By convention, every element is connected to the vertex representing the identity of the group. See Figure 3 and Figure 4 for the graph of $\mathbb{Z}_4$ and $\mathbb{Z}_5$, respectively.

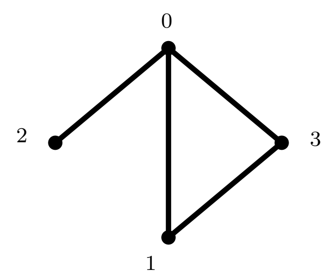


Figure 3: Graph of $\mathbb{Z}_4$

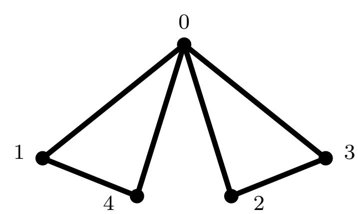


Figure 4: Graph of $\mathbb{Z}_5$

# 3 Cayley Digraphs

This study focuses on the marriage of graph theory and group theory. Specifically, the concept of Cayley digraphs, which allows us to use our knowledge about both topics to learn more about one another. Let $\Delta = \{g_1, g_2, \cdots g_k\}$ be a generating set of a finite group $\mathcal{G}$. We can construct a **Cayley digraph** of $\mathcal{G}$ with respect to $\Delta$, denoted $\overrightarrow{Cay}_{\Delta}(\mathcal{G})$, as follows:

1. Each element of $\mathcal{G}$ is represented by a vertex.

2. If $g_1 \in \Delta$ is represented by a directed edge $\longrightarrow$, then for $a, b \in \mathcal{G}$, $a \longrightarrow b$ means that $ag_1 = b$ in $\mathcal{G}$.

3. Each element of $\Delta$ is represented by a different style, or color, of directed edge. For example $g_1$ might be represented by a solid directed edge $\longrightarrow$, while $g_2$ might be represented by a dashed directed edge $\dashrightarrow$.

4. If $g_1 a = b$ and $g_1 b = a$, then the arrow is omitted from the edge adjacent to $a$ and $b$.

When creating a Cayley digraph, two of the defining features, order and size, are determined by the group and generating set chosen, respectively. For a cyclic group, we could choose one of its generators to construct its corresponding Cayley digraph. A **minimal generating set** is a set with the smallest number of elements required to generate a group. For example, $\{(1, 1)\}$ is a minimal generating set for $\mathbb{Z}_2 \times \mathbb{Z}_3$. Notice $\{(1, 0), (0, 1)\}$ also generates $\mathbb{Z}_2 \times \mathbb{Z}_3$ but this is not a minimal generating set. Choosing a generating set other than the minimal generating set often produces a more complex and interesting graph.

Consider the group $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_3$. Let $\Delta_1 = \{(1, 1)\}$ and let $\Gamma$ be the Cayley digraph of $\mathcal{G}$ with respect to $\Delta_1$. Then $\Gamma = \overrightarrow{Cay}_{\Delta_1}(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is the following graph.

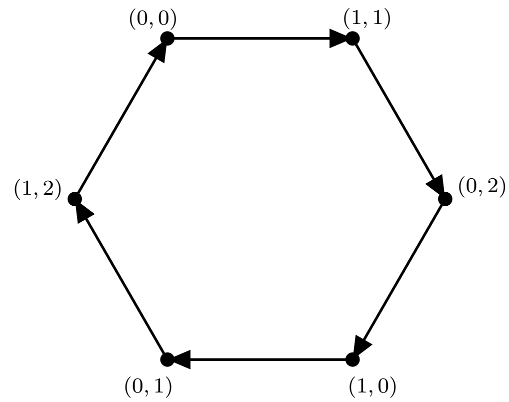


Figure 5: $\Gamma$ with $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\Delta_1$

Now consider the same group $\mathcal{G}$ with the generating set $\Delta_2 = \{(1, 0),\ (0, 1)\}$. We represent $(0, 1)$ with a solid directed edge and $(1, 0)$ with a dotted directed edge. Recall that when one generator maps two vertices to each other, the arrow is omitted. Then we have the following graph.

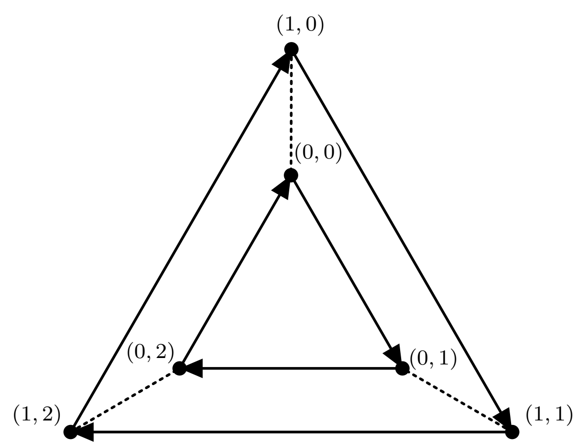


Figure 6: $\Gamma$ with $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\Delta_2$

The graphs $\overrightarrow{Cay}_{\Delta_1}(\mathcal{G})$ and $\overrightarrow{Cay}_{\Delta_2}(\mathcal{G})$ look very different. We see then that $\Gamma$ with $\Delta_1$ is a directed $6-$cycle while $\Gamma$ with $\Delta_2$ is a prism.

A **Cayley graph** is the undirected underlying graph of a Cayley digraph of $\mathcal{G}$ with respect to $\Delta$, denoted $\text{Cay}_\Delta(\mathcal{G})$. In 1969, Lászlo Lovász launched a conjecture that remains open to this day. Throughout the years, variations of the conjecture have surfaced; the version we used for this study is the following.

**Conjecture 3.1.** *Every finite connected Cayley graph is Hamiltonian.*

As mentioned earlier, determining the Hamiltonicity of graphs is a **NP-complete** problem. One might think that the structure a group provides would make Hamiltonicity easier to determine on Cayley graphs, but this has proven to be just as difficult. Several papers have been written proving the specific cases of Cayley graphs that are Hamiltonian. However, little work has been done over the Hamiltonicity of Cayley digraphs. The added restriction of directed edges makes this an even harder problem to solve and prove.

Let $\mathcal{G}$ be the group $\mathcal{G} = \mathbb{Z}_4 \times \mathbb{Z}_3$ with the generating set $\Delta_1 = \{(0,1),(1,0)\}$. Notice, $\Delta_1$ is not a minimal generating set of $\mathcal{G}$. In $\text{Cay}_{\Delta_1}(\mathcal{G})$, we denote the edges created by applying (0,1) to each element with a dotted line, and the edges created by applying (1,0) to each element with a solid line. See Figure 7. The edges that make up a Hamiltonian cycle are colored red. Starting at any vertex, we can follow the red path to visit each vertex and arrive back to the vertex with which we started.

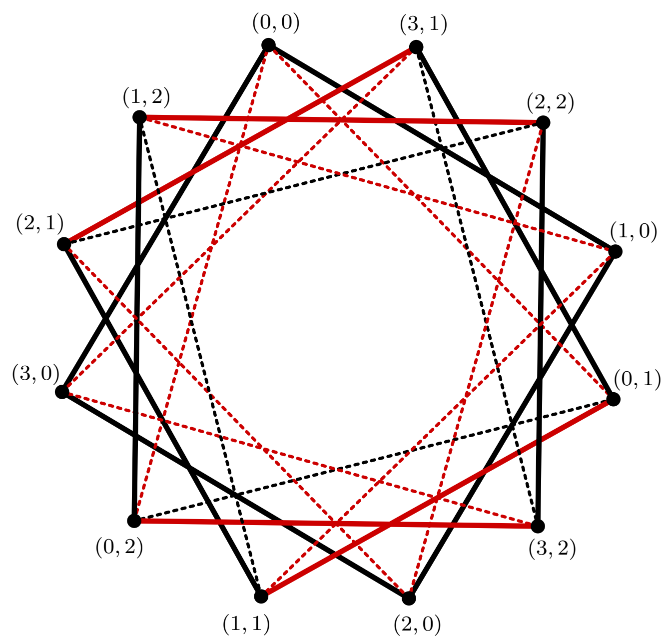


Figure 7: $\text{Cay}_{\Delta_1}(\mathcal{G})$

The figure below is of $\text{Cay}_\Delta(\mathbb{Z}_4 \times \mathbb{Z}_5)$ with the generating set $\Delta = \{(0,1),(1,0)\}$. The edges that make up a Hamiltonian cycle are colored red. Again, we can start at any vertex exactly once, follow the red path to visit each vertex and arrive to the vertex with which we started. It is not important to differentiate between elements of a generator when searching for a Hamiltonian cycle in a Cayley graph or digraph. For this reason, we represent both elements of $\Delta$ with solid directed edges. We will do so for the rest of the study.

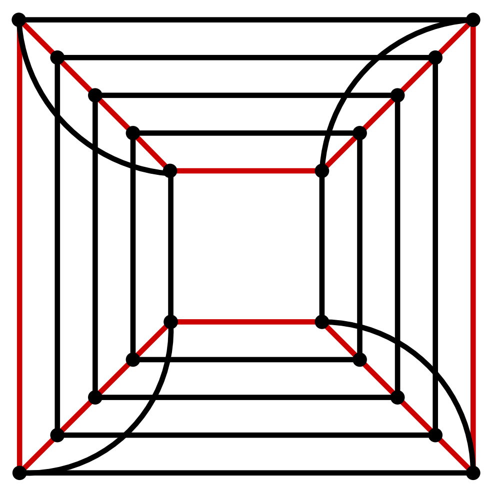

Figure 8: $\text{Cay}_{\Delta_2}(\mathbb{Z}_4 \times \mathbb{Z}_5)$

In a survey over Hamiltonicity of Cayley graphs [4], very few of the many results provide sufficient conditions or necessary conditions for a Cayley digraph to be Hamiltonian. Only one of the listed results provides both necessary and sufficient conditions for a Cayley digraph to be Hamiltonian. We will see this result in the next chapter. The following theorem is by G. Lanel [4].

**Theorem 3.2.** *Let $\mathcal{G} = \mathbb{Z}_m \times \mathbb{Z}_n$. Then the directed cycles $C_m \times C_n \cong \overrightarrow{Cay}_\Delta(\mathcal{G})$ when $\Delta = \{(0,1),(1,0)\}$.*

The cycles and Cayley digraph in Theorem 3.2 are directed, but it is worth noting we can intuitively deduce that the underlying cycles are isomorphic to the underlying

Cayley graph. Figure 5 and Figure 6 are examples of Theorem 3.2, while Figures 7 and 8 illustrate the undirected version of Theorem 3.2.

The following result was proven by A. Heus in 2008 [4].

**Theorem 3.3** ([4]). *The cartesian product $C_{n_1} \times \cdots \times C_{n_k}$ of $k$ cycles is Hamiltonian.*

Notice this result is in regards to the undirected Cartesian product of cycles. In this type of graph we have the freedom of traveling in any direction over its edges and this makes it easier to identify a Hamiltonian cycle that works for all such graphs. The graphs from Figure 7 and Figure 8 are examples that corroborate this theorem.

The following result was found and proven by Mary Stelow in 2017[3].

**Theorem 3.4** ([3]). *For any minimal generating set $A$ of a finite Dedekind group $\mathcal{G}$, there exists a Hamiltonian $A$-path on $\mathcal{G}$. Thus all Cayley digraphs of $\mathcal{G}$ have a Hamiltonian path.*

A **Dedekind** group is a group in which every subgroup is normal. We previously stated all abelian groups are Dedekind groups. The converse is not true. For example, Hamiltonian groups are non-abelian groups whose subgroups are all normal. Let $\mathcal{G}$ be a group of finite order with minimal generating set $A$. A sequence $S$ of elements in $\mathcal{G}$, $(s_1, s_2, \cdots, s_n)$, is called a **Hamiltonian $A$-path** on $\mathcal{G}$ if $s_1, s_2, \cdots, s_n$ are elements of $A$, and if the partial products of sequential elements, $s_1, s_1s_2, \cdots, \prod_{i=1}^{n} s_i = 1$, are all unique elements of $\mathcal{G}$ [3]. Using the minimal generating set and the structure that normal subgroups provide, the proof for the previous theorem succeeds in finding a Hamiltonian path for each group that fits its requirements. Notice though this theorem does not indicate Dedking groups are Hamiltonian. Nevertheless, identifying a Hamiltonian path in this type of graph is quite impressive.

The survey also contains results in regards to special-ordered groups. The following theorem was produced by K Kutnar and D. Marusic in 2012 [4].

**Theorem 3.5** ([4]). *Let $\mathcal{G}$ be a finite group. Every connected Cayley graph on $\mathcal{G}$ has a Hamiltonian cycle if $|\mathcal{G}|$ has any of the following forms (where $p, q,$ and $r$ are distinct primes):*

- $kp$, *where* $1 \le k < 32$,
- $kpq$, *where* $1 \le k \le 5$,
- $pqr$,
- $kp^2$, *where* $1 \le k \le 4$,
- $kp^3$, *where* $1 \le k \le 2$.

This theorem served as inspiration for the case we focus on in this study. This will be further explored in the following chapter. It is worth noting that this theorem affirms Hamiltonicity of the underlying Cayley graph, and that $\mathcal{G}$ need not be abelian.

# 4 Main Result

The next two results helped lead us to the specific case and approach of this study. In [3], Mary Stelow proved the following lemma.

**Lemma 4.1.** *If $\mathcal{G} = \mathbb{Z}_m \times \mathbb{Z}_n$, and $\Gamma$ is a Cayley digraph of $\mathcal{G}$ on the generators $(1, 0)$ and $(0, 1)$, $\Gamma$ has a Hamiltonian cycle if $m|n$.*

By Theorem 3.2 we know $C_m \times C_n \cong \overrightarrow{Cay}_\Delta(\mathbb{Z}_m \times \mathbb{Z}_n)$ when $\Delta = \{(0, 1), (1, 0)\}$. Notice, the condition in Lemma 4.1 is $m|n$. We sought to change this condition in hopes of determining Hamiltonicity for a different set of Cayley digraphs, one that depends on the order of $\mathcal{G}$.

In 1978, Trotter and Erdös provided the first result that exposed the necessary conditions for $C_n \times C_m$ to be Hamiltonian. Theorem 4.2 is that powerful result [4].

**Theorem 4.2.** *The direct product $C_n \times C_m$ of directed cycles is Hamiltonian if and only if the greatest common divisor, $d$, of $n$ and $m$ is at least two and there exist positive integers $d_1, d_2$ so that $d_1 + d_2 = d$ and gcd$(n, d_1)$=gcd$(m, d_2)$=1.*

The proof for this theorem uses concepts from number theory, such as the Diophantine Equation, to identify a Hamiltonian cycle that works for all such direct products. We offer a different approach; one that relies more on the properties of the graph itself.

In this study, we look at the case when $\mathcal{G} = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ where $p$ and $q$ are distinct primes and $q \neq 2$. Notice our case differs from that of Theorem 3.5 for two reasons: Theorem 3.5 is in regards to the underlying Cayley graph, not digraph, and secondly, the form $p^2q$ is unlike any stated in the theorem. Our case differs from Lemma 4.1 in that $m \nmid n$. For this study, we pick the generating set, $\Delta$, such that $\Delta = \{(1, 0), (0, 1)\}$. There are few theorems that determine Hamiltonicity for cyclic groups using the minimal or standard generating set. There are even less results using the generating set we use here. As previously mentioned, the generating set chosen determines the size of a graph. From now on, we will let $\Gamma$ be the Cayley digraph of $\mathcal{G}$ with respect to $\Delta$.

Figure 9 (below) is $\Gamma_1$ when $\mathcal{G} = \mathbb{Z}_4 \times \mathbb{Z}_3$. See Figure 7 for the Hamiltonian, underlying graph of $\Gamma$.

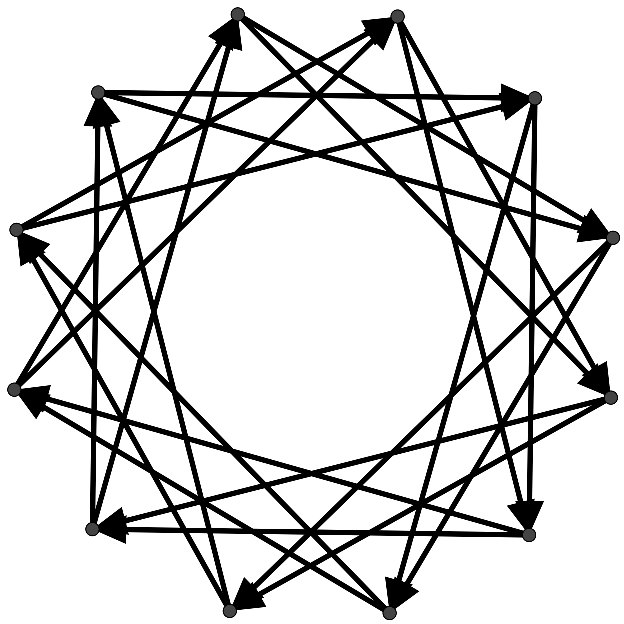

Figure 9: $\Gamma_1$ when $\mathbb{Z}_4 \times \mathbb{Z}_3$

The graph below is $\Gamma_2$ where $\mathcal{G} = \mathbb{Z}_4 \times \mathbb{Z}_5$. We can see in Figures 9 and 7 a key feature in this type of Cayley digraph that helps us obtain our final result. Notice that each vertex is part of both a directed $p^2-$cycle and a directed $q-$cycle. Moreover, each vertex is unique to these cycles. In fact, it is the case that this is true for

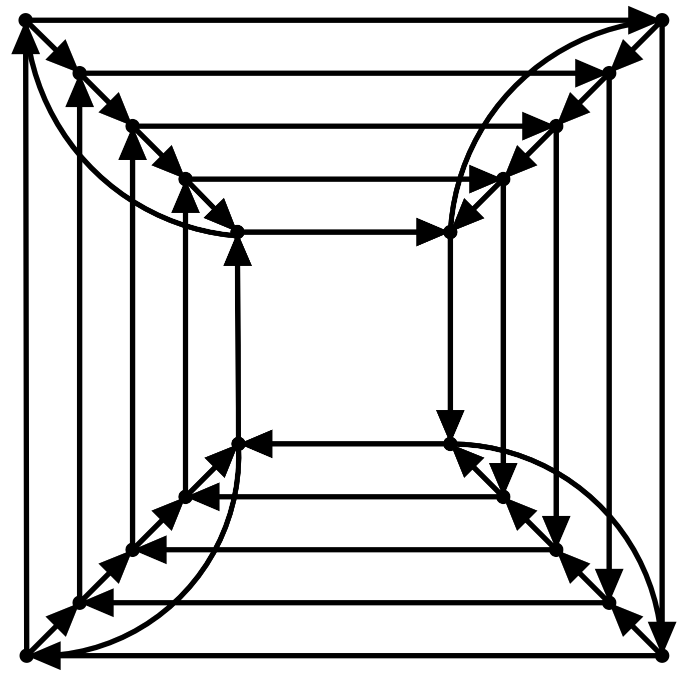

Figure 10: $\Gamma$ where $\mathcal{G} = \mathbb{Z}_4 \times \mathbb{Z}_5$

**Lemma 4.3.** *Let* $\Delta = \{(1,0),(0,1)\}$ *and* $\mathcal{G} = \mathbb{Z}_m \times \mathbb{Z}_n$ *where* $m, n > 2$*. Then each vertex of* $\Gamma = \overrightarrow{Cay}_\Delta(\mathcal{G})$ *is unique to one m-cycle and one n-cycle.*

*Proof.* Let $v_1, v_2 \in V(\mathcal{G})$ such that $v_1 \neq v_2$. Suppose $v_1$ and $v_2$ belong to the same $m-$cycle and $n-$cycle in $\Gamma$. Let $v_1 = (a_1, b_1)$. Let us apply (1,0) to $v_1$ and the resulting vertices until we have the directed $m-$cycle to which $v_1$ belongs. This directed cycle consists of the vertices $\{(a_i, b_1)\}_{i=1}^m$. Since $v_2$ belongs to the same cycle, we have that $v_2 = (a_j, b_1)$ for some $1 < j \leq m$. Now we apply $(0,1)$ to $v_1$ and the resulting vertices to find the $n-$cycle to which $v_1$ belongs. The vertices in this directed cycle are $\{(a_1, b_i)\}_{i=1}^n$. Since $v_2$ belongs to this cycle, we see that $v_2 = (a_1, b_k)$ for some $1 < k \leq n$. But $v_2 = (a_j, b_1) \neq (a_1, b_i)$, a contradiction. □

We will show that $\Gamma = \overrightarrow{Cay}_\Delta(\mathcal{G})$ when $\mathcal{G} = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ with the above requirements, is non-Hamiltonian. Lemma 4.3 plays an important role in our result. It is what gives us the necessary contradiction.

**Theorem 4.4.** *Let $\Gamma = \overrightarrow{Cay}_{\Delta}(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$, with distinct primes $p$ and $q$ such that $q \neq 2$, where $\Delta = \{(1,0),(0,1)\}$. Then $\Gamma$ is not Hamiltonian.*

*Proof.* Suppose to the contrary that $\Gamma$ is Hamiltonian. Then there exists a directed $n$-cycle in $\Gamma$, $\overrightarrow{C_n}$, where $n = p^2q$. From this directed cycle we will build $\Gamma$. Let $\{v_1, v_2, \cdots v_n\}$ be the set of vertices of $\overrightarrow{C_n}$ with vertex sequence $(v_1, v_2, \cdots, v_n, v_1)$. Then $\exists\, c \in \mathbb{N}$ such that $p^2q + 1 = c(p^2 + q - 2) + 1 + r$ where $0 \leq r < p^2 + q - 2$.

Suppose $r = 0$, then $p^2q + 1 = c(p^2 + q - 2) + 1$ or $p^2 = \frac{c(q-2)}{q-c}$. Note $q - 2$ is odd. If $c$ is odd, then $q - c$ is even and $c(q-2)$ is odd. But then $\frac{c(q-2)}{q-c} = p^2 \notin \mathbb{Z}$. Thus $c$ is even. Therefore $q - c$ is odd and $c(q-2)$ is even. We have that $p^2 = \frac{c(q-2)}{q-c} \in 2\mathbb{Z}$. Then $p^2$ must be 4. So we see $4q + 1 = c(4 + q - 2) + 1$ or $q = \frac{2c}{4-c}$. Since $c$ is even, and $c \neq 4$, then $c = 2$. Then we get $q = 2$, a contradiction. Therefore, $r \neq 0$ and $0 < r < p^2 + q - 2$.

Each vertex is uniquely a part of each $p^2-$cycle and $q-$cycle, by Lemma 4.3. Thus we add an arc, WLOG, to form the $p^2$-cycle to which $v_1$ belongs, from $v_{p^2}$ to $v_1$. Then to create the $q$-cycle to which $v_{p^2}$ belongs, we must add an arc from $v_k$ to $v_{p^2}$ where $k = p^2 + q - 1$. Similarly, to form the $p^2$-cycle to which $v_k$ belongs, we are forced to add an arc from $v_l$ to $v_k$ where $l = 2p^2 + q - 2$. Then to create the $q$-cycle to which $v_l$ belongs, we must add an arc from $v_m$ to $v_l$ where $m = 2p^2 + 2q - 3$. We continue to add arcs in this forced manner until we have $c$ $p^2$-cycles and $c$ $q$-cycles. The forced arc we added last is the arc from $v_g$ to $v_f$ where $f = cp^2 + (c-1)q - 2c$ and $g = cp^2 + cq - 2c + 1$. This arc creates the $q$-cycle to which $v_g$ belongs. On $\overrightarrow{C_n}$, $(v_g, v_{g+1}, \cdots, v_1)$ forms a path of length $r$.

Notice if $r \leq p^2$, then adding the forced arc from $v_h$ to $v_g$ where $h = (c+1)p^2 + cq - 2c$ creates a $p^2-$cycle. We have then that $v_h = i \mathrm{mod} n$ for some $1 \leq i < p^2$. Thus, $v_1$ is a part of this $p^2$-cycle as well, a contradiction.

This means $p^2 < r < p^2 + q - 2$. To create the $p^2$-cycle to which $v_g$ belongs, we must add an arc from $v_h$ to $v_g$ where $h = (c+1)p^2 + cq - 2c$. Then, to create the $q-$cycle to which $v_h$ belongs, we are forced to add an arc from $v_j$ to $v_h$ where $j = (c+1)p^2 + (c+1)q - 1 = i \bmod n$ for some $2 \leq i < q$. But then $v_1$ and $v_2$ form part of the same $p^2$-cycle and $q$-cycle, a contradiction.

$\therefore \Gamma$ is not Hamiltonian.

□

# 5 Conclusion

In regards to the Lovász Conjecture, it is my belief that this problem will remain open for several years to come. The large number of groups up to isomorphism with a given number of vertices is one of the things that influence this belief. As previously mentioned, if a Cayley digraph is Hamiltonian then its underlying graph is Hamiltonian. So if we make progress as a collective to determine Hamiltonicity for all Cayley digraphs, then we will of course make progress on the famed conjecture.

Additionally, I hope to prove that $\Gamma$ is non-Hamiltonian where $\mathcal{G} = \mathbb{Z}_m \times \mathbb{Z}_n$, $\Delta = \{(1,0),(0,1)\}$, and $\gcd(m,n) = 1$. We know these Cayley digraphs are non-Hamiltonian by Theorem 4.2. Later on, I would like to prove the Caylye digrpah of this group is non-Hamiltonian regardless of what the generating set is.

However, the nuance of proving that a Cayley digraph of a given group is non-Hamiltonian versus proving that a similar Cayley digraph *is* Hamiltonian has proven to be an interesting challenge. Most of the known results for Cayley digraphs provide conditions for Hamiltonicity. The results we hope to achieve in the future would provide conditions for non-Hamiltonicity. It is my hope to use a similar technique of attempting to construct $\Gamma$ from the directed $mn-$cycle and finding a contradiction. I believe this can work for $\Gamma$ when $G = \mathbb{Z}_p \times \mathbb{Z}_q$ but the proof for $\Gamma$ when $G = \mathbb{Z}_m \times \mathbb{Z}_n$ does not seem as intuitive.

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