Two Approaches to Clifford's Theorem

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ABSTRACT

This is a comparison of two approaches to Clifford's theorem where a representation of a normal subgroup of a group is induced up to the group and then restricted down to the normal subgroup. The first approach utilized was a character based approach and the second was a vector space approach. Each approach will be followed by several results, and a comparison between the two approaches will be made. Several examples of finite groups will illustrate the character approach of Clifford's theorem. Finally, a key result of the character approach will be used to find irreducible representations of a subgroup of $GL_2(\mathbb{F}_q)$.

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1 Introduction

This paper contains an exploration of Clifford's theorem and its results by using a vector space approach and a character approach. It is common to see a character approach to Clifford's theorem, however, it is uncommon to see an approach using the vector space of a representation and its decomposition. Clifford's theorem involves taking a normal subgroup of a finite group, G, and shows what happens when an irreducible representation of the normal subgroup is induced up to G and then restricted to the normal subgroup.

In chapter 2 we will introduce necessary definitions, theorems, propositions, and lemmas, that will aid us in many proofs in later sections. This section discusses what a representation and character are and how we can connect the two approaches.

Then, in chapter 3 we will explore the character approach to Clifford's theorem, results that follow this theorem, and examples. Clifford's theorem, from this approach, shows what happens when taking an irreducible character of a group G and restricting that character to a normal subgroup H, how the restricted character will break apart into the sum of irreducible constituents of a character of H. A large result that follows from this theorem will give the tools to find irreducible representations of a group from inducing representations from, what will be later defined, the inertia group. We will illustrate the use of this result later in chapter 5.

In chapter 4, we will take a different approach to Clifford's theorem and instead of looking at the characters themselves, we will look at the representations and the vector space of the representation. This approach will discuss what happens when we take an irreducible representation of a normal subgroup $N$of a group G and then restrict the induced representation to N. Also in this section, we will make a connection between the two approaches.

Finally, chapter 5 explores what happens when we apply Clifford's theorem to a specific group. We will be taking a subgroup of $GL_2(\mathbb{F}_q)$, which is made up of upper triangular matrices, and a normal subgroup of this group and look at two specific cases. We will explore the case where $q = 3$ and where $q = 5$, and then conclude what happens when $q = p$, for a prime p.

2 Background

In this chapter we will explore the necessary background information to compare Clifford's Theorem between a vector space approach and a character approach. This chapter will include basic definitions, theorems, and several remarks. Some proofs will be included in this chapter, while some proofs will be omitted. This chapter is primarily based on [4]. However, some theorems, remarks, and other necessary background are based on different courses taken and helpful notes from advisors.

2.1 Representation Theory Background

This section will discuss the definition of a representation, Maschke's Theorem followed by a proof, Schur's Lemma followed by a proof, and the discussion of Frobenius Reciprocity.

Definition 2.1. A representation of a group G is a homomorphism $\mu : G \to GL(V)$ for some complex vector space V. The degree of μ is the dimension of V.

Note, since V is a complex vector space, $V \cong \mathbb{C}^n$.

Remark 2.2. The group $GL_n(\mathbb{C})$ is isomorphic to $GL(\mathbb{C}^n)$.

Definition 2.3. Let $\mu : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations of G. The two representations, μ and σ, are equivalent if there exists an isomorphism of vector spaces $T : V \to W$ such that

$$\sigma(g) = T\mu(g)T^{-1}$$

for all $g \in G$. We denote two equivalent representations by $\phi \cong \sigma$.

Definition 2.4. Let $\mu : G \to GL(V)$ be a representation and $W \leq V$. The W is G-invariant if, for all $g \in G$ and $w \in W$, $\mu(g)(w) \in W$.

Definition 2.5. Let $\mu : G \to GL(\mathbb{C}^n)$ be a non-zero representation of G. Then, μ is irreducible if the only G-invariant subspaces of $\mathbb{C}^n$ are $\mathbb{C}^n$ and $\{0\}$. If μ is not irreducible, we say μ is reducible.

Lemma 2.6. Let $\mu : G \to GL(V)$ be a representation of G. If μ is equivalent to an irreducible representation of G, then μ is irreducible.

If $\mu : G \to GL(V)$ is a representation of G and W is a subspace of V where W is a G-invariant subspace, we can restrict the representation μ to W to get $\mu \mid_W : G \to GL(W)$ by defining

$$\mu \mid_W (g)(w) = \mu(g)(w)$$

Also, if $\mu : G \to GL(V)$ is a representation of G and H is a subgroup of G, we can restrict μ to H, by $\operatorname{Res}_H^G \mu : H \to GL(V)$, by

$$\operatorname{Res}_H^G \mu(h)(v) = \mu(h)$$

for $h \in H$. Note, $\operatorname{Res}_H^G \mu$ is a representation of H.

Definition 2.7. Let $\mu : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations of G. Then the direct sum of μ and σ, $\mu \oplus \sigma : G \to GL(V \oplus W)$, is given by,

$$(\mu \oplus \sigma)(g)(v, w) = (\mu(g)(v), \sigma(g)(w))$$

for $v \in V$ and $w \in W$.

Definition 2.8. Let $\mu : G \to GL(\mathbb{C}^n)$ be a representation for a group G. Then, μ is said to be completely reducible if $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where V_i is a G-invariant subspace and $\mu \mid_{V_i}$ is irreducible for all $i = 1, \ldots, n$.

Similarly to the previous lemma, we also have a lemma regarding a representation equivalent to a completely reducible representation.

Lemma 2.9. Let $\mu : G \to GL(V)$ be a representation of G. If μ is equivalent to a completely reducible representation of G, then μ is completely reducible.

Theorem 2.10. *Every representation of a finite group is completely reducible.*

Proof. Let $\mu : G \to GL(V)$ be a representation and let G be a finite group. If $\dim(V) = 1$, then μ has no non-zero proper subspaces, which impliesμ is irreducible. Assume true for $\mu : G \to GL(V)$ with $\dim(V) = n$. We will show the result is true for $\mu : G \to GL(V)$ with $\dim(V) = n + 1$. If μ is irreducible, then we are done. Otherwise, μ is decomposable by

Corallary. So, let μ be decomposable. Then $V = V_1 \oplus V_2$, where $V_1, V_2 \neq 0$. By definition, V_1 and V_2 are G-invariant subspaces and $\dim(V_1), \dim(V_2) < \dim(V)$. So, $\mu \cong (\mu\mid_{V_1} \oplus \mu\mid_{V_2})$. Let $V_1 = U_1 \oplus \cdots \oplus U_s$ and $V_2 = W_1 \oplus \cdots \oplus W_r$. U_i, W_j are G-invariant subspaces and the subrepresentations $\mu\mid_{U_i}, \mu\mid_{W_j}$ are irreducible representations. So

$$\begin{aligned} V &= V_1 \oplus V_2 \\ &= U_1 \oplus \cdots \oplus U_s \oplus W_1 \oplus \cdots \oplus W_r \end{aligned}$$

Hense μ is completely reducible. So every representation of a finite group is completely reducible. □

Based on Maschke's theorem, since all representations of finite groups are completely reducible, we must find the irreducible representations of a group to understand all representations of a group.

Definition 2.11. If $\mu \cong m_1\sigma_1 \oplus \cdots \oplus m_n\sigma_n$, then m_i is called the multiplicity of σ_i in μ. If $m_i > 0$, then σ_i is an irreducible constituent of μ.

Definition 2.12. Let $\mu : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations of G. A morphism from μ to σ is a linear map $T : V \to W$ such that

$$T\mu(g) = \sigma(g)T$$

for all $g \in G$. The set of all morphisms from μ to σ is denoted $\mathrm{Hom}_G(\mu, \sigma)$.

In this next example we will determine all irreducible representations of $\mathbb{Z}_n$. Later on, we will take these irreducible representation and induce them up to a subgroup of $GL_2(\mathbb{F}_q)$.

Example 2.13. To determine all irreducible representations of $\mathbb{Z}_n$, we will first determine all homomorphisms of $\mathbb{Z}_n$. Let $\mu_m : \mathbb{Z}_n \to \mathbb{C}^*$. Note $\mathbb{Z}_n$ is cyclic. So, once we know where the generator is mapped to, we can find the rest of the map. Now, say $\langle x \rangle = \mathbb{Z}_n$. Then, $x^n = 1$ so, $z^n = 1 \implies z = e^{2\pi i/n}$ for

$m = 0, 1, \ldots, n-1$ where $z \in \mathbb{C}^*$. Consider, $\mathbb{Z}_n \to \mathbb{C}^*$ are

$$\mu_m(x^k) = e^{2\pi i mk/n} \quad m = 0, 1, \ldots, n-1$$

This gives us n maps and permutes the generator of $\mathbb{Z}_n$ around to each nth root of unity. So, all homomorphisms $\mu : \mathbb{Z}_n \to \mathbb{C}^*$ are of the form,

$$\mu_m(x^k) = e^{2\pi i mk/n} \quad m = 0, 1, \ldots n-1$$

and therefore, all irreducible representations of $\mathbb{Z}_n$ are of this form.

The next theorem we will look at, Schur's Lemma, will make a connection between equivalent representations and the hom space of the representations.

Theorem 2.14. *Let μ, σ be irreducible representations and $T \in \mathrm{Hom}_G(\mu, \sigma)$. Define $\mu : G \to GL(V)$, $\sigma : G \to GL(W)$. Let $T : V \to W$ be given by $T(\mu(g)v) = \sigma(g)T(v)$. Then T is invertible or $T = 0$ if and only if*
a.) If $\mu \ncong \sigma$, then $\mathrm{Hom}_G(\mu, \sigma) = \{0\}$
b.) If $\mu \cong \sigma$, then T is invertible. So $T = \lambda I$ for $\lambda \in \mathbb{C}$.

Proof. Let $\mu : G \to GL(V)$, $\sigma : G \to GL(W)$ and $T : V \to W$. Let $T \in \mathrm{Hom}_G(\mu, \sigma)$. If $T = 0$ we are done. So, say $T \neq 0$. We want to show T is invertible. $kerT = \{v \in V | T(v) = 0\} \subseteq V$ and $kerT \subseteq V$ is G-invariant. Since V is irreducible that implies $kerT = V$ or $kerT = 0$. If $kerT = V$, then $T = 0$. This is a contradiction since we assumed $T \neq 0$. So, $kerT = 0$, implying T is injective. The $Im(T) = \{T(v) | v \in V\}$ is a G-invariant subspace of W and W is an irreducible representation. So, $Im(T) = W$ or $Im(T) = 0$. If $Im(T) = 0$, then $T = 0$. Again, this is a contradiction. So, $Im(T) = W$. So, T is surjetive. Hence, T is invertible.

Now, assume T is invertible and $\dim(V) = \dim(W)$. Let $\mu \cong \sigma$. Note $\mathrm{Hom}_G(\mu, \sigma) = \mathrm{Hom}_G(V, W)$. $V \cong W$, so we will treat $V = W$ and $\mu = \sigma$. So, $T \in \mathrm{Hom}_G(\mu, \sigma) =$

$\mathrm{Hom}_G(\mu, \mu)$. Now,

$$\begin{aligned} I &\in \mathrm{Hom}_G(\mu, \mu) \\ \lambda T &\in \mathrm{Hom}_G(\mu, \mu) \\ \lambda I - T &\in \mathrm{Hom}_G(\mu, \mu) \end{aligned}$$

but $\lambda I - T$ is not invertible, implying

$$\begin{aligned} \lambda I - T &= 0 \\ T = \lambda I \end{aligned}$$

□

Remark 2.15. Say μ and σ are irreducible representations. Then, $\mu \cong \sigma$ if and only if $\dim(\mathrm{Hom}_G(\mu, \sigma)) = 1$.

Corollary 2.16. For any abelian group G, all irreducible representations of G have degree one.

Proof. Say $\mu : G \to GL(V)$ is irreducible. For $h \in G$, let $T = \mu(h)$. Then,

$$\begin{aligned} T\mu(g) &= \mu(h)\mu(g) \\ &= \mu(hg) \\ &= \mu(gh) \\ &= \mu(g)\mu(h) \\ &= \mu(g)T \end{aligned}$$

for all $g \in G$. So, by Schur's lemma, $\mu(h) = \lambda_h I$ for some $\lambda_h \in \mathbb{C}$. Now, for $v \in V$ and $k \in \mathbb{C}$,

$$\mu(h)(kv) = \lambda_h I(kv) = \lambda_h kv$$

Note, $\lambda_h kv \in \mathbb{C}v$. So, $\mathbb{C}v$ is a G-invariant subspace. Now μ is an irreducible representation, thus $V = \mathbb{C}v$, and so $\dim(V) = 1$. $\square$

Definition 2.17. Let H be a subgroup of G and $\mu : H \to GL(V)$ be a representation of H. Then, $\mu^G : G \to GL(V^G)$ defined by,

$$\left(\mu^G(g)f\right)(g') = f(g'g)$$

is the induced representation from H to G, where

$$V^G = \{f : G \to V \mid f(hg) = \mu(h)f(g) \ \ \forall h \in H, g \in G\}$$

We will now show that μ^G is a representation. First we will show $\mu^G(g)f \in V^G$. Let $g, g' \in G$ and $h \in H$. Then,

$$\begin{aligned}
\left(\mu^G(g)f\right)(hg') &= f(hg'g) \\
&= \mu(h)f(g'g) \\
&= \mu(h)\left(\mu^G(g)f\right)(g')
\end{aligned}$$

So, $\mu^G(g)f \in V^G$. Now show $\left(\mu^G, V\right)$ is a representation. Let $g, g', g'' \in G$. Then,

$$\begin{aligned}
\left(\mu^G(gg')f\right)(g'') &= f(g''gg') \\
&= \left(\mu^G(g')f\right)(g''g) \\
&= \left(\mu^G(g)\left(\mu^G(g')f\right)\right)(g'')
\end{aligned}$$

Thus, μ^G is a homomorphism. We now need to show that $\mu^G(g) : V^G \to V^G$ is a linear map. Let $f_1, f_2 \in V^G$ and $c \in \mathbb{C}$. Then,

$$\begin{aligned}
\mu^G(g)(cf_1 + f_2)(g') &= (cf_1 + f_2)(g'g) \\
&= c(f_1(g'g)) + f_2(g'g) \\
&= c\mu^G(g)(f_1)(g') + \mu^G(g)(f_2)(g')
\end{aligned}$$

So, $\mu^G(g)$ is a linear map. Lastly, we need to show $\mu^G(g)$ is an invertible map. We will do so by showing the $ker(\mu^G(g)) = \{0\}$. Now, $ker(\mu^G(g)) = \{f \in V^G \mid \mu^G(g)(f) = 0\}$. So let $f \in ker(\mu^G(g))$. Then,

$$\begin{aligned}
\mu^G(g)(f)(g') &= f(g'g) \\
&= 0
\end{aligned}$$

Now, $\mu^G(g)$ is a linear transformation which maps the identity to the identity. Thus, $f = 0$. So, $\mu^G(g) \in GL(V^G)$. Therefore, μ^G is a representation. Note, we also denote the induced representation by $\operatorname{Ind}_H^G \mu$.

Theorem 2.18. *Say $N \leq H \leq G$ and $\sigma : N \to GL(V)$ is representation of N, we have the following property of the induced representation,*

$$\operatorname{Ind}_H^G \operatorname{Ind}_N^H(\sigma) \cong \operatorname{Ind}_N^G(\sigma)$$

This is called the transitivity of induction.

The next theorem, Frobenius Reciprocity, will describe the relationship between induced representations and restricted representations of a group and its subgroup.

Theorem 2.19. *Let H be a subgroup of G, $\mu : H \to GL(V)$ be a representation of H, and $\sigma : G \to GL(V)$ be a representation of G. Then,*

$$\operatorname{Hom}_G\left(\sigma, \operatorname{Ind}_H^G \mu\right) \cong \operatorname{Hom}_H\left(\operatorname{Res}_H^G \sigma, \mu\right)$$

2.2 Character Theory Background

This section will discuss basic definitions, theorems, and lemmas that we will use to describe the character theory approach to Clifford's Theorem. This section will include the definition of a character and Frobenius Reciprocity with a proof.

Definition 2.20. For $\mu : G \to GL_n(\mathbb{C})$ a representation, we define the character of μ by $\varphi_\mu : G \to \mathbb{C}$ where,

$$\varphi_\mu(g) = Tr\left(\mu(g)\right)$$

When the representation is understood, we will drop the subscript and denote the character of the representation by just φ. The character function is a well defined function. To see φ_μ is well defined, for $g, g' \in G$, say $g = g'$. Then,

$$\begin{aligned} & g = g' \\ \implies \quad & \mu(g) = \mu(g') \\ \implies \quad & Tr(\mu(g)) = Tr(\mu(g')) \\ \implies \quad & \varphi_\mu(g) = \varphi_\mu(g') \end{aligned}$$

Thus, the character function is a well defined function.

Definition 2.21. Let $(\mu, V), (\sigma, W)$ be representations with characters $\varphi_\mu, \varphi_\sigma$. Then, the inner product of the characters is

$$\langle \varphi_\mu, \varphi_\sigma \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi_\mu(g) \overline{\varphi_\sigma(g)}$$

The following proposition says the character of a representation depends on the equivalence class of the representation.

Proposition 2.22. Let $\mu, \sigma : G \to GL(\mathbb{C}^n)$ be equivalent representations. Then $\varphi_\mu = \varphi_\sigma$.

Proof. Let $\mu, \sigma : G \to GL(\mathbb{C}^n)$ be equivalent representations. Then, $\exists\ T : \mathbb{C}^n \to \mathbb{C}^n$ such that for all $g \in G$

$$\begin{aligned} \mu(g)T &= T(\sigma(g)) \\ \implies \mu(g) &= T\sigma(g)T^{-1} \end{aligned}$$

So,

$$\begin{aligned} \varphi_\mu(g) = Tr(\mu(g)) &= Tr(T\sigma T^{-1}) \\ &= Tr(T^{-1}T\sigma(g)) \\ &= Tr(\sigma(g)) \\ &= \varphi_\sigma(g) \end{aligned}$$

□

A similar proof results in characters being constant on conjugacy classes. This leads to our next proposition.

Proposition 2.23. Let $\mu : G \to GL_n(\mathbb{C})$ be a representation. Then for all $g, h \in G$

$$\varphi_\mu(g) = \varphi_\mu(hgh^{-1})$$

Proof. Let $\mu : G \to GL_n(\mathbb{C})$ be a representation. Let $g, h \in G$. Then,

$$\begin{aligned} \varphi_\mu(hgh^{-1}) &= Tr(\mu_{hgh^{-1}}) \\ &= Tr(\mu_h\mu_g\mu_{h^{-1}}) \\ &= Tr(\mu_{h^{-1}}\mu_h\mu_g) \\ &= Tr(\mu_g) = \varphi_\mu(g) \end{aligned}$$

□

Definition 2.24. A function $f : G \to \mathbb{C}$ is a class function if $f(g) = f\left(hgh^{-1}\right)\ \forall\, g, h \in G$. Equivalently f is constant on conjugacy classes.

By Maschke's Theorem, we know all representations of finite groups are completely reducible and therefore, isomorphic to the direct sum of irreducible representations. So we need to know how to find the characters of a direct sum of representations.

Proposition 2.25. Let μ be a representation of a group G, and $\mu \cong \sigma_1 \oplus \sigma_2$. Then $\varphi_\mu = \varphi_{\sigma_1} + \varphi_{\sigma_2}$.

Proof. Let μ be a representation of a group G, and $\mu \cong \sigma_1 \oplus \sigma_2$. Then, for $g \in G$,

$$\begin{aligned}
\varphi_\mu(g) &= Tr(\mu(g)) \\
&= Tr(\sigma_1(g) + \sigma_2(g)) \\
&= Tr(\sigma_1(g)) + Tr(\sigma_2(g)) \\
&= \varphi_{\sigma_1} + \varphi_{\sigma_2}
\end{aligned}$$

Thus, $\varphi_\mu = \varphi_{\sigma_1} + \varphi_{\sigma_2}$. $\square$

Note, if $\mu \cong m_1\sigma_1 \oplus \cdots \oplus m_n\sigma_n$ is the complete set of irreducible representations of μ,

$$\varphi_\mu = m_1\varphi_{\sigma_1} + \cdots + m_n\varphi_{\sigma_n}$$

where m_i is the multiplicity of σ_i in μ.

Definition 2.26. If $\varphi = \sum_{i=1}^{k} n_i\varphi_i$ is a character, where n_i is the multiplicity of φ_i, then those φ_i with $n_i > 0$ are called the irreducible constituents of φ.

Theorem 2.27. *Let μ and σ be irreducible representations of G. Then*

$$\langle \varphi_\mu, \varphi_\sigma \rangle = \begin{cases} 1 & \text{if } \mu \cong \sigma \\ 0 & \text{if } \mu \ncong \sigma \end{cases}$$

Hence, the irredicible characters of G form an orthonormal set of class functions.

Proposition 2.28. Let μ be a representation. μ is irreducible if and only if $\langle \varphi_\mu, \varphi_\mu \rangle = 1$.

Proof. Let $\mu : G \to GL(V)$ be a representation. Let $\mu_1, \mu_2, \dots, \mu_{(i)}$ be a complete set of irreducible representations of G. Let $\varphi_i = \varphi_{\mu_i}$. So, there exists a unique $m_i \in \mathbb{Z}, m_i \geq 0$ such that

$$\begin{aligned} \mu &= m_1\mu_1 \oplus m_2\mu_2 \oplus \cdots \oplus m_s\mu_s \\ \varphi_\mu &= m_1\varphi_1 + m_2\varphi_2 + \cdots + m_s\varphi_s \end{aligned}$$

μ is irreducible if and only if $\mu \cong \mu_i$ for some i by the previous theorem. So,

$$\begin{aligned} \langle \varphi_\mu, \varphi_\mu \rangle &= \langle m_1\varphi_1 + m_2\varphi_2 + \cdots + m_s\varphi_s,\ m_1\varphi_1 + m_2\varphi_2 + \cdots + m_s\varphi_s \rangle \\ &= m_1^2\langle \varphi_1, \varphi_1 \rangle + m_1m_2\langle \varphi_1, \varphi_2 \rangle + \cdots + m_s^2\langle \varphi_s, \varphi_s \rangle \\ &= m_1^2 + \cdots + m_s^2 \end{aligned}$$

If μ is irreducible, $\varphi_\mu = \varphi_{\mu_i}$ for some i. So $\mu \cong \mu_i$ implying $m_1 = 0, \dots, m_i = 1, \dots m_s = 0$. So $\langle \varphi_\mu, \varphi_\mu \rangle = 1$. If $\langle \varphi_\mu, \varphi_\mu \rangle = 1$, then there exists some i such that $m_i = 1$ and $m_j = 0$ $\forall j \neq i$. So, $\varphi_\mu = \varphi_{\mu_i}$ implying $\mu \cong \mu_i$. Hence, μ is irreducible if and only if $\langle \varphi_\mu, \varphi_\mu \rangle = 1$. □

If φ is a character of a representation of G and H is a subgroup of G. We denote the restriction of φ to a character of H by, φ_H.

Theorem 2.29. *Let H be a subgroup of G and φ be a character of a representation of H. Then, φ^G given by,*

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1})$$

where

$$\varphi^\circ(h) = \begin{cases} \varphi(h) & h \in H \\ 0 & h \notin H \end{cases}$$

is the induced character from H to G.

The following theorem, Frobenius Reciprocity, will discuss the relationship between the induced character and the restricted character of a group and its subgroup. This theorem is the character version as the Frobenius Reciprocity stated earlier.

Lemma 2.30. Let $H \leq G$ and suppose φ is a character of H and that ρ is a character of G. Then,

$$\langle \varphi, \rho_H \rangle = \langle \varphi^G, \rho \rangle$$

Proof. Let $H \leq G$ and suppose φ is a character of H and that ρ is a character of G. Then,

$$\begin{aligned} \langle \varphi^G, \rho \rangle &= \frac{1}{G} \sum_{g \in G} \varphi^G(g) \overline{\rho(g)} \\ &= \frac{1}{G} \frac{1}{H} \sum_{g \in G} \sum_{x \in G} \varphi^{\circ}(xgx^{-1}) \overline{\rho(g)} \end{aligned}$$

Now, let $y = xgx^{-1}$ and note $\rho(g) = \rho(y)$. So,

$$\begin{aligned} \frac{1}{G} \frac{1}{H} \sum_{g \in G} \sum_{x \in G} \varphi^{\circ}(xgx^{-1}) \overline{\rho(g)} &= \frac{1}{G} \frac{1}{H} \sum_{y \in G} \sum_{x \in G} \varphi^{\circ}(y) \overline{\rho(y)} \\ &= \frac{1}{H} \sum_{y \in H} \varphi(y) \overline{\rho(y)} \\ &= \langle \varphi, \rho_H \rangle \end{aligned}$$

Thus, $\langle \varphi^G, \rho \rangle = \langle \varphi, \rho_H \rangle$. □

3 Character Approach

In this section we will explore the first approach to Clifford's theorem using a character based approach [2]. This chapter will discuss taking a character of a representation of a group and restricting the character to a normal subgroup. We will show how this character breaks apart into characters of the normal subgroup. Then several results of Clifford's theorem will be discuss. Lastly, we will take the groups S_3 and S_5 and show what Clifford's theorem will look like.

3.1 Clifford's Theorem

Let H be a normal subgroup of G and let $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ is the set of irreducible characters of G. If φ is a class function of H and $g \in G$, the conjugate of φ in G, denoted φ^g, is defined by the map $\varphi^g : H \mapsto \mathbb{C}$ given by,

$$\varphi^g(h) = \varphi(ghg^{-1})$$

Note, the congujate of φ in G is well-defined.

Theorem 3.1. *Let $H \trianglelefteq G$, and let φ, ρ be class functions of H. For $x, y \in G$*

(a) φ^x is a class function.

(b) $(\varphi^x)^y = \varphi^{xy}$.

(c) $\langle \varphi^x, \rho^x \rangle = \langle \varphi, \rho \rangle$.

(d) $\langle \chi_H, \varphi^x \rangle = \langle \chi_H, \varphi \rangle$ for a class function χ of G.

(e) φ^x is a character if φ is.

Proof. (a) Let $H \trianglelefteq G$, φ, ρ be class functions of H, and let $x, y \in G$. To show φ^x is a class function, we want to show $\varphi^x(h) = \varphi^x(aha^{-1})$ for all $a, h \in H$. Note, for $a \in H$ and $x \in G$,

$xa = a'x$ for some $a' \in H$. So,

$$\begin{aligned}
\varphi^x(aha^{-1}) &= \varphi(xaha^{-1}x^{-1}) \\
&= \varphi((xa)h(xa)^{-1}) \\
&= \varphi(a'xhx^{-1}(a')^{-1}) \\
&= \varphi(xhx^{-1}) \\
&= \varphi^x(h)
\end{aligned}$$

Note, $\varphi(a'xhx^{-1}(a')^{-1}) = \varphi(xhx^{-1})$ since φ is a class function and $xhx^{-1} \in H$. Therefore, φ^x is a class function and (a) is proved. (b) Next, we want to show $(\varphi^x)^y = \varphi^{xy}$. Let $h \in H$. Then,

$$\begin{aligned}
\varphi^{xy}(h) &= \varphi((xy)h(xy)^{-1}) \\
&= \varphi(xyhy^{-1}x^{-1}) \\
&= \varphi^x(yhy^{-1}) \\
&= (\varphi^x)^y(h)
\end{aligned}$$

Thus, $\varphi^{xy} = (\varphi^x)^y$ and (b) is proved. (c) To show $\langle \varphi^x, \rho^x \rangle = \langle \varphi, \rho \rangle$, we will compute the inner product.

$$\begin{aligned}
\langle \varphi^x, \rho^x \rangle &= \frac{1}{|H|} \sum_{h \in H} \varphi^x(h)\overline{\rho^x(h)} \\
&= \frac{1}{|H|} \sum_{xhx^{-1} \in H} \varphi^{(}xh^{-1})\overline{\rho(xhx^{-1})} \\
&= \frac{1}{|H|} \sum_{h \in H} \varphi(h)\overline{\rho(h)} \\
&= \langle \varphi, \rho \rangle
\end{aligned}$$

Hence, $\langle \varphi^x, \rho^x \rangle = \langle \varphi, \rho \rangle$ and (c) is proved. (d) Similarly, we can show (d), $\langle \chi_H, \varphi^x \rangle =$

$\langle \chi_H, \varphi \rangle$ for a class function χ of G. Let $h \in H$. Then,

$$\begin{aligned}
\langle \chi_H, \varphi \rangle &= \frac{1}{|H|} \sum_{h \in H} \chi_H(h) \overline{\varphi(h)} \\
&= \frac{1}{|H|} \sum_{xhx^{-1} \in H} \chi_H(xhx^{-1}) \overline{\varphi(xhx^{-1})} \\
&= \frac{1}{|H|} \sum_{h \in H} \chi_H(h) \overline{\varphi^x(h)} \\
&= \langle \chi_H, \varphi^x \rangle
\end{aligned}$$

(e) Finally, we will show φ^x is a character if φ is. Assume φ is a character. Then,

$$\begin{aligned}
\varphi^g(h) &= \varphi\left(ghg^{-1}\right) \\
&= Tr\left(\varphi\left(ghg^{-1}\right)\right) \\
&= Tr(\varphi^g(h))
\end{aligned}$$

Thus, by definition, φ^g is a character. □

Theorem 3.2. *Let $H \trianglelefteq G$ and let χ be an irreducible character of G. Let φ be an irreducible constituent of χ_H and suppose $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_t$ are the distinct conjugates of φ in G. Then,*

$$\chi_H = e \sum_{i=1}^{t} \varphi_i$$

where $e = \langle \chi_H, \varphi \rangle$

Proof. Let φ^G be the induced character of H to G. Then for $h \in H$,

$$\begin{aligned}
\varphi^G(h) &= \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xhx^{-1}) \\
&= \frac{1}{|H|} \sum_{x \in G} \varphi(xhx^{-1}) \\
&= \frac{1}{|H|} \sum_{x \in G} \varphi^x(h)
\end{aligned}$$

Now we will restrict φ^G to H,

$$\left(\varphi^G\right)_H = \frac{1}{|H|}\sum_{x\in G}\varphi^x \quad\Longrightarrow\quad |H|\left(\varphi^G\right)_H = \sum_{x\in G}\varphi^x$$

If $\rho \in Irr(H)$ and ρ is not a conjugate of φ, then $\langle\left(\varphi^G\right)_H = \sum\varphi^x, \rho\rangle = 0$. Note χ is a constituent of φ^G, since $\langle\varphi, \chi_H\rangle = \langle\varphi^G, \chi\rangle$ by Frobenius Reciprocity. It follows that $\langle\chi_H, \rho\rangle = 0$. Since any irreducible character of H that is not conjugate to φ is not an irreducible constituent of χ_H, all irreducible constituents of χ_H are among the φ_i. So,

$$\chi_H = \sum_{i=1}^{t}\langle\chi_H, \varphi_i\rangle\varphi_i$$

By 3.1, $\langle\chi_H, \varphi_i\rangle = \langle\chi_H, \varphi\rangle$. Hence,

$$\begin{aligned}
\chi_H &= \sum_{i=1}^{t}\langle\chi_H, \varphi_i\rangle\varphi_i \\
&= \sum_{i=1}^{t}\langle\chi_H, \varphi\rangle\varphi_i \\
&= \langle\chi_H, \varphi\rangle\sum_{i=1}^{t}\varphi_i \\
&= e\sum_{i=1}^{t}\varphi_i \quad \text{where } e = \langle\chi_H, \varphi\rangle
\end{aligned}$$

□

3.2 Results

In this section, several results that follow from Clifford's theorem are presented. In particular 3.6 will be used frequently in section 5 to find irreducible representations of a group from inducing specific representations of a subgroup.

Corollary 3.3. Let $H \trianglelefteq G$ and suppose that $\chi \in \mathrm{Irr}(G)$ and $\langle\chi_H, 1_H\rangle \neq 0$. Then $H \subseteq ker(\chi)$.

Lemma 3.4. Let $H \trianglelefteq G$ and suppose $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{Irr}(H)$ with $\langle\chi_H, \varphi\rangle \neq 0$. Then

$\varphi(1) \mid \chi(1)$.

Definition 3.5. Let $\varphi \in \mathrm{Irr}(H)$. Then the inertia group, $I_G(\varphi)$, is defined by,

$$I_G(\varphi) = \{g \in G \mid \varphi^g(h) = \varphi(h) \ \ \forall h \in H\}$$

Theorem 3.6. *Let* $H \trianglelefteq G$*,* $\varphi \in \mathrm{Irr}(H)$*, and* $T = I_G(\varphi)$*. Let*

$$A = \{\psi \in \mathrm{Irr}(T) \mid \langle \psi_H, \varphi \rangle \neq 0\}, \qquad B = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi_H, \varphi \rangle \neq 0\}$$

Then,

(a) If $\psi \in A$*, then* ψ^G *is irreducible.*

(b) The map $\psi \mapsto \psi^G$ *is a bijecion of* A *onto* B*.*

(c) If $\psi^G = \chi$*, with* $\psi \in A$*, then* ψ *is the unique irreducible constituent of* χ_T *which lies in* A*.*

(d) If $\psi^G = \chi$*, with* $\psi \in A$*, then* $\langle \psi_H, \varphi \rangle = \langle \chi_H, \varphi \rangle$*.*

Proof. Let $\psi \in A$ and say χ is an irreducible constituent of ψ^G. By Frobenius Reciprocity,

$$\langle \chi, \psi^G \rangle = \langle \chi_T, \psi \rangle$$

Thus, ψ is a constituent of χ_T. So, $\chi_T = \sum n_i \psi_i$ for $n_i > 0$, and note, φ is a constituent of ψ_H. So $\psi_H = \sum m_i \varphi_i$ for $m_i > 0$. Therefore,

$$\chi_H = (\chi_T)_H = \sum n_i (\psi_i)_H = \sum n_i \sum m_i \varphi_i = \sum n_i m_i \varphi_i$$

Therefore, φ is a constituent of χ_H and $\langle \chi_H, \varphi \rangle \neq 0$. So, $\chi \in B$. Let $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_t$ be the distinct conjugates of φ in G. Now T is the stabilizer of φ in the action of G on $Irr(H)$. By the orbit-stabilizer theorem, we have $t = |G : T|$. By Clifford's Theorem,

$$\chi_H = e \sum_{i=1}^{t} \varphi_i \quad e = \langle \chi_H, \varphi \rangle$$

Note, for $g \in T$, $\varphi^g(h) = \varphi(h)$, $\forall h \in H$. So, φ is T-invariant. By Clifford's Theorem,

$$\psi_H = f \sum_{i=1}^{t} \varphi_i \quad f = \langle \psi_H, \varphi \rangle$$

Since φ is T-invariant, $\psi_H = f \sum_{i=1}^{t} \varphi_i = f\varphi$. Now, $f \leq e$ (not sure of reasoning). So, we have

$$et\varphi(1) = \chi(1) \leq \psi^G(1) = t\psi(1) = tf\varphi(1) \leq et\varphi(1)$$

Since $\chi(1) = \psi^G(1)$ and χ is an irreducible constituent of ψ^G, we can conclude $\chi = \psi^G$. Now χ is an irreducible constituent of G, so ψ^G is irreduicble and (a) is proved. Also following from the equality from above,

$$e = \langle \chi_H, \varphi \rangle = \rangle \psi_H, \varphi \rangle = f$$

Thus, (d) is proven. Now, we want to show that ψ is the unique irreduicble constituent of χ_T, which lies in A. Say $\psi_1 \, inA$ where $\psi \neq \psi_1$. Note, ψ_1 is a constituent of χ_T by Frobenius Reciprocity. Then,

$$\langle \chi_H, \varphi \rangle \geq \langle (\psi + \psi_1)_H, \varphi \rangle = \langle \psi_H, \varphi \rangle + \langle (\psi_1)_H, \varphi \rangle$$

Note, $\psi_1 \in A$, so $\langle (\psi_1)_H, \varphi \rangle \neq 0$. So,

$$\langle \psi_H, \varphi \rangle + \langle (\psi_1)_H, \varphi \rangle > \langle \psi_H, \varphi \rangle$$

However, this is a contradiction. Thus, ψ is unique and (c) is proven. Finally, we want to show the map $\psi \mapsto \psi^G$ is a bijection of A onto B. By (a), the map is well defined and by part (d) the image of the map lies in B. By part (c), ψ is unique, and thus the map is injective. Let $\chi \in B$. Now φ is a constituent of χ_H, so there must be some irreducible constituent ψ of χ_T such that $\langle \chi_H, \varphi \rangle \neq 0$. Therefore, $\psi \in A$ and χ is a constituent of ψ^G since,

$$\langle \chi_T, \psi \rangle = \langle \chi, \psi^G \rangle$$

Thus, $\chi = \psi^G$ and the map is onto. Hence, the map $\psi \mapsto \psi^G$ is bijective. $\square$

3.3 Examples

In this section we will explore several examples explicitly demonstrating what Clifford's theorem looks like. The first example we will take the symmetric group of order 6 and its normal subgroup, the alternating group of order 3, and show how characters of S_3 break apart into characters of A_3 when restricted to A_3. Then, we will look at the case where our group is S_5. In this example we will see how characters of S_5 break apart into irreducible characters of A_5 when restricted to A_5.

Example 3.7. Consider $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$. Now, $A_3 = \{(1), (123), (132)\}$ is a normal subgroup of S_3. The character tables for S_3 and A_3 are the following,

Table 1: S_3 Character Table

S_3	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 2: A_3 Character Table

A_3	(1)	(123)	(132)
φ_1	1	1	1
φ_2	1	ω	ω^2
φ_3	1	ω^2	ω

In the character table of A_3, $\omega = e^{2\pi i/3}$. Now, φ_2 is an irreducible constituent of $\chi_3 \mid_{A_3}$.

The conjugates of φ_2 in A_3 are φ_2 and φ_3, and so, by Clifford's Theorem,

$$\begin{aligned}
\chi_3 \mid_{A_3} &= \langle \chi_3 \mid_{A_3}, \varphi_2 \rangle \sum_{i=2}^{3} \varphi_i \\
&= \frac{1}{3} \sum_{\sigma \in A_3} \chi_3 \mid_{A_3} (\sigma) \overline{\varphi_2(\sigma)} [\varphi_2 + \varphi_3] \\
&= \frac{1}{3} (2 + (-1)\omega + (-1)\omega^2)[\varphi_2 + \varphi_3] \\
&= \frac{3}{3} [\varphi_2 + \varphi_3] \\
&= \varphi_2 + \varphi_3
\end{aligned}$$

So, $\chi_3 \mid_{A_3} = \varphi_2 + \varphi_3$. This shows us how χ_3 restricted to A_3 breaks apart as irreducible characters of A_3.

Example 3.8. Consider $G = S_5$, and A_5 a normal subgroup of S_5. To use Clifford's theorem, we need to find the character tables for S_5 and A_5.

Table 3: S_5 Character Table

S_5	(1)	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0

Table 4: A_5 Character Table

A_5	(1)	(123)	(12345)	(12354)	(12)(34)
φ_1	1	1	1	1	1
φ_2	4	1	-1	-1	0
φ_3	5	-1	0	0	1
φ_4	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
φ_5	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1

Now, φ_2 is an irreducible constituent of $\chi_3 \mid_{A_5}$. The conjugate of φ_2 in A_5 is φ_2, and so, by Clifford's theorem,

$$\begin{aligned}
\chi_3 \mid_{A_5} &= \langle \chi_3 \mid_{A_5}, \varphi_2 \rangle \sum \varphi_2 \\
&= \frac{1}{}(4(4) + 20(1) + 12(1) + 12(1))\varphi_2 \\
&= \varphi_2
\end{aligned}$$

Now φ_4 is an irreducible constituent of $\chi_5 \mid_{A_5}$. The conjugates of φ_4 are φ_4 and φ_5. So, by Clifford's theorem,

$$\begin{aligned}
\chi_5 \mid_{A_5} &= \langle \chi_5 \mid_{A_5}, \varphi_4 \rangle \sum \varphi_2 \\
&= \frac{1}{}\left(6(3) + 12\left(\frac{1+\sqrt{5}}{2}\right)(1) + 12\left(\frac{1-\sqrt{5}}{2}\right)(1) + 15(2)\right)(\varphi_4 + \varphi_5) \\
&= \varphi_4 + \varphi_5
\end{aligned}$$

So, $\chi_5 \mid_{A_5} = \varphi_4 + \varphi_5$. This shows us how χ_5 restricted to A_5 breaks apart as irreducible characters of A_5.

4 Vector Space Approach

This chapter will discuss the second approach we will take to Clifford's theorem, which will use a vector space approach [1]. We will explore how representations decompose into direct sums of irreducible constituents when we induce irreducible representations of a normal subgroup up to a group and then restrict back down to the normal subgroup. In this chapter we will also compare the two approaches by making several connections between Clifford's theorem using the character approach and Clifford's theorem using the vector space approach. A comparison of a key result between the two approaches will also be made.

4.1 Clifford's Theorem

Let G be a group and N be a normal subgroup of G. We will denote the set of all irreducible representations of G by $\hat{G}$. Likewise, we will denote the set of all irreducible representations of N by $\hat{N}$.

Definition 4.1. Let $\sigma \in \hat{N}$ and $g \in G$. Then,

$$\hat{G}(\sigma) = \{\theta \in \hat{G} \mid \sigma \leq Res_N^G(\theta)\}$$

Definition 4.2. Let $\sigma \in \hat{N}$. For a $g \in G$, the g-conjugate of σ is the representation ${}^g\sigma \in \hat{N}$ defined by

$${}^g\sigma(n) = \sigma(g^{-1}ng)$$

for all $n \in N$.

Proposition 4.3. If $\sigma, \sigma' \in \hat{N}$, then, for $g \in G$, ${}^g(\sigma \oplus \sigma') \cong {}^g\sigma \oplus {}^g\sigma'$.

Proof. Let $\sigma, \sigma' \in \hat{N}$, $g \in G$ and $n \in N$. Then,

$$\begin{aligned} {}^{g}(\sigma \oplus \sigma')(n) &= (\sigma \oplus \sigma')(g^{-1}ng) \\ &= (\sigma(g^{-1}ng), \sigma'(g^{-1}ng)) \\ &= \sigma(g^{-1}ng) \oplus \sigma'(g^{-1}ng) \\ &= {}^{g}\sigma(n) \oplus {}^{g}\sigma'(n) \end{aligned}$$

Thus, ${}^{g}(\sigma \oplus \sigma') \cong {}^{g}\sigma \oplus {}^{g}\sigma'$. □

Definition 4.4. For $\sigma, {}^{g}\sigma \in \hat{N}$, the subgroup,

$$I_G(\sigma) = \{g \in G \mid {}^{g}\sigma \cong \sigma\}$$

is called the inertia subgroup of G.

The g-conjugate defines an action of G on $\hat{N}$. We can see this by, for $g_1, g_2 \in G$,

$$\begin{aligned} {}^{g_1 g_2}\sigma(n) &= \sigma((g_1 g_2)^{-1} n (g_1 g_2)) \\ &= \sigma(g_2^{-1} g_1^{-1} n g_1 g_2) \\ &= {}^{g_2}\sigma(g_1^{-1} n g_1) \\ &= {}^{g_1}({}^{g_2}\sigma(n)) \end{aligned}$$

So, g-conjugate defines an action of G on the irreducible representations of N. Now, $I_G(\sigma)$ is the stabilizer of σ in the action of G on $\hat{N}$. To see this, say $g \in I_G(\sigma)$. Then,

$$\begin{aligned} {}^{g}\sigma(n) &= \sigma(gng^{-1}) \\ &= \sigma(n) \end{aligned}$$

Thus, $I_G(\sigma)$ is the stabilizer of σ in the action of G on $\hat{N}$.

Let R be a set of representatives for the right $I_G(\sigma)$-coset in G. That is,

$$G = \bigcup_{r \in R} r I_G(\sigma)$$

Let $H = K = N \trianglelefteq G$ and set $I_G(\sigma) = \bigcup_{q \in Q} qN$, where Q is a set of representatives for the right N-cosets in $I_G(\sigma)$. If $T = RQ$, by Mackey's theorem,

$$\begin{aligned} G &= \bigcup_{r \in R} r I_G(\sigma) \\ &= \bigcup_{r \in R} \bigcup_{q \in Q} rqN \\ &= \bigcup_{t \in T} tN \end{aligned}$$

which is the coset decomposition of G over N.

Theorem 4.5. *Suppose that N is a normal subgroup of G and let $\sigma \in \hat{N}$ and $\theta \in \hat{G}(\sigma)$. If R, Q and T are as above, then setting $d = [I_G(\sigma) : N] = |Q|$ and denoting the multiplicity of σ in $\operatorname{Res}^G_N \theta$ l, we have*

1.

$$\operatorname{Res}^G_N(\operatorname{Ind}^G_N \sigma) = \bigoplus_{t \in T} {}^t\sigma = d \bigoplus_{r \in R} {}^r\sigma$$

is the decomposition of $\operatorname{Res}^G_N(\operatorname{Ind}^G_N \sigma)$ into irreducible inequivalent subrepresentations.

2.

$$\operatorname{Hom}_G\left(\operatorname{Ind}^G_N \sigma, \operatorname{Ind}^G_N \sigma\right) \cong \mathbb{C}^d$$

3.

$$\operatorname{Res}^G_N \theta \cong l \bigoplus_{r \in R} {}^r\sigma$$

Proof. Let V_σ denote the representation space of σ. That is V_σ is the vector space of

$\sigma : N \to GL(V)$. For all $t \in T$, set

$$Z_t = \{f : G \to V_\sigma \mid f(t'n) = \delta_{t,t'}\sigma(n^{-1})f(t) \ \ \forall n \in N \ t' \in T\}$$

where

$$\delta_{t,t'} = \begin{cases} 0 & \text{if } t' \neq t \\ \\ 1 & \text{if } t = t' \end{cases}$$

First we prove, Z_t is a subspace of the induced space $\operatorname{Ind}_N^G V_\sigma$. Let $f_1, f_2 \in Z_t$. Then,

$$\begin{aligned} (f_1 + f_2)(t'n) &= f_1(t'n) + f_2(t'n) \\ &= \delta_{t,t'}\sigma(n^{-1})f_1(t) + \delta_{t,t'}\sigma(n^{-1})f_2(t) \\ &= \delta_{t,t'}\sigma(n^{-1})(f_1(t) + f_2(t)) \in Z_t \end{aligned}$$

So, $f_1 + f_2 \in Z_t$. Let $\lambda \in \mathbb{C}$ and $f \in Z_t$. Then,

$$\begin{aligned} (\lambda f(t'n)) &= f(\lambda t'n) \\ &= \delta_{t,t'}\sigma(n^{-1})f(\lambda t) \\ &= \lambda\delta_{t,t'}\sigma(n^{-1})f(t) \end{aligned}$$

So, $\lambda f \in Z_t$. Thus, Z_t is a subspace of the induced space $\operatorname{Ind}_N^G V_\sigma$. Now we will show

$$\operatorname{Ind}_N^G V\sigma = \bigoplus_{t \in T} Z_t$$

is a direct sum. Let $f \in \operatorname{Ind}_N^G V_\sigma$, and $f_t \in Z_t$. Now, for $n \in N$ and $g \in G$, $ng = n(t'n') =$

$t'n*$ for some $n^* \in N$ and $t' \in T$. So,

$$\begin{aligned} f(ng) &= f(t'n^*) \\ &= \sigma((n^*)^{-1})f(t') \\ &= f_{t_1}(t'n^*) + f_{t_2}(t'n^*) + \cdots + f_{t_m}(t'n^*) \end{aligned}$$

where $f_{t_i}(t'n^*) = 0$ for all $t_i \neq t'$ and there is a $t_j = t'$ such that $f(t'n^*) = f_{t_j}(t'n^*)$. Therefore, we can write elements of $\operatorname{Ind}_N^G V_\sigma$ as the sum of elements of Z_t. So,

$$\operatorname{Ind}_N^G V_\sigma = \sum_{t \in T} Z_t$$

Hence, $\operatorname{Ind}_N^G V_\sigma = \bigoplus_{t \in T} Z_t$. Let $\tilde{L_t} : V_\sigma \to Z_t$ be given by,

$$[\tilde{L_t}v](t'n) = \delta_{t,t'}\sigma(n^{-1})v$$

for any $v \in V_\sigma$. The claim is $\tilde{L_t}$ is a linear isomorphism. We will show $\tilde{L_t}$ is a linear transformation. Let $v_1, v_2 \in V_\sigma$. Then,

$$\begin{aligned} \tilde{L_t}(v_1 + v_2)(t'n) &= \delta_{t,t'}\sigma(n^{-1})(v_1 + v_2) \\ &= \delta_{t,t'}[\sigma(n^{-1})(v_1) + \sigma(n^{-1})(v_2)] \\ &= \delta_{t,t'}\sigma(n^{-1})(v_1) + \delta_{t,t'}\sigma(n^{-1})(v_2)] \\ &= \tilde{L_t}(v_1)(t'n) + \tilde{L_t}(v_2)(t'n) \end{aligned}$$

Let $\lambda \in \mathbb{C}$ and $v \in V_\sigma$. Then,

$$\begin{aligned}
\tilde{L_t}(\lambda v)(t'n) &= \delta_{t,t'}\sigma(n^{-1})(\lambda v) \\
&= \lambda\delta_{t,t'}\sigma(n^{-1})(v) \\
&= \lambda\tilde{L_t}(v)(t'n)
\end{aligned}$$

So, $\tilde{L_t}$ is a linear mapping. Now, we will show $\tilde{L_t}$ is a bijection. First, we will show $\tilde{L_t}$ is injective. Let $v_1, v_2 \in V_\sigma$ and say $\tilde{L_t}(v_1) = \tilde{L_t}(v_2)$. Then,

$$\begin{aligned}
\tilde{L_t}(v_1)(t'n) &= \tilde{L_t}(v_2)(t'n) \\
\Longrightarrow \delta - t, t'\sigma(n^{-1})(v_1) &= \delta - t, t'\sigma(n^{-1})(v_2) \\
\Longrightarrow v_1 &= v_2
\end{aligned}$$

Now we will show $\tilde{L_t}$ is surjective. Note, for $f \in Z_t$ there exists a $v \in V_\sigma$ such that $f(t) = v$. So,

$$\begin{aligned}
\tilde{L_t}(v)(t'n) &= \delta_{t,t'}\sigma(n^{-1})v \\
&= \delta_{t,t'}\sigma(n^{-1})f(t) \in Z_t
\end{aligned}$$

Therefore, $\tilde{L_t}$ is a linear isomorphism and $V_\sigma \cong Z_t$. If we let $\lambda = Ind_N^G \sigma$, then

$$\begin{aligned}
[\lambda(n)\tilde{L_t}v](t_1n_1) &= \tilde{L_t}(v)(n^{-1}t_1n_1) \\
&= \tilde{L_t}(v)(t_1t_1^{-1}n^{-1}t_1n_1) \\
&= \delta_{t,t_1}\sigma((t_1t_1^{-1}n^{-1}n_1)^{-1})v \\
&= \delta_{t,t_1}\sigma(n_1^{-1}t_1^{-1}nt_1)v \\
&= \delta_{t,t_1}\sigma(n_1^{-1}(t_1^{-1}nt_1))v \\
&= \delta_{t,t_1}\sigma(n^{-1})\sigma(t_1^{-1}nt_1)v \\
&= \delta_{t,t_1}\sigma(n^{-1})^{t_1}\sigma(n)v \\
&= \tilde{L_t}(^{t_1}\sigma(n)v)(t_1n_1)
\end{aligned}$$

for all $v \in V_\sigma$, $t_1 \in T$, $n, n_1 \in N$. Since $\lambda(n)\tilde{L_t} = \tilde{L_t}{}^t\sigma(n)$, $\tilde{L_t}$ is an intertwining operator and therefore,

$$\operatorname{Ind}_N^G \sigma \cong {}^t\sigma$$

Now since this is true for all $n \in N$, we have

$$\left(\operatorname{Res}_N^G \operatorname{Ind}_N^G \sigma, Z_t\right) \cong \left({}^t\sigma, V_\sigma\right)$$

Hence, $\operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma)$ is equivalent to $\bigoplus_{t\in T} {}^t\sigma$. Now,

$$\bigoplus_{t\in T} {}^t\sigma = \bigoplus_{r\in R}\bigoplus_{q\in Q} {}^{rq}\sigma = |Q| \bigoplus_{r\in R} {}^r\sigma$$

So, $\operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma) = |Q| \bigoplus_{r\in R} {}^r\sigma = d \bigoplus_{r\in R} {}^r\sigma$ and (1) is proved. Now $\sigma \in \hat{N}$ and the multiplicity of σ in $\operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma)$ is equal to d. Then,

$$\operatorname{Hom}_N\left(\sigma, \operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma)\right) \cong \mathbb{C}^d$$

Note, by Frobenius reciprocity,

$$\mathrm{Hom}_N\left(\sigma, \mathrm{Res}_N^G(\mathrm{Ind}_N^G \sigma)\right) \cong \mathrm{Hom}_G\left(\mathrm{Ind}_N^G \sigma, \mathrm{Ind}_N^G \sigma\right)$$

Therefore, $\mathrm{Hom}_G\left(\mathrm{Ind}_N^G \sigma, \mathrm{Ind}_N^G \sigma\right) \cong \mathbb{C}^d$ and we have shown (2). Consider the map $\varphi : \mathrm{Hom}_N(\sigma, \mathrm{Res}_N^G \theta) \to \mathrm{Hom}_N({}^g\sigma, \mathrm{Res}_N^G \theta)$ given by,

$$\varphi(T)(v) = \theta(g)T(v)$$

We can show that φ is a linear transformation. Let $T_1, T_2 \in \mathrm{Hom}_N(\sigma, \mathrm{Res}_N^G \theta)$. Then

$$\begin{aligned}
\varphi(T_1 + T_2)(v) &= \theta(g)(T_1 + T_2)(v) \\
&= \theta(g)(T_1(v) + T_2(v)) \\
&= \theta(g)T_1(v) + \theta(g)T_2(v) \\
&= \varphi(T_1)(v) + \varphi(T_2)(v)
\end{aligned}$$

Let $\lambda \in \mathbb{C}$ and $T \in \mathrm{Hom}_N(\sigma, \mathrm{Res}_N^G \theta)$. Then,

$$\begin{aligned}
\varphi(\lambda T)(v) &= \theta(g)(\lambda T)(v) \\
&= \lambda\theta(g)T(v) \\
&= \lambda\varphi(T)(v)
\end{aligned}$$

So, φ is a linear map. Now, φ is a bijective map, and first we will show φ is injective. Let

$T_1, T_2 \in \operatorname{Hom}_N(\sigma, \operatorname{Res}_N^G \theta)$ and assume $\varphi(T_1) = \varphi(T_2)$. So,

$$\begin{aligned} \varphi(T_1)(v) &= \varphi(T_2)(v) \\ \Longrightarrow \theta(g)T_1(v) &= \theta(g)T_2(v) \\ \Longrightarrow T_1(v) &= T_2(v) \\ \Longrightarrow T_1 &= T_2 \end{aligned}$$

One can check that φ is surjective. Thus, φ is a linear isomorphism and $\operatorname{Hom}_N(\sigma, \operatorname{Res}_N^G \theta) \cong \operatorname{Hom}_N({}^g\sigma, \operatorname{Res}_N^G \theta)$. Since the multiplicity of σ in $\operatorname{Res}_N^G \theta$ is l, then ${}^g\sigma$ has multiplicity l in $\operatorname{Res}_N^G \theta$. By Frobenius reciprocity,

$$\operatorname{Hom}_N(\sigma, \operatorname{Res}_N^G \theta) \cong \operatorname{Hom}_G(\operatorname{Ind}_N^G \sigma, \theta)$$

Thus, $\operatorname{Ind}_N^G \sigma$ has exactly l copies of θ. So, every irreducible subrepresentation of $\operatorname{Res}_N^G \theta$ is also a subrepresentation of $\operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma)$. Recall, $\operatorname{Res}_N^G(\operatorname{Ind}_N^G \sigma)$ has subrepresentations of the form $\bigoplus_{r \in R} {}^r\sigma$. Hence,

$$\operatorname{Res}_N^G \theta \cong l \bigoplus_{r \in R} {}^r\sigma$$

So, (3) is proved and the proof is complete. $\square$

From this theorem, we now know if we induce an irreducible representation, σ, up to a group from a normal subgroup and and then restrict back to the normal subgroup, we get a direct sum of conjugates of σ, where the multiplicity is the index of the inertia subgroup over the normal subgroup. We also know the dimension of $\operatorname{Hom}_G(\operatorname{Ind}_N^G, \operatorname{Ind}_N^G)$. Since we know the dimension of this space we can then determine if the induced representation is irreducible.

4.2 Results

This section will discuss key results from Clifford's theorem.

Corollary 4.6. Let $\sigma, \sigma_1 \in \hat{N}$. Then $\operatorname{Ind}_N^G \sigma$ is irreducible if and only if $I_G(\sigma) = N$. Also, if $I_G(\sigma) = I_G(\sigma_1) = N$, then $Ind_N^G(\sigma) \cong \operatorname{Ind}_N^G(\sigma_1)$ if and only if σ is conjugate to σ_1.

Proof. Say $\operatorname{Ind}_N^G(\sigma)$ is irreducible. Then, by Schur's Lemma,

$$\dim \operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma\right) = 1$$

By the previous theorem, $\operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma\right) \cong \mathbb{C}^d$, where $d = [I_G(\sigma) : N]$. So, $[I_G(\sigma) : N] = 1$, which implies that $I_G(\sigma) = N$. Now, say $I_G(\sigma) = N$. Then, $[I_G(\sigma) : N] = 1$. So, by the previous theorem,

$$\operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma\right) \cong \mathbb{C}$$

By Schur's Lemma, since $\dim \operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma\right) = 1$, then Ind_N^G is irreducible. Hence, $\operatorname{Ind}_N^G \sigma$ is irreducible if and only if $I_G(\sigma) = N$. Now, for $\sigma, \sigma_1 \in \hat{N}$, assume $\operatorname{Ind}_N^G \sigma \cong \operatorname{Ind}_N^G \sigma_1$. Since $I_G(\sigma) = N = I_G(\sigma_1)$, $\operatorname{Ind}_N^G \sigma$ and $\operatorname{Ind}_G^N \sigma_1$ are irreducible. By the previous theorem, $\operatorname{Res}_N^G \operatorname{Ind}_N^G \sigma = \bigoplus_{t \in T} {}^t\sigma$. Now by Frobenius Reciprocity,

$$\dim \operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma_1, \operatorname{Ind}_N^G \sigma\right) = \dim \operatorname{Hom}_N \left(\sigma_1, \operatorname{Res}_N^G \operatorname{Ind}_N^G \sigma\right) = \dim \operatorname{Hom}_N \left(\sigma_1, \bigoplus_{t \in T} {}^t\sigma\right) = 1$$

Since $\dim \operatorname{Hom}_N \left(\sigma_1, \bigoplus_{t \in T} {}^t\sigma\right) = 1$, σ_1 must be equivalent to ${}^t\sigma$ for some $t \in T$. Now, assume σ is conjugate to σ_1. That is, there is some $t \in T$ such that $\sigma_1 \cong {}^t\sigma$. So, $\dim \operatorname{Hom}_N \left(\sigma_1, \bigoplus_{t \in T} {}^t\sigma\right) = 1$. By Frobenius Reciprocity,

$$\dim \operatorname{Hom}_N \left(\sigma_1, \bigoplus_{t \in T} {}^t\sigma\right) = \dim \operatorname{Hom}_N \left(\sigma_1, \operatorname{Res}_N^G \operatorname{Ind}_N^G \sigma\right) = \dim \operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma_1, \operatorname{Ind}_N^G \sigma\right) = 1$$

Since, $\dim \operatorname{Hom}_G \left(\operatorname{Ind}_N^G \sigma_1, \operatorname{Ind}_N^G \sigma\right) = 1$ by Schur's Lemma, $\operatorname{Ind}_N^G \sigma \cong \operatorname{Ind}_N^G \sigma_1$. Hence, $\operatorname{Ind}_N^G(\sigma) \cong \operatorname{Ind}_N^G(\sigma_1)$ if and only if σ is conjugate to σ_1. ☐

Lemma 4.7. Let N be a normal subgroup of G and set I to be the inertia group of $\sigma \in \hat{N}$. Then the set $\hat{I}(\sigma) = \{\mu \in \hat{I} \mid \mu \leq \operatorname{Ind}_N^G \sigma\}$. Let

$$\operatorname{Ind}_N^I \sigma = \bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu \mu$$

be the decomposition of $\operatorname{Ind}_N^I \sigma$ into I-irreducible representations, where m_μ is the multiplicity of μ in $\operatorname{Ind}_N^I \sigma$. Then,

1. $$\operatorname{Ind}_N^G \sigma = \bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu$$

   is the decomposition of $\operatorname{Ind}_N^G \sigma$ into G-irreducible components.

2. If $\theta \in \hat{G}(\sigma)$, then

   $$\theta = \operatorname{Ind}_I^G \mu$$

   for some unique $\mu \in \hat{I}(\sigma)$.

Proof. (1) Let $\mu' = Ind_N^I \sigma = \bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu \mu$, where $\sigma \in \hat{N}$ and $\mu \in \hat{I}$. Then, by 4.5,

$$\begin{aligned} \operatorname{Res}_I^G \operatorname{Ind}_I^G \mu &= \operatorname{Res}_I^G \operatorname{Ind}_I^G \left(\bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu \mu \right) \\ &= \bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu d \bigoplus_{r \in R} {}^r\mu \end{aligned}$$

where $d = [I_G(\sigma) : I_G(\sigma] = 1$ and R is a set of representatives of the right $I_G(\sigma)$-cosets in G. So,

$$\bigoplus_{\mu \in \hat{I}(\sigma)} m_\mu d \bigoplus_{r \in R} {}^r\mu = \bigoplus_{r \in R} \bigoplus_{\mu \in I(\sigma)} m_\mu \mu = \bigoplus_{r \in R} \mu'$$

By the transitivity of induction,

$$\operatorname{Hom}_G(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma) \cong \operatorname{Hom}_G(\operatorname{Ind}_I^G \operatorname{Ind}_N^I \sigma, \operatorname{Ind}_I^G \operatorname{Ind}_N^I \sigma) = \operatorname{Hom}_G(\operatorname{Ind}_I^G \mu', \operatorname{Ind}_I^G \mu')$$

So, by Frobenius reciprocity,

$$\begin{aligned}
\operatorname{Hom}_G(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_N^G \sigma) &\cong \operatorname{Hom}_G(\operatorname{Ind}_I^G \mu', \operatorname{Ind}_I^G \mu') \\
&\cong \operatorname{Hom}_I(\operatorname{Res}_I^G \operatorname{Ind}_I^G \mu', \mu') \\
&\cong \operatorname{Hom}\left(\bigoplus_{r\in R} {}^r\mu', \mu'\right)
\end{aligned}$$

By 4.5, we know ${}^r\mu'$ are not equivalent, and so,

$$\operatorname{Hom}\left(\bigoplus_{r\in R} {}^r\mu', \mu'\right) \cong \operatorname{Hom}_I(\mu', \mu') = \operatorname{Hom}_I(\operatorname{Ind}_N^I \sigma, \operatorname{Ind}_N^I \sigma)$$

Therefore,

$$\operatorname{Hom}_G\left(\bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu, \bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu\right) \cong \operatorname{Hom}_I(\operatorname{Ind}_N^I \sigma, \operatorname{Ind}_N^I \sigma) \cong \mathbb{C}^d = \mathbb{C}$$

Since $\dim \operatorname{Hom}_G\left(\bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \psi, \bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu\right) = 1$, $\operatorname{Ind}_I^G \mu$ is irreducible and inequivalent for each $\mu \in \hat{I}(\sigma)$. So we have,

$$\operatorname{Ind}_N^G \sigma \cong \operatorname{Ind}_I^G\left(\bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \psi\right) = \bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu$$

where each $\operatorname{Ind}_I^G \mu$ is G-irreducible and inequivalent. Hence, (1) is proved. Note, since $\theta \in \hat{G}(\sigma)$, $\sigma \leq \operatorname{Res}_N^G(\theta)$. By Frobenius Reciprocity,

$$\operatorname{Hom}_N\left(\sigma, \operatorname{Res}_N^G \theta\right) = \operatorname{Hom}_G\left(\operatorname{Ind}_N^G \sigma, \theta\right)$$

So, $\theta \leq \operatorname{Ind}_N^G \sigma$. By the first part of this lemma, we have that $\operatorname{Ind}_N^G \sigma = \bigoplus_{\mu\in\hat{I}(\sigma)} m_\mu \operatorname{Ind}_I^G \mu$. Now θ is an irreducible representation of G, so θ must be equivalent to one of the $\operatorname{Ind}_I^G \mu$ for some unique $\mu \in \hat{I}(\sigma)$. So, (2) is now proved and the proof is complete.

$\square$

Definition 4.8. The number $l = \dim \operatorname{Hom}_N \left(\sigma, \operatorname{Res}_N^G \theta\right)$ is called the inertia index of $\theta \in \hat{G}(\sigma)$ with respect to N.

Theorem 4.9. *Let N be a normal subgroup of G. Let $\sigma \in \hat{N}$, $I = I_G(\sigma)$ and $\hat{I}(\sigma) = \{\mu \in \hat{I} \mid \mu \leq \operatorname{Ind}_N^I \sigma\}$. Say $\varphi : \hat{I}(\sigma) \to \hat{G}(\sigma)$ is defined by mapping $\mu \mapsto Ind_I^G \mu$. Then, φ is a bijection. Moreover, the inertia index of $\mu \in \hat{I}(\sigma)$ with respect to N coincides with the inertia index of $\operatorname{Ind}_I^G \mu$ with respect to N and is equal to m_μ, where m_μ is the multiplicity of μ in $\operatorname{Ind}_N^I \sigma$. Also, $\operatorname{Res}_n^I \mu = m_\mu \bigoplus \sigma$.*

Proof. Let $\varphi : \hat{I}(\sigma) \to \hat{G}(\sigma)$ be defined by mapping $\mu \mapsto \operatorname{Ind}_I^G \mu$. To see that φ is injective, the previous lemma says that for $\mu \in \hat{I}(\sigma)$ and $\theta \in \hat{G}(\sigma)$, $\theta = \operatorname{Ind}_I^G \mu$ for some unique μ. So, φ is injective. To see that φ is surjective, take a $\theta \in \hat{G}(\sigma)$ and $\mu \in \hat{I}(\sigma)$. Then, $\varphi(\mu) = \operatorname{Ind}_I^G \mu = \theta$, by the previous lemma. Therefore, φ is a bijection. Now m_μ is the multiplicity of μ in $\operatorname{Ind}_N^I \sigma$. So, $\dim \operatorname{Hom} \left(\operatorname{Ind}_N^I \sigma, \mu\right) = m_\mu$. By Frobenius Reciprocity,

$$\dim \operatorname{Hom} \left(\sigma, \operatorname{Res}_N^I \mu\right) = \dim \operatorname{Hom} \left(\operatorname{Ind}_N^I \sigma, \mu\right) = m_\mu$$

Now, by the previous lemma, m_μ is the multiplicity of $\operatorname{Ind}_I^G \mu$ in $\operatorname{Ind}_N^G \sigma$. So,

$$\dim \operatorname{Hom} \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_I^G \psi\right) = m_\mu$$

By Frobenius Reciprocity,

$$\dim \operatorname{Hom} \left(\sigma, \operatorname{Res}_N^G \operatorname{Ind}_I^G \mu\right) = \dim \operatorname{Hom} \left(\operatorname{Ind}_N^G \sigma, \operatorname{Ind}_I^G \mu\right) = m_\mu$$

Thus, the inertia index of $\mu \in \hat{I}(\sigma)$ with respect to N coincides with the inertia index of $\operatorname{Ind}_I^G \mu$ with respect to N. Lastly, by the previous theorem, we have

$$\operatorname{Res}_N^I \mu \cong m_\mu \bigoplus \sigma$$

That is, $\operatorname{Res}_N^I \mu \cong \sigma \oplus \sigma \oplus \cdots \oplus \sigma$, m_μ times. $\square$

4.3 Connecting the Two Approaches

In this section, we will relate the character approach to the vector space approach. The main theorems and lemmas we will compare are 3.2 to 4.5, and 4.7 with 4.9 to 3.6.

First we will compare 3.2 to 4.5. From (3) of 4.5, we see how a representation of a group when restricted down to a normal subgroup breaks apart into a direct sum of irreducible constituents. To see how this is connected to the character approach from 3.2, recall the character of a representation that is isomorphic to the direct sum of irreducible constituents is the sum of the characters of the irreducible constituents, and the multiplicity of an irreducible constituent can be found by taking the inner product of the character of the representation and the character of the irreducible constituent. So, from 4.5, we have

$$\operatorname{Res}_N^G \theta \cong l \bigoplus_{r \in R} {}^{r}\sigma$$

and from 3.2, we have

$$\chi_N = \langle \chi_N, \varphi \rangle \sum_{i=1}^{t} \varphi_i$$

Let χ be the character of θ and φ_i be the character of ${}^{r_i}\sigma$ for $r_i \in R$. Then, $\langle \chi_N, \varphi_i \rangle$ is the multiplicity of σ in $\operatorname{Res}_N^G \theta$, which is l.

Now, we will compare 4.7 with 4.9 to 3.6. From 3.6, the set $A = \{\psi \in \operatorname{Irr}(T) \mid \langle \psi_H, \varphi \rangle \neq 0\}$ is equivalent to the set $\hat{I}(\sigma) = \{\mu \in \hat{I} \mid \psi \leq \operatorname{Ind}_N^I(\sigma)\}$ from 4.7. Similarly from 3.6, the set $B = \{\chi \in \operatorname{Irr}(G) \mid \langle \chi_H, \varphi \rangle \neq 0\}$ is equivalent to the set $\hat{G}(\sigma) = \{\theta \in \hat{G} \mid \sigma \leq \operatorname{Res}_N^G(\theta)\}$ from 4.9. Note in 3.6 the normal subgroup is H, but in 4.7 and 4.9 the normal subgroup is N. Since H and N are arbitrary groups, we will refer to the normal subgroup as N. Let ψ^G be the induced character of μ from I to G, then we have the irreducible character of $\operatorname{Ind}_I^G \mu$ and can make that connection between (a) of 3.6 and (1) of 4.7. We can also connect (2) of 4.7 to (c) of 3.6. Since ψ^G is the irreducible character of $\operatorname{Ind}_I^G \mu$, and we know from 3.6 $\operatorname{Ind}_I^G \mu$ is unique, then ψ^G is unique. Now, 4.9 defines a bijection from $\hat{I}(\sigma)$ to $\hat{G}(\sigma)$. Since we have made the connection between the sets of A to $\hat{I}(\sigma)$ and B to $\hat{G}(\sigma)$, the bijection between A and B defined in 3.6 is a bijection mapping characters of representations in $\hat{I}(\sigma)$ to characters of representations in $\hat{G}(\sigma)$.

5 $GL_2(\mathbb{F}_q)$ Example

In this chapter we will explore Clifford's theorem with the subgroup of $Gl_2(\mathbb{F}_q)$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \; a, c \neq 0), \right\}$ and the normal subgroup of B, $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \right\}$ [3]. In this example we will demonstrate that N is a normal subgroup, find the inertia subgroup of B, and show how we can induce characters of the inertia group up to B and when those induced characters will be irreducible. First, we will look at the case where $\mathbb{F}_q \cong \mathbb{Z}_3$ and then explore when $\mathbb{F}_q \cong \mathbb{Z}_5$.

Consider two subgroups of $GL_2(\mathbb{F}_q)$, where q is a power of a prime, $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \; a, c \neq 0), \right\}$ and $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \right\}$. If we let $b = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B$ and $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$. Then,

$$\begin{aligned} bnb^{-1} &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/ac \\ 0 & 1/c \end{pmatrix} \\ &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a & -b/ac \\ 0 & 1/c \end{pmatrix} \\ &= \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So, $bnb^{-1} \in N$ and therefore, N is a normal subgroup of B. The orders of each subgroup are the following, $|B| = q(q-1)^2$ and $|N| = q$. Since we want to find irreducible characters of B using Clifford's Theorem and results, we need to know the conjugacy classes of B. Now one type of element that is in B looks like, $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. So, if we conjugate this matrix by any element in B, we get the following,

$$\begin{aligned}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} &= \begin{pmatrix} ax & cx \\ 0 & bc \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} \\ &= \begin{pmatrix} x & ax(-c/ab)+cx/b \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\end{aligned}$$

So, a representative of this conjugacy class is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and there are $q-1$ conjugacy classes represented this way of size 1. Another type of element that is in B can look like, $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. When we conjugate this element by any element in B, we get the following,

$$\begin{aligned}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} &= \begin{pmatrix} ax & ay+bx \\ 0 & bx \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} \\ &= \begin{pmatrix} x & ax(-c/ab)+ay/b+x \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} x & -cx/b+ay/b+x \\ 0 & x \end{pmatrix}\end{aligned}$$

So, a representative of this conjugacy class is $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and there are $q-1$ conjugacy classes represented this way of size $q-1$. The final type of element that appears in B will look like, $\begin{pmatrix} x & d \\ 0 & y \end{pmatrix}$. When we conjugate this element by any element in B, we get the following,

$$\begin{aligned}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\begin{pmatrix} x & d \\ 0 & y \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} &= \begin{pmatrix} ax & ad+cy \\ 0 & by \end{pmatrix}\begin{pmatrix} 1/a & -c/ab \\ 0 & 1/b \end{pmatrix} \\ &= \begin{pmatrix} x & -cx/b+ad/b+cy/b \\ 0 & y \end{pmatrix}\end{aligned}$$

So, a representative of this conjugacy class is $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$ and there are $(q-1)(q-2)$ conjugacy classes represented this way of size q. We know we have found all conjugacy classes of B since,

$$\begin{aligned}(q-1)+(q-1)(q-1)+q(q-1)(q-2) &= (q-1)+q^2-2q+1+q^3-3q+2 \\ &= q^3-2q^2+q \\ &= q(q^2-2q+1) \\ &= q(q-1)^2 \\ &= |B|\end{aligned}$$

The following table describes all conjugacy classes of B.

Table 5: Conjugacy Classes of B

Representative	Size of Class	Number of Classes
$\begin{pmatrix} \pm a & 0 \\ 0 & \pm a \end{pmatrix}$	1	$q-1$
$\begin{pmatrix} \pm a & b \\ 0 & \pm a \end{pmatrix}$	$q-1$	$q-1$
$\begin{pmatrix} \pm a & x \\ 0 & \pm b \end{pmatrix}$	$(q-1)(q-2)$	q

Now, the inertia group, $I_B(\varphi)$, where $\varphi \in \mathrm{Irr}(N)$ will have the form

$$I_B(\varphi) = \{b \in B \mid \varphi^b = \varphi\} = \{b \in B \mid \varphi(bnb^{-1}) = \varphi(n)\}$$

Now, depending on φ, we want $bnb^{-1} = n$. So,

$$\begin{aligned} bnb^{-1} &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/ac \\ 0 & 1/c \end{pmatrix} \\ &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a & -b/ac \\ 0 & 1/c \end{pmatrix} \\ &= \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix} \\ \implies & \frac{ax}{c} = x \\ \implies & a = c \end{aligned}$$

Therefore, the inertia group of B is,

$$I_B(\varphi) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{F}_q \; a \neq 0 \right\}$$

Now, $|I_B(\varphi)| = q(q-1)$ and $I_B(\varphi)$ is abelian. Since the inertia group is abelian, we will want to determine whether $I_B(\varphi)$ is cyclic or not. For $\psi \in \mathrm{Irr}(I_B(\varphi))$ such that $\langle \psi_N, \varphi \rangle \neq 0$, Then, we have ψ^B is irreducible by Theorem 4.01. Using 3.6 we can find irreducible characters of B. We need to determine how many irreducible characters we can find using this Theorem. Note, $N \cong \mathbb{F}_q$ is an abelian group, and thus has q conjugacy classes and therefore, q irreducible characters. Also, N is a normal subgroup of $I_B(\varphi)$, so $\psi_N = \langle \psi_N, \varphi \rangle \sum_{i=1}^{t} \varphi_i$ where φ_i are the conjugates of φ for $i = 1, \ldots, t$, by Clifford's Theorem.

5.1 Case: $\mathbb{F}_q \cong \mathbb{Z}_3$

We can explore a specific case, where $q = 3$, so $\mathbb{F}_3 \cong \mathbb{Z}_3$. Our goal is to produce irreducible characters of B from inducing irreducible characters of the inertia group to B. First, note if $q = 3$,

$$B = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}, \begin{pmatrix}2&1\\0&2\end{pmatrix}, \begin{pmatrix}2&2\\0&2\end{pmatrix}, \begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}1&1\\0&2\end{pmatrix}, \begin{pmatrix}1&2\\0&2\end{pmatrix}, \begin{pmatrix}2&1\\0&1\end{pmatrix}, \begin{pmatrix}2&2\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&1\end{pmatrix} \right\}$$

So, we can find the conjugacy classes of B, which are the following,

$$\begin{aligned}
cl\begin{pmatrix}1&0\\0&1\end{pmatrix} &= \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}\\
cl\begin{pmatrix}1&0\\0&2\end{pmatrix} &= \left\{\begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}1&1\\0&2\end{pmatrix}, \begin{pmatrix}1&2\\0&2\end{pmatrix}\right\}\\
cl\begin{pmatrix}1&1\\0&1\end{pmatrix} &= \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}\right\}\\
cl\begin{pmatrix}2&1\\0&1\end{pmatrix} &= \left\{\begin{pmatrix}2&1\\0&1\end{pmatrix}, \begin{pmatrix}2&2\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&1\end{pmatrix}\right\}\\
cl\begin{pmatrix}2&1\\0&2\end{pmatrix} &= \left\{\begin{pmatrix}2&1\\0&2\end{pmatrix}, \begin{pmatrix}2&2\\0&2\end{pmatrix}\right\}\\
cl\begin{pmatrix}2&0\\0&2\end{pmatrix} &= \left\{\begin{pmatrix}2&0\\0&2\end{pmatrix}\right\}
\end{aligned}$$

Since B has six conjugacy classes, B has six irreducible characters. Now $q = 3$ implies $N \cong \mathbb{Z}_3$. So, we know all irreducible characters of N. N will have the following character table.

Table 6: N Character Table

N	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&2\\0&1\end{pmatrix}$
φ_1	1	1	1
φ_2	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$
φ_3	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$

The inertia group with respect to $\varphi_2 \in \mathrm{Irr}(N)$ is,

$$I_B(\varphi_2) = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}, \begin{pmatrix}2&1\\0&2\end{pmatrix}, \begin{pmatrix}2&2\\0&2\end{pmatrix} \right\}.$$

Note, that $I_B(\varphi_2) = I_B(\varphi_3)$ and note that $I_B(\varphi_2) = \left\langle \begin{pmatrix}2&2\\0&2\end{pmatrix} \right\rangle$. To see this,

$$\begin{aligned}
\begin{pmatrix}2&2\\0&2\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix} &= \begin{pmatrix}1&2\\0&1\end{pmatrix}\\
\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix} &= \begin{pmatrix}2&0\\0&2\end{pmatrix}\\
\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix} &= \begin{pmatrix}1&1\\0&1\end{pmatrix}\\
\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix} &= \begin{pmatrix}2&1\\0&2\end{pmatrix}\\
\begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix} &= \begin{pmatrix}1&0\\0&1\end{pmatrix}
\end{aligned}$$

So, $\left| \begin{pmatrix}2&2\\0&2\end{pmatrix} \right| = 6$, and so $I_B(\varphi_2) \cong \mathbb{Z}_6$. Note, the inertia group, $I_B(\varphi_2) = I_B(\varphi_3)$. So the character table for the inertia group is the following,

where $\omega = e^{\pi i/3}$. From 3.6, we can find the set $A = \{\psi \in \mathrm{Irr}(I_B(\varphi_2)) \mid \langle \psi_N, \varphi_2 \rangle \neq 0\}$.

Table 7: $I_B(\varphi_2)$ Character Table

$I_B(\varphi_2)$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$	$\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}$
ψ_1	1	1	1	1	1	1
ψ_2	1	ω^2	ω^4	ω	-1	ω^5
ψ_3	1	ω^4	ω^2	ω^2	1	ω^4
ψ_4	1	1	1	-1	-1	-1
ψ_5	1	ω^2	ω^4	ω^4	1	ω^2
ψ_6	1	ω^4	ω^2	ω^5	-1	ω

Now, $I_B(\varphi_2) = I_B(\varphi_3)$ restricted to N has the following character table,

Table 8: $I_B(\varphi_2)_N$ Character Table

$I_B(\varphi_2)_N$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$
ψ_1	1	1	1
ψ_2	1	ω^2	ω^4
ψ_3	1	ω^4	ω^2
ψ_4	1	1	1
ψ_5	1	ω^2	ω^4
ψ_6	1	ω^4	ω^2

To be in A, $\langle \psi_N, \varphi_2 \rangle \neq 0$. So, for $I_B(\varphi_2)$,

$$\begin{aligned}
\langle \psi_2, \varphi_2 \rangle &= \frac{1}{3}(1 + e^{2\pi i/3} * e^{-2\pi i/3} + e^{4\pi i/3} * e^{-4\pi i/3}) \\
&= \frac{1}{3}(1+1+1) \\
&= 1 \\
\langle \psi_5, \varphi_2 \rangle &= \frac{1}{3}(1 + e^{2\pi i/3} * e^{-2\pi i/3} + e^{4\pi i/3} * e^{-4\pi i/3}) \\
&= \frac{1}{3}(1+1+1) \\
&= 1
\end{aligned}$$

So, for φ_2, $A = \{\psi \in \mathrm{Irr}(I_B(\varphi_2)) \mid \langle \psi_N, \varphi_2 \rangle \neq 0\} = \{\psi_2, \psi_5\}$. To be in A, $\langle \psi_N, \varphi_3 \rangle \neq 0$. So, for $I_B(\varphi_3)$,

$$\begin{aligned}
\langle \psi_3, \varphi_3 \rangle &= \frac{1}{3}(1 + e^{4\pi i/3} * e^{-4\pi i/3} + e^{2\pi i/3} * e^{-2\pi i/3}) \\
&= \frac{1}{3}(1+1+1) \\
&= 1 \\
\langle \psi_6, \varphi_3 \rangle &= \frac{1}{3}(1 + e^{4\pi i/3} * e^{-4\pi i/3} + e^{2\pi i/3} * e^{-2\pi i/3}) \\
&= \frac{1}{3}(1+1+1) \\
&= 1
\end{aligned}$$

So, for φ_3, $A = \{\psi \in \mathrm{Irr}(I_B(\varphi_3)) \mid \langle \psi_N, \varphi_3 \rangle \neq 0\} = \{\psi_3, \psi_6\}$. By the theorem, for $\psi \in A$, then ψ^B is irreducible. Thus,

$$\begin{aligned}
\psi_2^B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \frac{1}{6}(6(\omega^2) + 6(\omega^4)) \\
&= -1 \\
\psi_2^B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{6}(12(1)) \\
&= 2 \\
\psi_2^B \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} &= \frac{1}{6}(6(\omega^4) + 6(\ \omega^2) \\
&= -1 \\
\psi_2^B \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \frac{1}{6}(12(-1)) \\
&= -2
\end{aligned}$$

Therefore, ψ_2^B is a character of B, To see ψ_2^B is irreducible, we will check the inner product

Table 9: B Character Table

B	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}$
ψ_2^B	2	-1	-2	-1	0	0

of ψ_2^B with itself.

$$\begin{aligned}\langle \psi_2^B, \psi_2^B \rangle &= \frac{1}{12}(4 + 2(1) + -2(-2) + 2(1))\\ &= \frac{1}{12}(4+2+4+2)\\ &= 1\end{aligned}$$

Therefore, ψ_2^B is an irreducible character of B. Now, $\psi_5 \in A$. So, ψ_5^B produces another irreducible character of B.

$$\begin{aligned}\psi_5^B\begin{pmatrix}1 & 1\\ \\ 0 & 1\end{pmatrix} &= \frac{1}{6}(6(\omega^2) + 6(\omega^4))\\ &= -1\\ \psi_5^B\begin{pmatrix}1 & 0\\ \\ 0 & 1\end{pmatrix} &= \frac{1}{6}(12(1))\\ &= 2\\ \psi_5^B\begin{pmatrix}2 & 2\\ \\ 0 & 2\end{pmatrix} &= \frac{1}{6}(6(\omega^4) + 6(\ \omega^2)\\ &= -1\\ \psi_5^B\begin{pmatrix}2 & 0\\ \\ 0 & 2\end{pmatrix} &= \frac{1}{6}(12(1))\\ &= 2\end{aligned}$$

So the we can build on the character table of B, To see that ψ_5^B is an irreducible character

Table 10: B Character Table

B	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$\begin{pmatrix}2&2\\0&2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&2\end{pmatrix}$	$\begin{pmatrix}2&1\\0&1\end{pmatrix}$
ψ_2^B	2	-1	-2	-1	0	0
ψ_5^B	2	-1	2	-1	0	0

of B, we will take the inner product of ψ_5^B with itself.

$$\begin{aligned}\langle\psi_5^B,\psi_5^B\rangle &= \frac{1}{12}(4+2(1)+2(2)+2(1))\\ &= \frac{1}{12}(4+2+4+2)\\ &= 1\end{aligned}$$

Therefore, ψ_5^B is an irreducible character of B.

5.2 Case: $\mathbb{F}_q \cong \mathbb{Z}_5$

Now we explore the case where $q = 5$, so $\mathbb{F}_5 \cong \mathbb{Z}_5$. Our goal is to produce irreducible characters of B from inducing irreducible characters of the inertia group to B. Now,

$$B = \left\{ \begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \mid a,b,c \in \mathbb{Z}_5, a,c \neq 0 \right\}$$

and $|B| = q(q-1)^2 = 80$. Since $N \cong \mathbb{Z}_5$, we know all irreducible representations of N. The following table is the character table of N, where $\omega = e^{2\pi i/5}$. To use Clifford's Theorem, we need to determine the inertia subgroup of B, and we need to determine the irreducible

Table 11: N Character Table

N	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&2\\0&1\end{pmatrix}$	$\begin{pmatrix}1&3\\0&1\end{pmatrix}$	$\begin{pmatrix}1&4\\0&1\end{pmatrix}$
φ_1	1	1	1	1	1
φ_2	1	ω	ω^2	ω^3	ω^4
φ_3	1	ω^2	ω^4	ω	ω^3
φ_4	1	ω^3	ω	ω^4	ω^2
φ_5	1	ω^4	ω^3	ω^2	ω

characters of the inertia subgroup. Previously, we determined that,

$$I_B(\varphi) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{F}_q \ a \neq 0 \right\}$$

for $\varphi \in \operatorname{Irr}(N)$, and that $||_B(\varphi)| = q(q-1)$. So, for $q = 5$, $|I_B(\varphi)| = 20$. Now, the inertia subgroup is an abelian group and therefore is either isomorphic to $\mathbb{Z}_{20}$, or is isomorphic to the direct sum of cyclic groups. We can find a generator for $I_B(\varphi)$, which is the element $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, and so, $I_B(\varphi) \cong \mathbb{Z}_{20}$. Now, $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is a generator for $I_B(\varphi)$, so the irreducible representations of $I_B(\varphi)$ will be of the form,

$$\rho_j \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^k \right) = e^{2\pi ijk/20} = e^{\pi ijk/10}$$

for $j = 1, \ldots, 20$. Since each ψ_j is a degree one representation, $\psi_j \in \operatorname{Irr}(I_B(\varphi))$ will be of the form,

$$\psi_j(D) = \left(e^{\pi ij/10}\right)^k$$

for $D \in I_B(\varphi)$ and $k = 0, \ldots, 19$. Note, for $\varphi_i \in Irr(N)$, $i = 2, \ldots, 5$, $I_B(\varphi_2) = I_B(\varphi_3) = I_B(\varphi_4) = I_B(\varphi_5)$. Now, $I_B(\varphi_i)$ restricted to N has the following partial character table,

Table 12: $I_B(\varphi_i)_N$ Character Table

$I_B(\varphi_2)_N$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$
ψ_1	1	1	1	1	1
ψ_2	1	$e^{6\pi i/5}$	$e^{2\pi i/5}$	$e^{8\pi i/5}$	$e^{4\pi i/5}$
ψ_3	1	$e^{2\pi i/5}$	$e^{4\pi i/5}$	$e^{6\pi i/5}$	$e^{8\pi i/5}$
ψ_4	1	$e^{8\pi i/5}$	$e^{6\pi i/5}$	$e^{4\pi i/5}$	$e^{2\pi i/5}$
ψ_5	1	$e^{4\pi i/5}$	$e^{8\pi i/5}$	$e^{2\pi i/5}$	$e^{6\pi i/5}$
ψ_6	1	1	1	1	1
ψ_7	1	$e^{6\pi i/5}$	$e^{2\pi i/5}$	$e^{8\pi i/5}$	$e^{4\pi i/5}$

Note, ψ_6 is where the characters of $I_B(\varphi)$ will start to repeat. To use 3.6, we must restrict $I_B(\varphi)$ to N and determine for which μ_j, $j = 1, \ldots 20$, $\langle \mu_j, \varphi_i \rangle \neq 0$. So, for φ_2,

$$\begin{aligned}\langle \psi_3, \varphi_2 \rangle &= \frac{1}{5}\left(1 + e^{2\pi i/5}e^{-2\pi i/5} + e^{4\pi i/5}e^{-4\pi i/5} + e^{6\pi i/5}e^{-6\pi i/5} + e^{8\pi i/5}e^{-8\pi i/5}\right) \\ &= \frac{1}{5}(5) \\ &= 1 \\ &\neq 0\end{aligned}$$

So, $\psi_3 \in A$. Along with ψ_3, $\langle \psi_8, \varphi_2 \rangle, \langle \psi_{13}, \varphi_2 \rangle, \langle \psi_{18}, \varphi_2 \rangle$ are not equal to zero, and so, $\psi_8, \psi_{13}, \psi_{18} \in A$. For φ_3,

$$\begin{aligned}\langle \psi_5, \varphi_3 \rangle &= \frac{1}{5}\left(1 + e^{4\pi i/5}e^{-4\pi i/5} + e^{8\pi i/5}e^{-8\pi i/5} + e^{2\pi i/5}e^{-2\pi i/5} + e^{6\pi i/5}e^{-6\pi i/5}\right) \\ &= \frac{1}{5}(5) \\ &= 1 \\ &\neq 0\end{aligned}$$

So, $\psi_5 \in A$. Along with ψ_5, $\langle \psi_{10}, \varphi_3 \rangle, \langle \psi_{15}, \varphi_3 \rangle, \langle \psi_{20}, \varphi_3 \rangle$ are not equal to zero, and so, $\psi_{10}, \psi_{15}, \psi_{20} \in A$. For φ_4,

$$\begin{aligned}\langle \psi_2, \varphi_4 \rangle &= \frac{1}{5}\left(1 + e^{6\pi i/5}e^{-6\pi i/5} + e^{2\pi i/5}e^{-2\pi i/5} + e^{8\pi i/5}e^{-8\pi i/5} + e^{4\pi i/5}e^{-4\pi i/5}\right) \\ &= \frac{1}{5}(5) \\ &= 1 \\ &\neq 0\end{aligned}$$

So, $\psi_2 \in A$. Along with ψ_2, $\langle \psi_7, \varphi_4 \rangle, \langle \psi_{12}, \varphi_4 \rangle, \langle \psi_{17}, \varphi_4 \rangle$ are not equal to zero, and so,

$\psi_7, \psi_{12}, \psi_{17} \in A$. For φ_5,

$$\begin{aligned}
\langle \psi_4, \varphi_5 \rangle &= \frac{1}{5}\left(1 + e^{8\pi i/5}e^{-8\pi i/5} + e^{6\pi i/5}e^{-6\pi i/5} + e^{4\pi i/5}e^{-4\pi i/5} + e^{2\pi i/5}e^{-2\pi i/5}\right) \\
&= \frac{1}{5}(5) \\
&= 1 \\
&\neq 0
\end{aligned}$$

So, $\psi_4 \in A$. Along with ψ_4, $\langle \psi_9, \varphi_5 \rangle, \langle \psi_{14}, \varphi_5 \rangle, \langle \psi_{19}, \varphi_5 \rangle$ are not equal to zero, and so, $\psi_9, \psi_{14}, \psi_{19} \in A$. By 3.6, ψ_2^B will be an irreducible character of B. First, we will find ψ_2^B.

$$\begin{aligned}
\psi_2^B\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} &= \frac{1}{20}(80(1))\\
&= 4\\
\psi_2^B\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} &= \frac{1}{20}(80(i))\\
&= 4i\\
\psi_2^B\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix} &= \frac{1}{20}(80(i))\\
&= 4i\\
\psi_2^B\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix} &= \frac{1}{20}(80(-1))\\
&= -4\\
\psi_2^B\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_2^B\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_2^B\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_2^B\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1
\end{aligned}$$

Now, we must check that ψ_2^B is irreducible. Note, by the definition of the induced character, all elements of B that are in conjugacy classes that do not contain elements from $I_B(\varphi_4)$,

$\psi_2^B(C) = 0$ for all $C \in B$ such that $C \notin I_B(\varphi_4)$.

$$\begin{aligned} \left\langle \psi_2^B, \psi_2^B \right\rangle &= \frac{1}{80}\big(4(4) + 4i(\overline{4i}) + 4i(\overline{4i} + -4(-4) + 4(1) + 4(1) + 4(1) + 4(1)\big) \\ &= \frac{1}{80}(80) \\ &= 1 \end{aligned}$$

Thus, ψ_2^B is an irreducible character of B. Since $\langle \psi_2, \varphi_4 \rangle \neq 0$ and $\langle \psi_7, \varphi_4 \rangle, \langle \psi_{12}, \varphi_4 \rangle, \langle \psi_{17}, \varphi_4 \rangle$ are not equal to zero, $\psi_2^B = \psi_7^B = \psi_{12}^B = \psi_{17}^B$. Similarly ψ_3^B will also be an irreducible character of B. We will now find ψ_3^B.

$$
\begin{aligned}
\psi_3^B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{20}(80(1)) \\
&= 4 \\
\psi_3^B \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \frac{1}{20}(80(i)) \\
&= 4i \\
\psi_3^B \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} &= \frac{1}{20}(80(i)) \\
&= -4 \\
\psi_3^B \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} &= \frac{1}{20}(80(-1)) \\
&= 4 \\
\psi_3^B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_3^B \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_3^B \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_3^B \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1
\end{aligned}
$$

Now, we must check that ψ_3^B is irreducible. Note, by the definition of the induced character, all elements of B that are in conjugacy classes that do not contain elements from $I_B(\varphi_4)$,

$\psi_2^B(C) = 0$ for all $C \in B$ such that $C \notin I_B(\varphi_2)$.

$$\begin{aligned}\left\langle \psi_3^B, \psi_3^B \right\rangle &= \frac{1}{80}\big(4(4) + 4i(\overline{4i}) + -4(-4) + 4(4) + 4(1) + 4(1) + 4(1) + 4(1)\big) \\ &= \frac{1}{80}(80) \\ &= 1\end{aligned}$$

So, ψ_3^B is an irreducible character of B. Since $\langle \psi_3, \varphi_2 \rangle \neq 0$ and $\langle \psi_8, \varphi_2 \rangle, \langle \psi_{13}, \varphi_2 \rangle, \langle \psi_{18}, \varphi_2 \rangle$ are not equal to zero, $\psi_3^B = \psi_8^B = \psi_{13}^B = \psi_{18}^B$. Next, we will find and show ψ_4^B is an irreducible character of B.

$$\begin{aligned}
\psi_4^B\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} &= \frac{1}{20}(80(1))\\
&= 4\\
\psi_4^B\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} &= \frac{1}{20}(80(i))\\
&= 4\\
\psi_4^B\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix} &= \frac{1}{20}(80(i))\\
&= 4i\\
\psi_4^B\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix} &= \frac{1}{20}(80(-1))\\
&= -4\\
\psi_4^B\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_4^B\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_4^B\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1\\
\psi_4^B\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix} &= \frac{1}{20}(20(1))\\
&= 1
\end{aligned}$$

Now, we will show ψ_4^B is irreducible. Note, by the definition of the induced character, all elements of B that are in conjugacy classes that do not contain elements from $I_B(\varphi_5)$,

$\psi_4^B(C) = 0$ for all $C \in B$ such that $C \notin I_B(\varphi_5)$.

$$\begin{aligned}\langle \psi_4^B, \psi_4^B \rangle &= \frac{1}{80}\big(4(4) + 4(4) + 4(\overline{4i}) + -4(-4) + 4(1) + 4(1) + 4(1) + 4(1)\big)\\ &= \frac{1}{80}(80)\\ &= 1\end{aligned}$$

Hence, ψ_4^B is an irreducible character of B. Since $\langle \psi_4, \varphi_5 \rangle \neq 0$ and $\langle \psi_9, \varphi_5 \rangle, \langle \psi_{14}, \varphi_5 \rangle, \langle \psi_{19}, \varphi_5 \rangle$ are not equal to zero $\psi_4^B = \psi_9^B = \psi_{14}^B = \psi_{19}^B$. Lastly, we will find and show ψ_5^B is an irreducible character of B. To find ψ_5^B,

$$\begin{aligned}
\psi_5^B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{20}(80(1)) \\
&= 4 \\
\psi_5^B \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \frac{1}{20}(80(i)) \\
&= 4i \\
\psi_5^B \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} &= \frac{1}{20}(80(i)) \\
&= 4 \\
\psi_5^B \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} &= \frac{1}{20}(80(-1)) \\
&= 4 \\
\psi_5^B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_5^B \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_5^B \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1 \\
\psi_5^B \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} &= \frac{1}{20}(20(1)) \\
&= 1
\end{aligned}$$

Now, we will show ψ_5^B is irreducible. Note, by the definition of the induced character, all elements of B that are in conjugacy classes that do not contain elements from $I_B(\varphi_3)$,

$\psi_5^B(C) = 0$ for all $C \in B$ such that $C \notin I_B(\varphi_3)$.

$$\begin{aligned}
\langle \psi_5^B, \psi_5^B \rangle &= \frac{1}{80}\big(4(4) + 4(\overline{4i}) + 4(4) + 4(4) + 4(1) + 4(1) + 4(1) + 4(1)\big) \\
&= \frac{1}{80}(80) \\
&= 1
\end{aligned}$$

So, ψ_5^B is an irreducible character of B. Since $\langle \psi_5, \varphi_3 \rangle \neq 0$ and ψ_5, $\langle \psi_{10}, \varphi_3 \rangle$, $\langle \psi_{15}, \varphi_3 \rangle$, $\langle \psi_{20}, \varphi_3 \rangle$ are not equal to zero, $\psi_5^B = \psi_{10}^B = \psi_{15}^B = \psi_{20}^B$. Therefore, we have found the irreducible characters of B by using Clifford's Theorem. The following table summarizes the irreducible characters of B found by this method. Note, we will exclude the conjugacy classes not contained in $I_B(\varphi_i)$, $i = 2, 3, 4, 5$, as the character value at those conjugacy classes are zero.

Table 13: Character Table of B

B	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$
ψ_2^B	4	$4i$	$4i$	-4	1	1	1	1
ψ_3^B	4	$4i$	-4	4	1	1	1	1
ψ_4^B	4	4	$4i$	-4	1	1	1	1
ψ_5^B	4	$4i$	4	4	1	1	1	1

5.3 Case: $\mathbb{F}_q \cong \mathbb{Z}_p$

The previous two examples explored specific cases of when $\mathbb{F}_q \cong \mathbb{Z}_p$. We saw when $p = 3$, the inertia subgroup of B was isomorphic to the group $\mathbb{Z}_6$, and when $p = 5$, the inertia subgroup was isomorphic to the group $\mathbb{Z}_{20}$. In this section we will show the inertia subgroup, when $\mathbb{F}_q \cong \mathbb{Z}_p$, is isomorphic to $\mathbb{Z}_{p(p-1)}$. Consider the case when $q = 3$, the generators of

$I_B(\varphi)$ are,

$$\begin{aligned}\left\langle \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right\rangle &= I_B(\varphi) \\ \left\langle \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\rangle &= I_B(\varphi)\end{aligned}$$

Note, $\langle 2 \rangle = \mathbb{Z}_3^\times$. Now, consider the case when $q = 5$, the generators of $I_B(\varphi)$ are,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$$

Note, $\langle 2 \rangle = \mathbb{Z}_5^\times$ and $\langle 3 \rangle = \mathbb{Z}_5^\times$. In both cases, generators had the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where a is a generator of $\mathbb{Z}_p^\times$. Now,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^m = \begin{pmatrix} a^m & m(a^{m-1}b) \\ 0 & a^m \end{pmatrix}$$

So, if

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $a^m = 1$ and $m(a^{m-1}b) = 0$. Since $a^m = 1$, this implies that $p - 1 \mid m$ and since $m(a^{m-1}b) = 0$, this implies $p \mid m$. Now p is a prime so, $gcd(p, p-1) = 1$. Therefore, $p(p-1) \mid m$ and $|I_B(\varphi)| = p(p-1)$, so $m = |I_B(\varphi)|$. Thus,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

where a is a generator of $\mathbb{Z}_p^\times$, is a generator of $I_B(\varphi)$. We can conclude that $I_B(\varphi) \cong \mathbb{Z}_{p(p-1)}$. Since we determined the inertia subgroup is isomorphic to $\mathbb{Z}_{p(p-1)}$, we know all irreducible representations of the inertia subgroup. From 4.5, we know $d = [I_B(\varphi) : N] = p - 1$ and

$\mathrm{Hom}_B\left(\mathrm{Ind}_N^B(\sigma), \mathrm{Ind}_N^B(\sigma)\right) \cong \mathbb{C}^d$. By 4.7, $\mathrm{Ind}_N^B(\sigma) = \bigoplus_{\mu \in \mathrm{Irr}(I_B(\sigma))} m_\mu \, \mathrm{Ind}_I^B(\mu)$, and so,

$$\mathrm{Hom}_B\left(\mathrm{Ind}_N^B(\sigma), \mathrm{Ind}_N^B(\sigma)\right) \cong \mathrm{Hom}_B\left(\mathrm{Ind}_I^B(\mu), \mathrm{Ind}_I^B(\mu)\right) \cong \mathbb{C}^d$$

Thus, there are $p-1$ irreducible representations of B found from using 3.6.

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