On the Existence of Solutions to Discrete, Nonlinear, Multipoint, Boundary Value Problems

by

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Abstract

In this manuscript we study a general family of discrete nonlinear boundary value problems of the form

$$x(t+1) = A(t)x(t) + f(x(t)) \quad \text{where } t \in \{0, 1, ..., N-1\} \tag{1}$$

subject to

$$B_0 x(0) + B_1 x(1) + \cdots + B_N x(N) = 0. \tag{2}$$

We concern ourselves with establishing conditions that are sufficient to guarantee the existence of solutions to these boundary value problems in two distinct cases.

First, we will examine this problem as a scalar, discrete, nonlinear, mulitpoint boundary value problem. We then tackle this problem in the setting of a full $n \times n$ system. In both of these cases, if the associated linear homogeneous boundary value problem only has trivial solutions we are able to show the existence of solutions when the nonlinear term exhibits sublinear growth.

We then allow for the solution space of the associated linear problem to be of dimension one. In this instance, we introduce a projection scheme in order to relate the solution space of the linear problem to the nonlinearity. We then leverage this relationship to establish a framework that is sufficient to guarantee the existence of a solution.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

In this manuscript we will examine the solvability for a well-established family of discrete, nonlinear, multipoint boundary value problems. More specifically, we will study equations of the form

$$x(t+1) = A(t)x(t) + f(x(t)) \quad \text{where } t \in \{0, 1, ..., N-1\} \tag{1.1}$$

subject to

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0. \tag{1.2}$$

For the duration of this work we will assume that $A(t)$ is an invertible $n \times n$ matrix for each t and $x(t)$ is a vector in $\mathbb{R}^n$. Furthermore, we impose the condition that B_i is a real-valued $n \times n$ matrix for all i, f is a continuous function on $\mathbb{R}^n$, and N is a fixed integer larger than two. In the third chapter we require f to be continuous on $\{0, 1, \cdots N\} \times \mathbb{R}^n$.

Multipoint boundary value problems are known to arise in a wide array of disciplines in the physical sciences. One notable example in which these boundary value problems appear is in determining an optimal size for large bridges with multipoint supports [2].

This is not the only scenario in which these types of problems occur, in fact, any physical problem in which N states are recorded at N times will force N boundary conditions into the resulting system. It then becomes clear why research into the existence of solutions to these systems is warranted.

We will look at (1.1)-(1.2) in two separate settings. In chapter two, we first examine scalar, discrete, nonlinear, multipoint boundary value problems of the form

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \cdots + a_0(t)y(t) = g(y(t+m-1) \tag{1.3}$$

for $t = 0, 1, \ldots N-1$, subject to

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \sum_{j=1}^{n} b_{ij}(2)y(j+1) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y(j+N-1) = 0 \tag{1.4}$$

where n is a positive integer and $i = 1, 2, \ldots, n$. While in the third chapter, we will remove the scalar hypothesis and consider a full system at resonance. That is, our problem takes the form of

$$x(t+1) = A(t)x(t) + f(t, x(t)) \quad \text{where } t \in \{0, 1, ..., N-1\} \tag{1.5}$$

under the N-point boundary condition

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0. \tag{1.6}$$

In both examined cases we take a similar approach. We will reformulate the above problems into equivalent equations comprised of linear and nonlinear operators defined on Banach spaces. We then examine two sub-cases, the first in which we assume the linearity is invertible, and the second where the linearity is noninvertible, but has a one dimensional kernel.

In the case of the nonsingular linearity we will assume the nonlinearity exhibits sub-linear growth. We then rely on the Brouwer Fixed Point Theorem to show that we are able to solve the problem at hand. In the second sub-case, we assume the nonlinearity is bounded, and through the use of a projection scheme, seek to relate the large argument behavior of the nonlinearity to the solution space of the associated homogeneous linear problem. In the scalar setting we will again invoke the Brouwer Fixed Point Theorem. In the environment of the full system, we will utilize an argument centered on topological degree to obtain the desired result.

This manuscript servers as extension of my undergraduate project in which I examined the solvability of discrete, nonlinear, two-point boundary value problems. The approach taken follows techniques used in [3],[9],[10].

1.2 Preliminaries

The idea of degree theory plays an essential role throughout this text, either through direct use as we see in chapter three, or as background machinery to the Brouwer Fixed Point Theorem used in chapter two. Therefore, it is essential to familiarize the reader with some fundamental results from the field. In short, degree theory can be thought of as a way to gather information on the existence of zeros to equations of the form $y = f(x)$ [8]. A familiar analog to the degree of a given mapping is the winding number of a closed curve in the complex plane. In fact, the degree serves as a generalization of the winding number observed in complex analysis and differential geometry. The interested reader may find more information relating to degree theory, including proofs of the following propositions and theorems in [8],[11].

To start our brief summary of degree theory, we first state some basic assumptions that we will use to define the topological degree. We will denote the closure of a set A by $\overline{A}$ and will denote the boundary of a set A by ∂A. Now suppose $D \subseteq \mathbb{R}^n$ is open and bounded, $f : D \to \mathbb{R}^n$ is continuous and satisfies $f(x) \neq 0$ for all $x \in \partial D$. We now define $A_f = \{x \in \overline{D} \mid f(x) = 0\}$. We will use this set as an indexing set, as such we will introduce definitions and properties to ensure that A_f is finite.

Defintion 1.1. Let $D \subseteq \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a function such that each of its first partial derivatives exist. Denote $J_f(x)$ to be the determinant of the Jacobian matrix of f and set $B_f = \{x \in D \mid J_f(x) = 0\}$. Then we say f is a non-degenerate mapping if $A_f \cap B_f = \emptyset$.

Proposition 1.2. *Let $D \subseteq \mathbb{R}^n$ be open and bounded, $f : D \to \mathbb{R}^n$ be continuous such that $f(x) \neq 0$ for all $x \in \partial D$. If f is of class C^1 and is non-degenerate, then A_f is finite.*

Defintion 1.3. Let $D \subseteq \mathbb{R}^n$ be open and bounded, $f : D \to \mathbb{R}^n$ be of class C^1 such that $f(x) \neq 0$ for all $x \in \partial D$. The topological degree of f with respect to D and 0, $d\big[f, D, 0\big]$ is defined by

$$d\big[f, D, 0\big] = \sum_{x \in A_f} \text{sign}\big(J_f(x)\big)$$

As it stands, our current definition of the degree of a mapping is very limited in scope due to the strong conditions imposed on the function f. Using Sard's Lemma and the Weierstrass Approximation Theorem, we are able to extend the definition of the topological degree to a function that is degenerate and is continuous. Formal statements and proofs of these results can be found in the text by Rouche and Mahwin [11].

Our use of degree theory focuses on two very powerful properties of the degree. The first property relates the value of the degree with the existence of solutions to the equation of interest, and the second is the homotopy invariance of the degree. Before we formally state these properties, first consider the following definition.

Defintion 1.4. Let X and Y be subsets of $\mathbb{R}$, and f and g be continuous functions from X to Y. If there exists a continuous map $H : [0, 1] \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ we say f and g are homotopic and the map H is a homotopy.

That is, we can consider a homotopy to be a continuous deformation between two continuous functions. We now state the two properties of interest.

Theorem 1.5 (Kronecker Existence Theorem)**.** *Let $D \subseteq \mathbb{R}^n$ be open and bounded, $f : D \to \mathbb{R}^n$ be of class C^1 such that $f(x) \neq 0$ for all $x \in \partial D$. If* $\boldsymbol{d}\big[f, D, 0\big] \neq 0$, *then there exists an $x \in D$ such that $f(x) = 0$.*

Theorem 1.6 (Theorem of Invariance with Respect to Homotopy)**.** *Let $F : \bar{D}\times[0,1] \to \mathbb{R}^n$, $(x, \lambda) \mapsto F(x, \lambda)$, be a continuous mapping such that $F(x, \lambda) \neq 0$ for all $x \in \partial D$ and $\lambda \in [0, 1]$. Define: $f_\lambda : D \to \mathbb{R}^n$, $x \mapsto F(x, \lambda)$. Then* $\boldsymbol{d}\big[f_\lambda, D, 0\big]$ *is well defined for all $\lambda \in [0, 1]$ and is independent of λ.*

The proof of the Kronecker Existence Theorem and Theorem of Invariance with Respect to Homotopy can be found in [8]. The homotopic invariance of the degree will prove to be the key to showing our problem has a solution. If we are able to create a homotopy between a map with nonzero degree and the map we are studying, then we will know each map to have the same degree. Of possible note to the reader is the axiomization of the topological degree. As cited in the text by Outerelo and Ruiz, the development of degree theory culminated with the axiomatic characterization of the degree. This process began with the work of Mitio Nagumo in 1951 and was completed by Lutz Führer in 1971. More information regarding this axiomatic approach can be found in [8].

Chapter 2

Scalar Problem

2.1 Introduction

In this chapter we will focus on scalar, nonlinear, discrete, multipoint boundary value problems of the form

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \cdots + a_0(t)y(t) = g(y(t+m-1)) \tag{2.1}$$

for $t = 0, 1, \ldots N-1$, subject to

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \sum_{j=1}^{n} b_{ij}(2)y(j+1) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y(j+N-1) = 0 \tag{2.2}$$

where $i = 1, 2, \ldots, n$. We will assume that n is a positive integer, $g : \mathbb{R} \to \mathbb{R}$ is continuous, $m \in \{1, 2, \ldots, n\}$, N is an integer larger than two, and that the coefficients b_{ij} and the functions $a_0, a_1, \cdots, a_{n-1}$ are all real valued where $a_0(t) \neq 0$ for all t. We will analyze this problem as the system

$$x(t+1) = A(t)x(t) + f(x(t)) \quad \text{where } t \in \{0, 1, ..., N-1\} \tag{2.3}$$

under the boundary condition

$$B_0 x(0) + B_1(1)x(1) + \cdots + B_N x(N) = 0. \tag{2.4}$$

Each $n \times n$ matrix B_k is given by $B_k = [b_{ij}(k)]$. The $n \times n$ matrix $A(t)$ is defined as

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{bmatrix}.$$

We will define the vector function

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+n-1) \end{bmatrix},$$

and define $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_m) \end{bmatrix}.$$

We will study the boundary value problem (2.3)-(2.4) by way of operators. To aid in this discussion we introduce the following finite dimensional spaces:

$$Z = \{\phi : \{0, 1, ..., N\} \to \mathbb{R}^n\}$$
$$X = \{\phi \in Z \mid B\phi(0) + B_1(1)\phi(1) + \cdots + B_N\phi(N) = 0\}$$
$$Y = \{\phi : \{0, 1, ..., N-1\} \to \mathbb{R}^n\}.$$

We will define the norms on X and Y by

$$\|x\| = \sup_{t=0,1,\cdots,N} |x(t)|,$$
$$\|y\| = \sup_{t=0,1,\cdots,N-1} |y(t)|,$$

where $|\cdot|$ is any norm on $\mathbb{R}^n$. Using these spaces, we can define the operators:

$$L : X \to Y \text{ defined by } (Lx)(t) = x(t+1) - A(t)x(t)$$
$$F : X \to Y \text{ defined by } (F(x))(t) = f(x(t)).$$

We make note that the operator L is linear and that x is a solution to (2.3)-(2.4) if and only if $Lx = F(x)$.

We will use these operators as leverage to determine conditions sufficient to guarantee solutions to the boundary value problem exist. To this end, the kernel of L will prove instrumental to our procedure. If the kernel of L is trivial, we will find a fixed point of the map $L^{-1}F$. If L is singular, we will examine the case when the kernel of L is one dimensional. In this case, we will use projections to help understand the behavior of the nonlinearity.

2.2 The Case of Nonsingular L

We seek to show that solutions to the nonlinear scalar discrete boundary value problem (2.1) - (2.2) exist when the associated operator L is invertible and g exhibits sublinear growth. To begin the analysis of the problem, we will first examine the linear homogeneous scalar problem of the form

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \cdots + a_0(t)y(t) = 0 \tag{2.5}$$

for $t = 0, 1, \ldots N-1$, subject to

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \sum_{j=1}^{n} b_{ij}(2)y(j+1) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y(j+N-1) = 0 \tag{2.6}$$

where $i = 1, 2, \ldots, n$. To this end, we introduce a widely known solution to the linear homogeneous system

$$x(t+1) = A(t)x(t). \tag{2.7}$$

Define

$$\Phi(t) = \begin{cases} I & t = 0 \\ A(t-1)A(t-2)\cdots A(0) & t = 1, 2, \cdots \end{cases}$$

then Φ is a fundamental matrix solution of (2.7) [6].

Proposition 2.1. *The solution space of the linear homogeneous scalar problem (2.5) - (2.6) has the same dimension of* $ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$.

Proof. x is in the solution space of (2.5)-(2.6) if and only if $Lx = 0$ and $B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0$. This is true if and only if $x(t+1) = A(t)x(t)$ and $B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0$. So we have $x \in ker(L)$ if and only if $x(t) = \Phi(t)d$ and $B_0d +$

$B_1\Phi(1)d + \cdots + B_N\Phi(N)d = 0$ for some $d \in \mathbb{R}^n$. Therefore the solution space of the linear homogeneous scalar problem (2.5)-(2.6) has the same dimension as $ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$. $\square$

We now turn our attention to the problem stated at the beginning of the section by exploring the relationship between the inevitability of the operator L and the matrix $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$.

Proposition 2.2. *The linear map L is invertible if and only if $B_0+B_1\Phi(1)+\cdots+B_N\Phi(N)$ is invertible.*

Proof. Let $h \in Y$. Now $Lx = h$ if and only if there is an $x \in X$ such that $x(t+1) = A(t)x(t) + h(t)$ and $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N) = 0$. By Variation of Parameters [6] we have $Lx = h$ if and only if

$$x(t) = \Phi(t)x(0) + \sum_{i=0}^{t-1} \Phi(t)\Phi^{-1}(i+1)h(i) \tag{2.8}$$

and the boundary condition is satisfied. Substituting (2.8) into the boundary condition yields $Lx = h$ if and only if

$$B_0x(0) + B_1\Big[\Phi(1)x(0) + h(0)\Big] + \cdots + B_N\Big[\Phi(N)x(0) + \Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\Big] = 0.$$

That is $Lx = h$ if and only if

$$\Big[B_0+B_1\Phi(1)+\cdots+B_N\Phi(N)\Big]x(0) = -\Big[B_1h(0)+\cdots+\Phi(N)x(0)+\Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\Big] \tag{2.9}$$

Now suppose that L is invertible. Then there is only one x that satisfies $Lx = h$. Therefore there exists a unique $x(0)$ satisfying (2.9). Hence, $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible. Conversely, suppose that $B_0+B_1\Phi(1)+\cdots+B_N\Phi(N)$ is invertible. Then there is one and only one $x(0)$ satisfying (2.9). By using the Variation of Parameters formula we can build the unique x that satisfies $Lx = h$. Thus L is invertible. $\square$

Theorem 2.3. *Suppose $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible and that there exists positive real numbers α, M_1, M_2 with $0 \leq \alpha < 1$ such that $|g(x)| \leq M_1\|x\|^\alpha + M_2$ for all $x \in \mathbb{R}^n$. Then there exists at least one solution to (2.1)-(2.2).*

Proof. Define the sets X and Y and the maps L and F as done previously. Since $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible L is also invertible by proposition 2.2. Therefore $Lx = F(x)$ is equivalent to $x = L^{-1}F(x)$. We now define $\Gamma : X \to X$ by $\Gamma x = L^{-1}F(x)$. We will show that the map Γ has a fixed point by utilizing the Brouwer Fixed Point Theorem. To this end, we define the set $B = \{x \in X \mid \|x\| \leq M\}$ where M is a positive real number. By construction, B is a closed, bounded, and convex set, and Γ is continuous by composition. Thus all that is left to show is that Γ maps B into B. First, observe that for each $t \in \{0, 1, \ldots, N-1\}$

$$\begin{aligned} |f(x(t))| = |g(x_m(t))| &\leq M_1|x_m(t)|^\alpha + M_2 \\ &\leq M_1|x(t)|^\alpha + M_2 \\ &\leq M_1\|x\|^\alpha + M_2. \end{aligned}$$

Therefore we have

$$\|F(x)\| = \sup_{t=0,\ldots N-1} |f(x(t))| \leq M_1\|x\|^\alpha + M_2.$$

Now for $x \in B$ we have

$$\|\Gamma x\| = \|L^{-1}F(x)\| \leq \|L^{-1}\|\|F(x)\| \leq \|L^{-1}\|\left[M_1\|x\|^\alpha + M_2\right]$$

That is

$$\frac{\|\Gamma x\|}{M} \leq \|L^{-1}\| \left(\frac{M_1}{M^{1-\alpha}} + \frac{M_2}{M} \right)$$

For M sufficiently large, $\dfrac{\|\Gamma x\|}{M} \leq 1$, and so $\|\Gamma x\| \leq M$. Therefore Γ is a map from B into B. By the Brouwer Fixed Point Theorem there exists $\overline{x} \in X$ such that $\overline{x} = L^{-1}F(\overline{x})$. Thus $L\overline{x} = F(\overline{x})$ and so a solution to (2.1)-(2.2) exists. $\square$

2.3 The Case of Singular L

We now assume that the kernel of L is one dimensional and begin our analysis of the linear operator L by defining the map $S : \{0, 1, \cdots, N-1\} \to \mathbb{R}^n$ by

$$S(t) = \Phi(t)d$$

where $d \in ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$ is a unit vector. Using this map S we can state a useful corollary to proposition 2.1.

Corollary 2.4. *If the solution space of the linear homogeneous scalar problem is nontrivial then $x \in ker(L)$ if and only if $x(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}$.*

Through the use of this corollary we can define the following projection on the set X.

Proposition 2.5. *The map $P : X \to X$ defined by $(Px)(t) = S(t)d^T x(0)$ is a projection onto the kernel of L.*

Proof. Linearity and boundedness is clear. It must now be shown that $P^2 = P$ and

$Im(P) = ker(L)$. To see that $P^2 = P$ consider

$$\begin{aligned}(P^2x)(t) = (P(Px))(t) &= P(S(\cdot)d^T x(0))(t) \\ &= S(t)d^T S(0)d^T x(0) \\ &= S(t)d^T(\Phi(0)d)d^T x(0) \\ &= S(t)d^T x(0) \\ &= (Px)(t).\end{aligned}$$

Therefore P is a projection. Now, let $x \in Im(P)$. Then there exists $y \in X$ such that

$$x(t) = (Py)(t) = S(t)d^T y(0) = S(t)\alpha, \text{ where } \alpha = d^T y(0).$$

That is $x = S\alpha \in ker(L)$. Therefore $Im(P) \subseteq ker(L)$. Now let $x \in ker(L)$. Then $x = S\alpha$ for some $\alpha \in \mathbb{R}$. Applying P yields

$$\begin{aligned}(Px)(t) &= P(S(\cdot)\alpha)(t) \\ &= S(t)d^T S(0)\alpha \\ &= S(t)d^T d\alpha \\ &= S(t)\alpha = x(t).\end{aligned}$$

Thus we have $x \in Im(P) \subseteq ker(L)$. Therefore $Im(P) = ker(L)$ and P is a projection onto the $ker(L)$. □

We note the boundedness of P implies that P is continuous. Furthermore, setting $X_P = Im(P)$ and $X_{I-P} = Im(I - P)$ allows us to write $X = X_P \oplus X_{I-P}$.

Proposition 2.6. *Suppose* $ker\Big(\sum_{i=0}^{N}[B_i\Phi(i)]^T\Big) = span\{c\}$ *for some vector* $c \in \mathbb{R}^n$. *Then*

h is in the image of L if and only if

$$\sum_{i=0}^{N-1} h^T(i) \sum_{k=i+1}^{N} \Big[B_k\Phi(k)\Phi^{-1}(i+1)\Big]^T c = 0.$$

Proof. By proposition 2.2, h is in the image of L if and only if

$$\sum_{i=0}^{N} B_i\Phi(i)x(0) = -\Big(B_1h(0)+B_2\sum_{i=0}^{1}\Phi(2)\Phi^{-1}(i+1)h(i)+\cdots+B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\Big).$$

This is true if and only if

$$B_1h(0)+\cdots+B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i) \in Im\Big(\sum_{i=0}^{N} B_i\Phi(i)\Big).$$

Since $Im\Big(\sum_{i=0}^{N} B_i\Phi(i)\Big) = ker\Big(\Big[\sum_{i=0}^{N} B_i\Phi(i)\Big]^T\Big)^{\perp}$ we have that

$$\Big[B_1h(0)+\cdots+B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\Big]^T\beta = 0$$

for $\beta \in ker\Big(\sum_{i=0}^{N}[B_i\Phi(i)]^T\Big)$. Since c spans $ker\Big(\sum_{i=0}^{N}[B_i\Phi(i)]^T\Big)$ we have h is in the image if L if and only if

$$\Big[B_1h(0)+\cdots+B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\Big]^T c = 0.$$

Rearranging, we obtain

$$\sum_{i=0}^{N-1} h^T(i) \sum_{k=i+1}^{N} \Big[B_k\Phi(k)\Phi^{-1}(i+1)\Big]^T c = 0.$$

$\square$

Defintion 2.7. Suppose $ker\Big(\sum_{i=0}^{N}[B_i\Phi(i)]^T\Big) = span\{c\}$ for some vector $c \in \mathbb{R}^n$. Define

$\psi : \{0, 1, \ldots, N-1\} \to \mathbb{R}^n$ by

$$\psi(t) = \sum_{i=t+1}^{N} \Big[B_i \Phi(i) \Phi^{-1}(t+1) \Big]^T c.$$

Using this definition we are able to say h is in the image of L if and only if $\sum_{i=0}^{N-1} h(i)^T \psi(i) = 0$. Note that this condition is equivalent to $\sum_{i=0}^{N-1} \psi(i)^T h(i) = 0$. This map ψ along with the previous result are key in building the projection onto the image of L. In order to define this projection we need to make use of the following lemma.

Lemma 2.8. *ψ is the zero map if and only if* $\bigcap_{i=0}^{N} ker\big(B_i^T\big) \neq \{0\}$.

Proof. Suppose $v \in \bigcap_{i=0}^{N} ker\big(B_i^T\big)$. Then we have $[B_i \Phi(i)]^T v = 0$ for all $0 \leq i \leq N$. Thus

$$\Big[\sum_{i=0}^{N} B_i \Phi(i) \Big]^T v = \sum_{i=0}^{N} [B_i \Phi(i)]^T v = 0.$$

Therefore $v \in ker\Big(\Big[\sum_{i=0}^{N} B_i \Phi(i) \Big]^T\Big)$. That is $v = \alpha c$ for some real number α. Now

$$\begin{aligned} \psi(t) &= \sum_{i=t+1}^{N} \Big[B_i \Phi(i) \Phi^{-1}(t+1) \Big]^T c \\ &= \sum_{i=t+1}^{N} \Big[B_i \Phi(i) \Phi^{-1}(t+1) \Big]^T \frac{v}{\alpha} \\ &= 0. \end{aligned}$$

Now suppose $\psi \equiv 0$. Then we have

$$\psi(N-1) = \Big[B_N\Phi(N)\Phi^{-1}(N)\Big]^T c = B_N^T c = 0.$$
$$\psi(N-2) = B_{N-1}^T c + \Big[B_N\Phi(N)\Phi^{-1}(N-1)\Big]^T c = B_{N-1}^T c = 0.$$
$$\vdots$$
$$\psi(0) = B_1^T c + \cdots + \Big[B_N\Phi(N)\Phi^{-1}(1)\Big]^T c = B_1^T c = 0.$$

Thus $c \in \bigcap_{i=1}^{N} ker\big(B_i^T\big)$. To see $c \in ker(B_0^T)$, consider $\Big(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)\Big)^T c$. Since $c \in ker\Big(\Big[\sum_{i=0}^{N} B_i\Phi(i)\Big]^T\Big)$ we have

$$\begin{aligned}
0 &= \Big(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)\Big)^T c \\
&= B_0^T c + (B_1\Phi(1))^T c + \cdots + (B_N\Phi(N))^T c \\
&= B_0^T c.
\end{aligned}$$

Therefore $c \in ker(B_0^T)$ and so $c \in \bigcap_{i=0}^{N} ker\big(B_i^T\big)$.

□

Now that we have ψ is not identically zero when $\bigcap_{i=0}^{N} ker\big(B_i^T\big) = \{0\}$, we can define the map $W : Y \to Y$ by

$$(Wh)(t) = \psi(t)\Big(\sum_{i=0}^{N-1} |\psi(i)|^2\Big)^{-1} \sum_{i=0}^{N-1} \psi(i)^T h(i).$$

Proposition 2.9. *Assume* $\bigcap_{i=0}^{N} ker\big(B_i^T\big) = \{0\}$. *Then* $E = I - W$ *is a projection onto the image of* L.

Proof. Linearity and boundedness is clear. It must now be shown that $E^2 = E$ and $Im(E) = Im(L)$. To see that $E^2 = E$ it is sufficient to show that $W^2 = W$. Now consider

$$
\begin{aligned}
(W(Wh))(t) &= W\Big(\psi(\cdot)\Big[\sum_{i=0}^{N-1}|\psi(i)|^2\Big]^{-1}\sum_{i=0}^{N-1}\psi(i)^T h(i)\Big) \\
&= \psi(t)\Big[\sum_{k=0}^{N-1}|\psi(k)|^2\Big]^{-1}\sum_{k=0}^{N-1}\psi^T(k)\psi(k)\Big[\sum_{i=0}^{N-1}|\psi(i)|^2\Big]^{-1}\sum_{i=0}^{N-1}\psi(i)^T h(i) \\
&= \psi(t)\Big(\sum_{i=0}^{N-1}|\psi(i)|^2\Big)^{-1}\sum_{i=0}^{N-1}\psi(i)^T h(i) \\
&= (Wh)(t).
\end{aligned}
$$

Thus W is a projection and so $E = I - W$ is also a projection. Now, to show that E is a projection onto the image of L let $y \in Im(E)$. That is $y = Eh$ for some $h \in Y$. Then

$$
\begin{aligned}
\sum_{i=0}^{N-1}\psi(i)^T(Eh)(i) &= \sum_{i=0}^{N-1}\psi(i)^T h(i) - \psi(i)^T(Wh)(i) \\
&= \sum_{i=0}^{N-1}\psi(i)^T h(i) - \psi(i)^T\psi(i)\Big(\sum_{k=0}^{N-1}|\psi(k)|^2\Big)^{-1}\sum_{k=0}^{N-1}\psi^T(k)h(k) \\
&= \sum_{i=0}^{N-1}\psi(i)^T h(i) - \sum_{k=0}^{N-1}\psi^T(k)h(k) \\
&= 0.
\end{aligned}
$$

Therefore $y \in Im(L)$ and so $Im(E) \subseteq Im(L)$. Conversely, let $y \in Im(L)$. Then

$$
\begin{aligned}
(Ey)(t) &= y(t) - (Wy)(t) \\
&= y(t) - \psi(t)\Big(\sum_{i=0}^{N-1}|\psi(i)|^2\Big)^{-1}\sum_{i=0}^{N-1}\psi(i)^T y(i) \\
&= y(t).
\end{aligned}
$$

Hence $Ey = y$ and so $y \in Im(E)$. Thus $Im(L) \subseteq Im(E)$ and so $Im(E) = Im(L)$. Therefore E is a projection onto the image of L. $\square$

Since we have $X = X_P \oplus X_{I-P}$, we can restrict L to $\overline{L} : X_{I-P} \to Im(L)$ such that $\overline{L}$ is a bijection. Therefore, there exists a bounded linear map $M : Im(L) \to X_{I-P}$ satisfying

(i) $\overline{L}Mh = h$ for all $h \in Im(L)$;

(ii) $M\overline{L}x = x_{I-P}$ for all $x \in X$.

Proposition 2.10. *$Lx = F(x)$ if and only if there exists a real number α such that $x = S\alpha + MEF(x)$ and $\sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n = 0$ where $[v]_k$ denotes the k^{th} row of the vector v.*

Proof.

$$\begin{aligned}
Lx = F(x) &\iff Lx - F(x) = 0 \\
&\iff E(Lx - F(x)) = 0 \text{ and } W(Lx - F(x)) = 0 \\
&\iff Lx = EF(x) \text{ and } (I - E)Lx - WF(x) = 0 \\
&\iff x_{I-P} = MEF(x) \text{ and } WF(x) = 0 \\
&\iff x = x_P + MEF(x) \text{ and } F(x) \in Im(L) \\
&\iff x = S\alpha + MEF(x) \text{ for a real number } \alpha \text{ and } \sum_{i=0}^{N-1} [(F(x))(i)]^T \psi(i) = 0 \\
&\iff x = S\alpha + MEF(x) \text{ and } \sum_{i=0}^{N-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_m(i)) \end{bmatrix}^T \psi(i) = 0 \\
&\iff x = S\alpha + MEF(x) \text{ and } \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n = 0.
\end{aligned}$$

□

To conclude our analysis of the operator suppose that $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ exist, and let $\lim_{x\to\infty} g(x) = g(\infty)$ and $\lim_{x\to-\infty} g(x) = g(-\infty)$. We introduce the following partition

$$\mathcal{O}_0 = \{i \in \{0, 1, \ldots, N-1\} \mid [S(i)]_m = 0\}$$
$$\mathcal{O}_1 = \{i \in \{0, 1, \ldots, N-1\} \mid [S(i)]_m > 0\}$$
$$\mathcal{O}_2 = \{i \in \{0, 1, \ldots, N-1\} \mid [S(i)]_m < 0\}.$$

and then define the following numbers

$$J_1 = g(\infty) \sum_{i\in\mathcal{O}_1} [\psi(i)]_n + g(-\infty) \sum_{i\in\mathcal{O}_2} [\psi(i)]_n$$
$$J_2 = g(-\infty) \sum_{i\in\mathcal{O}_1} [\psi(i)]_n + g(\infty) \sum_{i\in\mathcal{O}_2} [\psi(i)]_n.$$

Proposition 2.11. *Suppose g is continuous, $g(\infty)$, $g(-\infty)$ exist, $\mathcal{O}_0$ is empty, and $J_1, J_2 \neq 0$. Then there exists some real number $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ J_1 has the same sign as K_1 and J_2 has the same sign as K_2 where*

$$K_1 = \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n$$
$$K_2 = \sum_{i=0}^{N-1} g\big([S(i)(-\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n$$

Proof. Let $\epsilon > 0$. Since S and MEF are linear maps on finite dimensional spaces we have that S and MEF are bounded. That is, there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$

$$g(\infty) - \epsilon < g\big([S(i)\alpha + (MEF(x))(i)]_m\big) < g(\infty) + \epsilon$$

where $i \in \mathcal{O}_1$, and

$$g(-\infty) - \epsilon < g\big([S(i)\alpha + (MEF(x))(i)]_m\big) < g(-\infty) + \epsilon$$

where $i \in \mathcal{O}_2$. Now define the sets

$$\mathcal{U}_0 = \{i \in \mathcal{O}_1 \mid [\psi(i)]_n = 0\}$$
$$\mathcal{U}_1 = \{i \in \mathcal{O}_1 \mid [\psi(i)]_n > 0\}$$
$$\mathcal{U}_2 = \{i \in \mathcal{O}_1 \mid [\psi(i)]_n < 0\}.$$

Then for $\alpha \geq \alpha_0$

$$\sum_{i=\mathcal{O}_1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n =$$
$$\sum_{i=\mathcal{U}_1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n + \sum_{i=\mathcal{U}_2} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n.$$

This gives way to

$$\begin{aligned}
(g(\infty) - \epsilon) \sum_{i=\mathcal{U}_1}[\psi(i)]_n + (g(\infty) + \epsilon) \sum_{i=\mathcal{U}_2}[\psi(i)]_n \\
< \sum_{i=\mathcal{O}_1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n \\
< (g(\infty) + \epsilon) \sum_{i=\mathcal{U}_1}[\psi(i)]_n + (g(\infty) - \epsilon) \sum_{i=\mathcal{U}_2}[\psi(i)]_n.
\end{aligned}$$

Similarly, define the sets

$$\mathcal{W}_0 = \{i \in \mathcal{O}_2 \mid [\psi(i)]_n = 0\}$$
$$\mathcal{W}_1 = \{i \in \mathcal{O}_2 \mid [\psi(i)]_n > 0\}$$
$$\mathcal{W}_2 = \{i \in \mathcal{O}_2 \mid [\psi(i)]_n < 0\}.$$

Again, we have

$$\sum_{i=\mathcal{O}_2} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n =$$

$$\sum_{i=\mathcal{W}_1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n + \sum_{i=\mathcal{W}_2} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n.$$

for $\alpha \geq \alpha_0$. Thus

$$\begin{aligned}
(g(-\infty) - \epsilon) \sum_{i=\mathcal{W}_1} [\psi(i)]_n + (g(-\infty) + \epsilon) \sum_{i=\mathcal{W}_2} [\psi(i)]_n \\
< \sum_{i=\mathcal{O}_2} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n \\
< (g(-\infty) + \epsilon) \sum_{i=\mathcal{W}_1} [\psi(i)]_n + (g(-\infty) - \epsilon) \sum_{i=\mathcal{W}_2} [\psi(i)]_n.
\end{aligned}$$

Adding these inequalities yields

$$\begin{aligned}
g(\infty) \sum_{i=\mathcal{U}_1} [\psi(i)]_n + g(\infty) \sum_{i=\mathcal{U}_2} [\psi(i)]_n + g(-\infty) \sum_{i=\mathcal{W}_1} [\psi(i)]_n + g(-\infty) \sum_{i=\mathcal{W}_2} [\psi(i)]_n \\
-\epsilon \bigg(\sum_{i=\mathcal{U}_1} [\psi(i)]_n - \sum_{i=\mathcal{U}_2} [\psi(i)]_n + \sum_{i=\mathcal{W}_1} [\psi(i)]_n - \sum_{i=\mathcal{W}_2} [\psi(i)]_n \bigg) \\
< \sum_{i=\mathcal{O}_1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n + \sum_{i=\mathcal{O}_2} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n \\
< g(\infty) \sum_{i=\mathcal{U}_1} [\psi(i)]_n + g(\infty) \sum_{i=\mathcal{U}_2} [\psi(i)]_n + g(-\infty) \sum_{i=\mathcal{W}_1} [\psi(i)]_n + g(-\infty) \sum_{i=\mathcal{W}_2} [\psi(i)]_n \\
+\epsilon \bigg(\sum_{i=\mathcal{U}_1} [\psi(i)]_n - \sum_{i=\mathcal{U}_2} [\psi(i)]_n + \sum_{i=\mathcal{W}_1} [\psi(i)]_n - \sum_{i=\mathcal{W}_2} [\psi(i)]_n \bigg).
\end{aligned}$$

Simplifying this inequality results in

$$\begin{aligned}
g(\infty) \sum_{i \in \mathcal{O}_1} [\psi(i)]_n + g(-\infty) \sum_{i \in \mathcal{O}_2} [\psi(i)]_n - \epsilon \Bigg(\sum_{i=0}^{N-1} \Big| [\psi(i)]_n \Big| \Bigg) \\
< \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m \big) [\psi(i)]_n \\
< g(\infty) \sum_{i \in \mathcal{O}_1} [\psi(i)]_n + g(-\infty) \sum_{i \in \mathcal{O}_2} [\psi(i)]_n + \epsilon \Bigg(\sum_{i=0}^{N-1} \Big| [\psi(i)]_n \Big| \Bigg).
\end{aligned}$$

Therefore we have

$$J_1 - \epsilon \Bigg(\sum_{i=0}^{N-1} \Big| [\psi(i)]_n \Big| \Bigg) < K_1 < J_1 + \epsilon \Bigg(\sum_{i=0}^{N-1} \Big| [\psi(i)]_n \Big| \Bigg).$$

That is K_1 lies in an interval of radius $\epsilon \Bigg(\sum_{i=0}^{N-1} \Big| [\psi(i)]_n \Big| \Bigg)$ about J_1. Since ψ is not identically zero the radius of the interval is nonzero. Hence, by letting ϵ be arbitrarily small, J_1 and K_1 have the same sign for $\alpha \geq \alpha_0$. We obtain the result for J_2 and K_2 by utilizing a similar argument. $\square$

We can now state and prove the main theorem. We will accomplish this task using the Brouwer Fixed Point Theorem.

Theorem 2.12. *Suppose* $ker\Big(\sum_{i=0}^{N} [B_i \Phi(i)]^T \Big) = span\{c\}$ *for some vector* $c \in \mathbb{R}^n$ *and* $\bigcap_{i=0}^{N} ker\big(B_i^T\big) = \{0\}$. *If*

(i) $g : \mathbb{R} \to \mathbb{R}$ *is continuous;*

(ii) $g(\infty)$ *and* $g(-\infty)$ *exist;*

(iii) $\mathcal{O}_0$ *is empty;*

(iv) $J_1 J_2 < 0$.

Then there is at least one solution to (1)-(2).

Proof. Without loss of generality suppose $J_1 > J_2$. Let α_0 be sufficiently large such that for $\alpha \geq \alpha_0 > rN\|\psi\|$ where $r = \sup_{t\in\mathbb{R}} |g(t)|$, J_1 and K_1 have the same sign and J_2 and K_2 have the same sign.

We will proceed by using the Brouwer Fixed Point Theorem. To this end, we define the functions

$$
\begin{aligned}
&H_{_X} : X \times \mathbb{R} \to X \text{ by } H_{_X}(x,\alpha) = S\alpha + MEF(x),\\
&H_{_\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R} \text{ by } H_{_\mathbb{R}}(x,\alpha) = \alpha - \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n,\\
&H : X \times \mathbb{R} \to X \times \mathbb{R} \text{ by } H(x,\alpha) = (H_{_X}, H_{_\mathbb{R}}).
\end{aligned}
$$

It is clear that H is continuous. Now define

$$
B = \Big\{(x,\alpha) \;\Big|\; \|x\| \leq \|S\|\delta + \|ME\|r \text{ and } |\alpha| \leq \delta \text{ where } \delta = \alpha_0 + rN\|\psi\|\Big\}.
$$

Clearly, B is nonempty, closed, and convex. To invoke the Brouwer Fixed point theorem it is left to show that $H(B) \subseteq B$. We note that $J_1J_2 < 0$, and $J_1 > J_2$ implies

$$
\sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n > 0
$$

and

$$
\sum_{i=0}^{N-1} g\big([S(i)(-\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n < 0
$$

for all $x \in X$ and $\alpha \geq \alpha_0$. So for $\alpha \in [\alpha_0, \delta]$ we have

$$
H_{_\mathbb{R}}(x,\alpha) = \alpha - \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n \leq \alpha
$$

and

$$H_{\mathbb{R}}(x,-\alpha) = -\alpha - \sum_{i=0}^{N-1} g\big([S(i)(-\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n \geq -\alpha.$$

Since $\Big|\sum_{i=0}^{N-1} g\big([S(i)(\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n\Big| \leq rN\|\psi\|$, if $\alpha \in [\alpha_0, \delta]$ then

$$\begin{aligned}
H_{\mathbb{R}}(x,\alpha) &= \alpha - \sum_{i=0}^{N-1} g\big([S(i)\alpha + (MEF(x))(i)]_m\big)[\psi(i)]_n \\
&\geq \alpha - rN\|\psi\| \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
H_{\mathbb{R}}(x,-\alpha) &= -\alpha - \sum_{i=0}^{N-1} g\big([S(i)(-\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n \\
&\leq -\alpha + rN\|\psi\| \\
&\leq 0.
\end{aligned}$$

So for $\alpha \in [\alpha_0, \delta]$ we have $H_{\mathbb{R}}(x,\alpha) \in [-\delta, \delta]$. Now consider $\alpha \in [0, \alpha_0)$, then

$$\begin{aligned}
\Big|H_{\mathbb{R}}(x,\alpha)\Big| &= \Big|\alpha - \sum_{i=0}^{N-1} g\big([S(i)(\alpha) + (MEF(x))(i)]_m\big)[\psi(i)]_n\Big| \\
&\leq |\alpha| + rN\|\psi\| \\
&< \delta.
\end{aligned}$$

Identically, $|H_{\mathbb{R}}(x,-\alpha)| \leq \delta$. Therefore $|H_{\mathbb{R}}(x,\alpha)| \leq \delta$ for all $\alpha \in [-\delta, \delta]$. Now let

$(x, \alpha) \in B$. Then

$$\begin{aligned}\|H_x(x, \alpha)\| &= \|S\alpha + MEF(x)\| \\ &\leq \|S\|\delta + \|ME\|r.\end{aligned}$$

Thus we have $H(x, \alpha) \in B$ for $(x, \alpha) \in B$. Therefore, by the Brouwer Fixed Point theorem, there exists $(\overline{x}, \overline{\alpha}) \in B$ such that $H(\overline{x}, \overline{\alpha}) = (\overline{x}, \overline{\alpha})$. That is

$$\begin{aligned}&\overline{\alpha} - \sum_{i=0}^{N-1} g\big([S(i)\overline{\alpha} + (MEF(x))(i)]_m\big)[\psi(i)]_n = \overline{\alpha} \\ \implies &\sum_{i=0}^{N-1} g\big([S(i)\overline{\alpha} + (MEF(x))(i)]_m\big)[\psi(i)]_n = 0\end{aligned}$$

and

$$\overline{x} = S\alpha + MEF(\overline{x}).$$

Thus by proposition 2.10, $L\overline{x} = F\overline{x}$ and so $(\overline{x}, \overline{\alpha})$ is a solution to the scalar problem (2.1)-(2.2). $\square$

2.4 Example

Consider the scalar equation

$$y(t+2) - 2y(t+1) + y(t) = g(y(t+1)) \tag{2.10}$$

subject to

$$3y(0) - 10y(1) + 2y(6) + y(7) = 0 \tag{2.11}$$

$$-6y(0) - y(3) + 3y(4) = 0 \tag{2.12}$$

We note that in this example, $N = 6$, $n = 2$, and $m = 2$. Now we define

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \text{ for all t} \in \{0, 1, \cdots, 6\},$$

$$B_0 = \begin{bmatrix} 3 & -10 \\ -6 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix}, B_6 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix},$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix},$$

and

$$f(x) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ g(x_2) \end{bmatrix}.$$

Using matrices defined above, we can rewrite the problem as

$$x(t+1) = A(t)x(t) + f(x(t)) \tag{2.13}$$

subject to

$$B_0x(0) + B_3x(3) + B_6x(6) = 0. \tag{2.14}$$

Since $A(t)$ is constant for all t, we can write $A(t) = A$, and so we have $\Phi(t) = A^t$. Further we note that A^t has a closed form formula allowing us to write

$$\Phi(t) = A^t = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}.$$

Now we have

$$B_0 + B_3A^3 + B_6A^6 = \begin{bmatrix} -13 & 9 \\ -13 & 9 \end{bmatrix},$$

and so

$$ker(B_0 + B_3A^3 + B_6A^6) = span \left\{ \begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix} \right\}.$$

We note that $\begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix}$ is a unit vector under the max norm, and that $\begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix}$ plays the role of the unit vector d. We can now define the function S by

$$S(t) = A^t \begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{13} + \frac{4}{13}t \\ 1 + \frac{4}{13}t \end{bmatrix}$$

where $t = 0, 1, \ldots, 5$. Thus $\mathcal{O}_1 = \{1, 2, 3, 4, 5\}$, and $\mathcal{O}_0$ and $\mathcal{O}_2$ are empty. We also note that

$$ker(B_0^T) = span\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\},$$

$$ker(B_3^T) = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\},$$

$$ker(B_6^T) = span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

and so $\bigcap_{i=0}^{6} ker\big(B_i^T\big) = \{0\}$. We now seek to define the map ψ. To do so, consider

$$ker\Big([B_0 + B_3 A^3 + B_6 A^6]^T\Big) = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

Again, we note that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a unit vector, and mention that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ plays the role of the vector c. Therefore

$$\psi(t) = \sum_{i=t+1}^{6} [B_i A^{i-1-t}]^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{cases} \left([B_3 A^{2-t}]^T + [B_6 A^{5-t}]^T\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \text{for t = 0,1,2} \\ [B_6 A^{5-t}]^T \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \text{for t = 3,4,5} \end{cases}$$

$$= \begin{cases} \begin{bmatrix} t-8 \\ 9-t \end{bmatrix} & \text{for t = 0,1,2} \\ \begin{bmatrix} 3t-13 \\ 16-3t \end{bmatrix} & \text{for t = 3,4,5.} \end{cases}$$

Now we must ensure that J_1 and J_2 are of opposite sign. We note that

$$\begin{aligned} J_1 &= g(\infty) \sum_{i \in \mathcal{O}_1} [\psi(i)]_2 + g(-\infty) \sum_{i \in \mathcal{O}_2} [\psi(i)]_2 \\ &= g(\infty) \sum_{i \in \mathcal{O}_1} [\psi(i)]_2 \\ &= g(\infty) \Big[\sum_{i=0}^{2} 9 - i + \sum_{i=3}^{5} 16 - 3i \Big] \\ &= 36 g(\infty). \end{aligned}$$

Similarly we have $J_2 = 36g(-\infty)$. Therefore, to guarantee a solution to (2.10) - (2.12) we can select g to be any continuous function satisfying $g(\infty)g(-\infty) < 0$. For example, consider $g(x) = \dfrac{2}{\pi} \arctan(x)$. Then we have $g(\infty) = 1$ and $g(-\infty) = -1$, and so $J_1 J_2 < 0$. Thus there exists a solution to (2.10) - (2.12).

Chapter 3

Full System at Resonance

3.1 Introduction

In this chapter, we extend the results obtained from chapter two and examine the discrete, nonlinear system subject to a multipoint boundary condition. Namely, we seek sufficient conditions for the existence of solutions to the nonlinear, discrete, multipoint boundary value problem

$$x(t+1) = A(t)x(t) + f(t, x(t)) \quad \text{where } t \in \{0, 1, ..., N-1\} \tag{3.1}$$

under the boundary condition

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0. \tag{3.2}$$

Here $A(t)$ is an invertible $n \times n$ matrix for all $t \in \{0, 1, \ldots, N-1\}$, each B_k is a real valued $n \times n$ matrix, $x : \{0, 1, \ldots, N\} \to \mathbb{R}^n$, and $f : \{0, 1, \ldots N-1\} \times \mathbb{R}^n \to \mathbb{R}^n$.

The approach taken will resemble that done in the previous chapter. We will rely on the related linear homogeneous equation

$$x(t+1) = A(t)x(t) \text{ where } t \in \{0, 1, \ldots, N-1\}, \tag{3.3}$$

and the linear inhomogeneous equation

$$x(t+1) = A(t)x(t) + h(t) \text{ where } t \in \{0, 1, \ldots, N-1\}, h \in \mathbb{R}^n. \tag{3.4}$$

We will then examine the case in which the solution space to (3.3) subject to the boundary condition (3.2) is one dimensional. If 0 is the only solution to the problem, then theorem 2.3 guarantees a solution to (3.1) - (3.2) exists where f satisfies the sublinear growth condition established in chapter 2. In the case of a one dimensional solution space we will restrict the behavior of the nonlinearity $f(t, x(t))$ for large values of x. As done in the previous chapter, we will introduce a projection scheme and then we will use degree theory to attain the desired result.

This problem is a clear extension of the previous section as we no longer require scalar hypothesis. The methods in this section resemble previous work done by Pollack and Taylor [9].

3.2 Preliminaries

For the remainder of this chapter we will assume the following fundamental hypothesis:

H1: $\displaystyle\bigcap_{i=0}^{N} ker\left(B_i^T\right) = \{0\}$.

H2: $f(t, \cdot)$ is continuous on $\mathbb{R}^n$ for all $t \in \{0, 1, \cdots, N-1\}$ and there exists a real number b such that $|f(t, x)| \leq b$ for all $t \in \{0, 1, \cdots, N-1\}$ and $x \in \mathbb{R}^n$.

H3: There exists $\alpha_0 \geq 0$ and a decreasing function $\delta : [\alpha_0, \infty) \to [0, \infty)$ such that:

(i) $\displaystyle\lim_{\alpha \to \infty} \delta(\alpha) = 0$;

(ii) If $t \in \{0, 1, \cdots, N-1\}$, $|x_0| > \alpha_0 + s$ and $|x_1| \leq s$, then $|f(t, x_0) - f(t, x_0 + x_1) \leq \delta(|x_0| - s)s$.

H4: The linear homogeneous problem (3.3)-(3.2) has a one dimensional solution space.

As in the case of the nonlinear, scalar, discrete problem, the solution space of the homogeneous system is fundamental to our analysis of the nonlinear function f. We reiterate that a fundamental matrix solution to the linear homogeneous difference equation

$$x(t+1) = A(t)x(t) \text{ for } t = 0, 1, 2, \ldots N-1$$

is given by

$$\Phi(t) = \begin{cases} I & t = 0 \\ A(t-1)A(t-2)\cdots A(0) & t = 1, 2, \cdots. \end{cases}$$

By hypothesis $H4$, we know $ker\Big(\sum_{i=0}^{N} B_i\Phi(i)\Big) = span\{d\}$ for some $d \in \mathbb{R}^n$. Furthermore, $H4$ guarantees that $ker\Big(\sum_{i=0}^{N}[B_i\Phi(i)]^T\Big) = span\{c\}$ for a vector $c \in \mathbb{R}^n$. So as done in chapter two, we can define the map $S : \{0, 1, \cdots, N-1\} \to \mathbb{R}^n$ by

$$S(t) = \Phi(t)d,$$

and we can define the map $\psi : \{0, 1, \ldots, N-1\} \to \mathbb{R}^n$ by

$$\psi(t) = \sum_{i=t+1}^{N} \Big[B_i\Phi(i)\Phi^{-1}(t+1)\Big]^T c. \tag{3.5}$$

Using proposition 2.6 and definition 2.7, we have (3.4)-(3.2) has a solution if and only if $\sum_{i=0}^{N-1} \psi(i)^T h(i) = 0$. Lastly, we define $J : \mathbb{R} \to \mathbb{R}$ by

$$J(\alpha) = \sum_{i=0}^{N-1} \psi(i)^T f(i, S(i)\alpha). \tag{3.6}$$

H5: $J_1 = \lim_{\alpha \to \infty} J(\alpha)$ and $J_2 = \lim_{\alpha \to -\infty} J(\alpha)$ exist and are of differing sign.

Under these fundamental hypothesis, we can state the theorem at hand

Theorem 3.1. *If $A(t), B_k$, and f are as defined above and hypotheses $H1 - H5$ are satisfied, then there is a solution to $(3.1) - (3.2)$.*

To prove theorem 3.1, we will first state and prove a more general result about the existence of solutions to linear mappings that does not rely on the setting of difference equations. Theorem 3.1 will then be able to be proven as a consequence of this more general result.

3.3 General Setting

Theorem 3.2. *Let X and Y be finite dimensional, normed, linear spaces with the same dimension, $L : X \to Y$ be a linear map such that $ker(L) = span\{k\}$ where $\|k\| = 1$. Suppose $\psi^* : Y \to \mathbb{R}$ is a linear functional satisfying the property $y \in Im(L)$ if and only if $\psi^* y = 0$. Furthermore, suppose that $F : X \to Y$ is continuous on X and $\|F(x)\| \leq b$ for all $x \in X$. Then $Lx = F(x)$ has a solution if the following additional hypotheses are satisfied:*

H6: *For all positive real numbers, s, there exists $A_s \geq 0$ and $\mu_s : [A_s, \infty) \to [0, \infty)$ such that:*

(i) $\lim_{\alpha \to \infty} \mu_s(\alpha) = 0$

(ii) if $|\alpha| \geq A_s$ *and* $\|x_1\| \leq s$*, then* $\|F(\alpha k) - F(\alpha k + x_1)\| \leq \mu_s(|\alpha|)s$

H7: $\overline{J_1} = \lim_{\alpha \to \infty} \psi^* F(\alpha k)$ *and* $\overline{J_2} = \lim_{\alpha \to -\infty} \psi^* F(\alpha k)$ *exist and are of differing sign.*

Proof. Let $X_0 = ker(L) = span\{k\}$ where $\|k\| = 1$. Since X is a finite dimensional linear space, we know there exists a complement X_1 such that $X = X_0 \oplus X_1$. Let P_0 and P_1 be projections associated with X_0 and X_1. Again, since X is finite, the projections P_0 and P_1 exist and are bounded. Let $Y_1 = Im(L)$ and Y_0 be any complement of Y_1 such that $Y = Y_0 \oplus Y_1$. Define E_0 and E_1 to be projections on Y associated with Y_0 and Y_1. The finite dimensionality of Y ensures that these projections exist and are bounded. We can now restrict the linear map L to $\overline{L} : X_1 \to Y_1$. We note that $\overline{L}$ is a continuous bijection and so there exists a bounded inverse $\overline{M} : Y_1 \to X_1$. We make note of the relationship between the projections and the maps L and $\overline{M}$: for $x \in X$ we have $P_1 x \in X_1$, this implies

$$\overline{M}Lx = \overline{M}L(P_0 x + P_1 x) = \overline{M}LP_1 x = P_1 x.$$

That is $\overline{M}L = P_1$. In a similar manner, suppose $y \in Y$, then $E_1 y \in Y_1$. Therefore

$$L\overline{M}E_1 y = E_1 y,$$

thus $L\overline{M}E_1 = E_1$.

<u>Claim:</u> $Lx = F(x)$ if and only if there exists a real number α such that $x_1 = \overline{M}E_1 F(\alpha k + x_1)$ and $\psi^* F(\alpha k + x_1) = 0$.

<u>Proof:</u>

$$\begin{aligned}
Lx = F(x) &\iff Lx - F(x) = 0 \\
&\iff \psi^*(Lx - F(x)) = 0 \text{ and } \overline{M}E_1(Lx - F(x)) = 0 \\
&\iff \psi^* Lx - \psi^* F(x) = 0 \text{ and } \overline{M}E_1 Lx = \overline{M}E_1 F(x) \\
&\iff \psi^* F(\alpha k + x_1) = 0 \text{ and } x_1 = \overline{M}E_1 F(\alpha k + x_1)
\end{aligned}$$

$\blacksquare$

Therefore $Lx = F(x)$ is equivalent to the system

$$x_1 = \overline{M}E_1F(\alpha k + x_1) \tag{3.7}$$

$$0 = \psi^* F(\alpha k + x_1). \tag{3.8}$$

Now define the functions

$$H_{X_1} : X_1 \times \mathbb{R} \to X_1 \text{ by } H_{_X}(x,\alpha) = x_1 - \overline{M}E_1F(\alpha k + x_1),$$

$$H_{_\mathbb{R}} : X_1 \times \mathbb{R} \to \mathbb{R} \text{ by } H_{_\mathbb{R}}(x,\alpha) = \psi^* F(\alpha k + x_1),$$

$$H : X_1 \times \mathbb{R} \to X_1 \times \mathbb{R} \text{ by } H(x,\alpha) = (H_{_X}, H_{_\mathbb{R}}).$$

We note if $H(x_1, \alpha) = (0,0)$ then $x = \alpha k + x_1$ satisfies $Lx = F(x)$. So we seek to use degree theory to find a zero of the function H on $X_1 \times \mathbb{R}$. We will accomplish this goal by constructing an open set $B \subset X_1 \times \mathbb{R}$ where the topological degree $deg[H, B, 0] = 1$. By $H7$, $\overline{J_1}\,\overline{J_2} < 0$. Without loss of generality suppose $\overline{J_1} > 0$ and $\overline{J_2} < 0$. To create B, we select a real number s such that $\|\overline{M}E_1\|b \leq s$. Now choose $\alpha_1 \geq A_s$ satisfying:

(i) $0 < \dfrac{\overline{J_1}}{2} < \psi^* F(\alpha_1 k)$;

(ii) $\psi^* F(-\alpha_1 k) < \dfrac{\overline{J_2}}{2} < 0$;

(iii) $\alpha_1 > \max\{\overline{J_1}, -\overline{J_2}\}$;

(iv) $\mu_s(\alpha_1) \leq \dfrac{\min\{\overline{J_1}, -\overline{J_2}\}}{2\|\psi^*\|s}$.

Now define $B \subseteq X_1 \times \mathbb{R}$ to be

$$B = \Big\{(x_1, \alpha) \;\Big|\; \|x_1\| < s \text{ and } |\alpha| < \alpha_1\Big\}.$$

Let $\overline{B}$ denote the closure of B. We seek to use the invariance of topological degree with respect to homotopy to show that the map H has at least one zero. To this end, we define the functions

$G_{X_1} : \overline{B} \times [0,1] \to X_1$ by $G_{X}(x_1, \alpha, \lambda) = x_1 - \lambda \overline{M} E_1 F(\alpha k + x_1)$,

$G_{\mathbb{R}} : \overline{B} \times [0,1] \to \mathbb{R}$ by $G_{\mathbb{R}}(x_1, \alpha, \lambda) = (1-\lambda)\alpha + \lambda \psi^* F(\alpha k) - \lambda \psi^* \big[F(\alpha k) - F(\alpha k + x_1)\big]$,

$G : \overline{B} \times [0,1] \to \mathbb{X} \times \mathbb{R}$ by $G(x_1, \alpha, \lambda) = (G_{X_1}, G_{\mathbb{R}})$.

For each $\lambda \in [0,1]$ let $G_\lambda = G(\cdot, \cdot, \lambda)$. We note that G is continuous and $G_0(x_1, \alpha) = (x_1, \alpha)$ and $G_1(x, \alpha) = H(x, \alpha)$. Since $(0,0) \in B$, we know $deg[G_0, B, 0] = 1$. It is left to show that $G_\lambda(x, \alpha)$ is nonzero along the boundary of B for all $\lambda \in [0,1]$. Let $(x_1, \alpha) \in \partial B$ and $\lambda \in [0,1]$. Then we have $\|x_1\| = s$. Thus

$$
\begin{aligned}
\|G_{X_1}(x_1, \alpha, \lambda)\| &= \|x_1 - \lambda \overline{M} E_1 F(\alpha k + x_1)\| \\
&\geq \|x_1\| - \lambda \|\overline{M} E_1 F(\alpha k + x_1)\| \\
&\geq s - \|\overline{M} E_1\| b \\
&> 0.
\end{aligned}
$$

That is $G_{X_1}(x_1, \alpha, \lambda) \neq 0$ on the boundary of B for all $\lambda \in [0,1]$. Again, suppose $(x_1, \alpha) \in \partial B$ and $\lambda \in [0,1]$. Thus $|\alpha| = \alpha_1$. Now consider

$$
\begin{aligned}
\big|G_{\mathbb{R}}(x_1, \alpha, \lambda)\big| &= \big|(1-\lambda)\alpha + \lambda \psi^* F(\alpha k) - \lambda \psi^* [F(\alpha k) - F(\alpha k + x_1)]\big| \\
&\geq \big|(1-\lambda)\alpha + \lambda \psi^* F(\alpha k)\big| - \|\psi^*\| \big|F(\alpha k) - F(\alpha k + x_1)\big| \\
&\geq \min\{\alpha, \psi^* F(\alpha k)\} - \|\psi^*\| \mu_s(\alpha_1) s \\
&> \frac{\overline{J_1}}{2} - \|\psi^*\| \frac{\min\{\overline{J_1}, -\overline{J_2}\}}{2\|\psi^*\| s} s \\
&\geq 0.
\end{aligned}
$$

Similarly we obtain $\big|G_{\mathbb{R}}(x_1, \alpha, \lambda)\big| > 0$ when $\alpha = -\alpha$. Thus $G_{\mathbb{R}}(x_1, \alpha, \lambda) \neq 0$ along the boundary of B and for all $\lambda \in [0,1]$. Therefore $G_\lambda \neq (0,0)$ on the boundary of B for all $\lambda \in [0,1]$. We then have

$$deg[H, B, 0] = deg[G_1, B, 0] = deg[G_0, B, 0] = 1.$$

Therefore there exists $(\overline{x}, \overline{\alpha}) \in B$ such that such that $H(\overline{x}, \overline{\alpha}) = (0,0)$, hence we have a solution to $Lx = F(x)$. $\square$

3.4 Difference Equation

We will now use Theorem 3.2 to prove our main result.

Theorem 3.1. *If* $A(t), B_k$*, and* f *are as defined above and hypotheses* $H1 - H5$ *are satisfied, then there is a solution to* $(3.1) - (3.2)$.

Proof. We seek to verify all hypotheses to Theorem 3.2 are satisfied. We begin by defining $Z = \{\phi : \{0, 1, ..., N\} \to \mathbb{R}^n\}$ and $\beta = B_0 x(0) + B_1(1)x(1) + \cdots + B_N x(N)$. We note that $H1$ is equivalent to $rank(\beta) = n$. Denote $X = \ker(\beta) \subset Z$ and let $Y = \{\phi : \{0, 1, ..., N-1\} \to \mathbb{R}^n\}$. We note that $dim(X) = nullity(\beta) = dim(Z) - rank(\beta) = (N+1)n - n = Nn = dim(Y)$. We define the following norms on the sets X and Y:

$$\|x\| = \sup_{t=0,1,\cdots,N} |x(t)|,$$

$$\|y\| = \sup_{t=0,1,\cdots,N-1} |y(t)|,$$

where $|\cdot|$ is any norm on $\mathbb{R}^n$. Lastly define the operators

$$L : X \to Y \text{ defined by } (Lx)(t) = x(t+1) - A(t)x(t);$$

$$F : X \to Y \text{ defined by } (F(x))(t) = f(t, x(t))$$

where $t \in \{0, 1, \cdots, N-1\}$. We note that $x \in X$ solves $x(t+1) = A(t)x(t)$ if and only if $x \in ker(L)$. Furthermore, x solves (3.1)-(3.2) if and only if $Lx = F(x)$. Hypothesis H4 guarantees that $ker(L) = span\{k\}$ where $\|k\| = 1$. If we define $\psi^* : Y \to \mathbb{R}$ by

$$\psi^* h = \sum_{i=0}^{N-1} \psi(i)^T h(i)$$

then we have $\psi^* h = 0$ if and only if $h \in Im(L)$ by proposition 2.6. Since $(F(x))(t) = f(t, x(t))$ we have F is continuous and $\|F(x)\| \leq b$.

The only remaining hypotheses to verify are H6 and H7. To verify H6, we set

$$m = \min_{t=0,1,\ldots,N} |k(t)| \tag{3.9}$$

where k is the nonzero, unit vector in $ker(L)$. Since $A(t)$ is invertible for each $t \in \{0, 1, \ldots, N\}$, it follows that $k(t) \neq 0$ for all $t \in \{0, 1, \ldots, N\}$. Therefore $0 < m \leq 1$. Let α_0 be the real number assured by H3 and set $A_s = \dfrac{\alpha_0 + s}{m}$. Define $\mu_s : [A_s, \infty) \to [0, \infty)$ by

$$\mu_s(\alpha) = \delta(m\alpha - s). \tag{3.10}$$

We note that μ_s is well defined and $\lim_{\alpha \to \infty} u_s(\alpha) = 0$. To verify the second condition of H6 suppose $|\alpha| \geq A_s$ and $\|x_1\| < s$. Then we have $|\alpha k(t)| \geq |\alpha| m \geq A_s m = \alpha_0 + s$ for $t \in \{0, 1 \ldots, N-1\}$. So by hypothesis H3 we have

$$\begin{aligned} \|F(\alpha k) - F(\alpha k + x_1)\| &= \sup_{t=0,1,\ldots,N-1} |f(t, \alpha k(t)) - f(t, \alpha k(t) + x_1(t))| \\ &\leq \sup_{t=0,1,\ldots,N-1} \delta(|\alpha k(t)| - s)s \\ &\leq \delta(|\alpha| m - s)s \\ &= \mu_s(|\alpha|)s. \end{aligned}$$

Therefore hypothesis H6 is satisfied. Lastly, if we set $S(t) = k(t)$ then we have

$$\overline{J_1} = \lim_{\alpha \to \infty} \psi^* F(\alpha k) = \lim_{\alpha \to \infty} \sum_{i=0}^{N-1} \psi(i)^T f(i, k(i)\alpha)$$

and

$$\overline{J_2} = \lim_{\alpha \to -\infty} \psi^* F(\alpha k) = \lim_{\alpha \to -\infty} \sum_{i=0}^{N-1} \psi(i)^T f(i, k(i)\alpha)$$

which exist and are of opposite sign by H5. Therefore all hypotheses of Theorem 3.2 have been satisfied and so there exists a solution to $Lx = F(x)$. $\square$

3.5 Example

Consider the autonomous nonlinear three-point boundary value problem:

$$x(t+1) = Ax(t) + f(x(t)) \tag{3.11}$$

where $t = 0, 1, \dots, 4$, subject to

$$B_0 x(0) + B_2 x(2) + B_5 x(5) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.12}$$

Here we define

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} -3 & -5 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix},$$

and $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \dfrac{(x_1 - x_2)^3}{1 + (x_1^2 + x_2^2)^{\frac{3}{2}}} + v_1 \\ \dfrac{(x_1 + x_2)^3}{1 + (x_1^2 + x_2^2)^{\frac{3}{2}}} + v_2 \end{bmatrix}.$$

where v_1 and v_2 are constants. We seek to show that (3.11)-(3.12) satisfies all the hypothesis of theorem 3.1. We begin by noting that A is an invertible constant matrix. Now, since A is constant we have $\Phi(t) = A^t$. Then we have

$$B_0 + B_2A^2 + B_5A^5 = \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix},$$

and so

$$ker(B_0 + B_2A^2 + B_5A^5) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

If we use the max norm on $\mathbb{R}^2$, we have that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a unit vector and so we can define the map $S : \{0, 1, \ldots, 5\} \to \mathbb{R}^2$ by

$$S(t) = A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We note that the map S spans the solution space of the associated linear problem and so H4 is satisfied. Next, we observe that

$$ker(B_0^T) = span\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\},$$

$$ker(B_2^T) = span\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\},$$

$$ker(B_5^T) = span\left\{\begin{bmatrix}-1\\0\end{bmatrix}\right\}$$

hence H1 is satisfied. To assist in the verification of H2 let

$$\tilde{f}(x) = \tilde{f}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}\dfrac{(x_1-x_2)^3}{1+(x_1^2+x_2^2)^{\frac{3}{2}}}\\ \dfrac{(x_1+x_2)^3}{1+(x_1^2+x_2^2)^{\frac{3}{2}}}\end{bmatrix}.$$

Since $\tilde{f}$ is continuous, $|\tilde{f}(x)| < \sqrt{2}^3$ for all x, $f(x) = \tilde{f}(x) + v$ and v is constant, the conditions of H2 are met.

We note that hypothesis H3 will be satisfied if f is differentiable and if $\|Df(x)\| \to 0$ as $|x| \to \infty$. Given our function, f, we have

$$Df(x) = \frac{1}{\sigma^2}\begin{bmatrix}3(x_1-x_2)^2\sigma - 3x_1\sqrt{x_1^2+x_2^2}(x_1-x_2)^3 & -3(x_1-x_2)^2\sigma - 3x_2\sqrt{x_1^2+x_2^2}(x_1-x_2)^3\\ 3(x_1+x_2)^2\sigma - 3x_1\sqrt{x_1^2+x_2^2}(x_1+x_2)^3 & 3(x_1+x_2)^2\sigma - 3x_2\sqrt{x_1^2+x_2^2}(x_1+x_2)^3\end{bmatrix}$$

where $\sigma = (x_1^2+x_2^2)^{\frac{3}{2}} + 1$ and $x \neq 0$. We note that the matrix norm compatible with the max norm on $\mathbb{R}^2$ is the infinity matrix norm defined as $\|A\|_\infty = \max\limits_{1\leq i\leq n}\sum\limits_{j=1}^{n}|a_{ij}|$. Thus we have

$$\|Df(x)\|_\infty \leq \frac{|3(x_1-x_2)^2\sigma - 3x_1\sqrt{x_1^2+x_2^2}(x_1-x_2)^3| + |3(x_1-x_2)^2\sigma + 3x_2\sqrt{x_1^2+x_2^2}(x_1-x_2)^3|}{\sigma^2}.$$

To help interpret this inequality we convert to polar coordinates by letting $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Doing so we obtain

$$\begin{aligned}
\|Df(x)\|_\infty & \\
&\leq \frac{|3r^2(\sin\theta-\cos\theta)^2[r^3+1] - 3r^5\cos\theta(\cos\theta-\sin\theta)^3|}{(r^3+1)^3} \\
&+ \frac{|3r^2(\sin\theta-\cos\theta)^2[r^3+1] + 3r^5\sin\theta(\cos\theta-\sin\theta)^3|}{(r^3+1)^3} \\
&\leq \frac{6r^2|r^3+1| + 3\sqrt{2}^3|r^5| + 6r^2|r^3+1| + 3\sqrt{2}^3|r^5|}{(r^3+1)^2} \\
&\leq \frac{30|r^5|+12r^2}{(r^3+1)^2} \\
&\leq \frac{30|r^5|+12r^2}{r^6+1}.
\end{aligned}$$

Therefore, if we define $\delta : [2,\infty) \to (0,\infty)$ by $\delta(\alpha) = \dfrac{30\alpha^5+12\alpha^2}{\alpha^6+1}$ we have $\|Df(x)\|_\infty \leq \delta(\alpha)$ for $2 \leq \alpha \leq |x|$. Furthermore, $\lim_{\alpha\to\infty}\delta(\alpha) = 0$ and so H3 is satisfied.

The only remaining hypothesis to verify is H5. To this end note

$$\lim_{\alpha\to\infty}\tilde{f}(\alpha x) = \lim_{\alpha\to\infty}\begin{bmatrix} \dfrac{\alpha^3(x_1-x_2)^3}{1+\alpha^3(x_1^2+x_2^2)^{\frac{3}{2}}} \\ \dfrac{\alpha^3(x_1+x_2)^3}{1+\alpha^3(x_1^2+x_2^2)^{\frac{3}{2}}} \end{bmatrix} = \begin{bmatrix} \dfrac{(x_1-x_2)^3}{(x_1^2+x_2^2)^{\frac{3}{2}}} \\ \dfrac{(x_1+x_2)^3}{(x_1^2+x_2^2)^{\frac{3}{2}}} \end{bmatrix}.$$

Let $\gamma(x) = \lim_{\alpha\to\infty}\tilde{f}(\alpha x)$. Note that γ is an odd function of x and that $\lim_{\alpha\to\infty} f(\alpha x) = \gamma(x)+v$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Now let us compute the numbers J_1 and J_2:

$$\begin{aligned} J_1 &= \lim_{\alpha \to \infty} \sum_{i=0}^{N-1} \psi(i)^T f(\alpha S(i)) \\ &= \sum_{i=0}^{N-1} \psi(i)^T \lim_{\alpha \to \infty} f(\alpha S(i)) \\ &= \sum_{i=0}^{N-1} \psi(i)^T (\gamma(S(i)) + v) \end{aligned}$$

and

$$\begin{aligned} J_2 &= \lim_{\alpha \to -\infty} \sum_{i=0}^{N-1} \psi(i)^T f(\alpha S(i)) \\ &= \sum_{i=0}^{N-1} \psi(i)^T \lim_{\alpha \to -\infty} f(\alpha S(i)) \\ &= \sum_{i=0}^{N-1} \psi(i)^T (-\gamma(S(i)) + v). \end{aligned}$$

Therefore, we have

$$J_1 J_2 = -\left(\sum_{i=0}^{N-1} \psi(i)^T \gamma(S(i)) \right)^2 + \left(\sum_{i=0}^{N-1} \psi(i)^T v \right)^2.$$

In order to guarantee that J_1 and J_2 are of differing signs we need

$$\left| \sum_{i=0}^{N-1} \psi(i)^T v \right| < \left| \sum_{i=0}^{N-1} \psi(i)^T \gamma(S(i)) \right|. \tag{3.13}$$

First, we must ensure that the right hand side of (3.13) is not zero. To do this we recruit the help of MATLAB to calculate the sum in question. The first step of calculating this sum is to examine the map ψ just as was done in the scalar example. We note

$$ker\Big([B_0 + B_2A^3 + B_5A^5]^T\Big) = span\left\{\begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix}\right\}.$$

furthermore, $\begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix}$ is a unit vector under the max norm. Using this vector we can define the map ψ in the following manner

$$\begin{aligned}
\psi(t) &= \sum_{i=t+1}^{5} [B_iA^{i-1-t}]^T \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix} \\
&= \begin{cases} \Big([B_2A^{1-t}]^T + [B_5A^{4-t}]^T\Big) \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix} & \text{for t = 0,1} \\ \\ [B_5A^{4-t}]^T \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix} & \text{for t = 2,3,4.} \end{cases}
\end{aligned}$$

With this piecewise definition of ψ we utilize the computational strength of MATLAB to calculate the sum in question. Doing so, we obtain

$$\left|\sum_{i=0}^{N-1} \psi(i)^T \gamma(S(i))\right| \approx 5.6490.$$

It remains to decipher inequality (3.13). To help us do so we introduce the following notation:

$$\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \text{ and } \gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \end{bmatrix}.$$

Now define

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \psi_1(1) \\ \psi_2(1) \\ \vdots \\ \psi_1(4) \\ \psi_2(4) \end{bmatrix}, \ \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_1(S(0)) \\ \gamma_2(S(0)) \\ \gamma_1(S(1)) \\ \gamma_2(S(1)) \\ \vdots \\ \gamma_1(S(4)) \\ \gamma_2(S(4)) \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_1 \\ v_2 \\ \vdots \\ v_1 \\ v_2 \end{bmatrix}$$

Using this new notation we note that (3.13) is equivalent to

$$|\boldsymbol{\Psi} \cdot \mathbf{V}| < |\boldsymbol{\Psi} \cdot \boldsymbol{\Gamma}|. \tag{3.14}$$

Since $\mathbf{V}$ is constant and does not influence $\boldsymbol{\Psi}$ and $\boldsymbol{\Gamma}$, we can choose v_1 and v_2 such that

$$|\mathbf{V}|_2 < \frac{|\boldsymbol{\Psi} \cdot \boldsymbol{\Gamma}|}{|\boldsymbol{\Psi}|_2}.$$

We note that $|\mathbf{V}|_2 = \sqrt{5}|v|_2$. Thus, choosing v_1 and v_2 such that

$$|v|_2 < \frac{|\boldsymbol{\Psi} \cdot \boldsymbol{\Gamma}|}{\sqrt{5}|\boldsymbol{\Psi}|_2}. \tag{3.15}$$

will guarantee that the crucial inequality holds and so J_1 and J_2 will be of differing signs. Again using MATLAB, we find that

$$\frac{|\boldsymbol{\Psi} \cdot \boldsymbol{\Gamma}|}{\sqrt{5}|\boldsymbol{\Psi}|_2} \approx \frac{5.6490}{11.7659} \approx 0.4801. \tag{3.16}$$

Therefore if we choose $v = \begin{bmatrix} \frac{1}{5} \\ \frac{\sqrt{3}}{5} \end{bmatrix}$ we have

$$|v|_2 = \frac{2}{5} < 0.4801 \approx \frac{|\mathbf{\Psi} \cdot \mathbf{\Gamma}|}{\sqrt{5}|\mathbf{\Psi}|_2}.$$

Thus the final hypothesis H5 is satisfied, guaranteeing a solution to (3.11)-(3.12).

Chapter 4

Conclusion

4.1 Future Directions

The results of this thesis show that under certain restrictions on the nonlinearity f we can establish the existence of solutions to the discrete, nonlinear, multipoint boundary value problem of study. One possible extension of this work is to lift the restriction of a bounded nonlinearity and examine a sublinear growth condition similar to the one imposed when the linear map was invertible. Another potential avenue of study is extending the dimensionality of the associated linear homogeneous problem. In the scalar setting, we supposed that the nonlinear term g only relied on y at a single time step, namely $y(t+m-1)$ for some m. An extension of this is to allow g to depend on y at more than one time step. We note that these results have been proven for difference equations, so a natural extension is to think about the applicability to differential equations. More generally, if we can extend the discussed theorems to a general time-scale, we will have results that hold for both difference equations and differential equations.

Bibliography

[1] ETHERIDGE, D. L., AND RODRIGUEZ, J. Scalar discrete nonlinear two-point boundary value problems. *Journal of Difference Equations and Applications* (1996), 127–144.

[2] GENG, F., AND CUI, M. Multi-point boundary value problem for optimal bridge design. *International Journal of Computer Mathematics 87* (2010), 1051 – 1056.

[3] HAUGHT, D. On the existence of solutions to discrete, two point, non-linear boundary value problems. Master's thesis, 2010.

[4] KREYSZIG, E. *Introductory Functional Analysis with Applications*. John Wiley & Sons, 1989.

[5] LEE, J. *Introduction to Topological Manifolds*. Springer, 2011.

[6] LUENBERGER, D. G. *Introduction to Dynamic Systems. Theory, Models, and Applications.* John Wiley & Sons, 1979.

[7] MARONCELLI, D., AND RODRIGUEZ, J. On the solvability of multipoint boundary value problems for discrete systems at resonance. *Journal of Difference Equations and Applications* (2013).

[8] OUTERELO, E., AND RUIZ, J. *Mapping Degree Theory*. American Mathematical Society, 2009.

[9] POLLACK, D., AND TAYLOR, P. Multipoint boundary value problems for discrete nonlinear systems at resonance. *International Journal of Pure and Applied Mathematics 63* (2010), 311 – 326.

[10] RODRIGUEZ, J., AND TAYLOR, P. Scalar discrete nonlinear multipoint boundary value problems. *Journal of Mathematical Analysis and Applications 330* (2006), 891 – 890.

[11] ROUCHE, N., AND MAHWIN, J. *Ordinary Differential Equations: Stability and Periodic Solutions*. Pitman Advanced Publishing, 1980.

Appendix A

MATLAB Code

```
%Define t as symbolic variable
syms t u v


%Choose Integer N
N = input('Select Integer N: ')

%Get dimension of matrix A
n = input('Select n to generate square matrices: ')


%Create Symbolic Matrix A
A = sym('a',n)

%Create Variable to store boundary indices
ind=zeros(1,N+1);

%Initialize Matrices B_i, BT_i, NB_i
```

```
B=cell(N+1,1);
NBT=cell(N+1,1);
for i=1:N+1
    B{i} = zeros(n);
    BT{i} = zeros(n);
end



%Get user input for matrix A(t)
for ro=1:n
      for co=1:n
          prompt=sprintf('Enter the value for A(%d,%d): ',ro,co)
          A(ro,co) = input(prompt);
      end
end
A

%Check for invertibility
for i=1:N
    if cond(subs(A,t,i-1))==inf
        Ainv = sprintf('The matrix is not invertible for all t. ...
            Hypothesis 1.1 is not satisfied.');
        break;
    else
        Ainv = sprintf('The matrix is invertible for all t. ...
            Hypothesis 1.1 is satisfied.');
    end
end
disp(Ainv)

%%% NOTE: You must enter B matrices as an array %%%
```

```
%%% i.e. [1 2; 3 4] will create the 2x2 matrix with entries 1 2 3 ...
    4 $$$

%Get B_i matrices
i=0
while i<N+1
    prompt =sprintf('Specify the index i : ')
    i =input(prompt);
    ind(i+1)=i;
    prompt2=sprintf("Enter the values for B_%d. Use a semicolon ...
        to specify a new row.",i)
    B{i+1} =input(prompt2);

    if i > N-1
        break
    end
end

B;
ind;


%Get Transpose of B_i
for i=1:N+1
    BT{i} = transpose(B{i});
end



%Calculate Kernel of (B_i)^T
for i=1:N+1
    %Calculates Null Space for nonzero B_i
    if ind(i)==i-1
```

```
        NBT{i}=null(BT{i});
        %Stores zero vector for matrices with nullity 0.
        if isempty(NBT{i}) == 1
           NBT{i}= zeros(1,n);
        end
    end
end


%Store null space of nonzero B_i^T
k=1
for i=1:N+1
    if ~isempty(NBT{i}) == 1
        KBT{k} = NBT{i};
        k=k+1;
    end
end
KBT


%Initialize matrix D
D = zeros(n);

%Initialize Dummy Matrix to Calculate State-Transisiton Matrix
AA = eye(n);

%Create Logical Test to determine if A is constant
K = isSymType(A,'constant');
if K == ones(n,n)
    %Calculate Matrix D for Constant A
        for i=1:N+1
             D = D+B{i}*A^(i-1);
        end
```

```
else
        %Calculate Matrix D for A(t)
        for i=1:N+1
             AA = subs(A,t,i-1)*AA;
             DD = DD+B{i}*AA;
        end
end
D

%Nullity of D
ND = null(D)
if numel(ND) == 2 && norm(ND) ~=0
    fprintf("Solution space is one dimensional. Hypothesis H1.5 ...
       is satisfied.")
end

NDT = null(transpose(D))



%Check intersection of ker(B_i)^T
%initialize dummy intersection
NN = size(KBT,2);
    DummyInt=cell(NN-1,1);
%Check First intersection
    DummyInt{1} = intersect(KBT{1},KBT{2});
%Loop
    for i=2:NN-1
       DummyInt{i} = intersect(DummyInt{i-1},KBT{i+1});
    end
    IKB = DummyInt{NN-1}

if IKB == 0
```

```
    fprintf("Intersection of Ker(B_i^T) is the zero vector. H1.2 ...
        is satisfied.")
end


%Input function and calculate Jacobian
g =@(u,v) [(u-v)^3/(1+(u^2+v^2)^(3/2));(u+v)^3/(1+(u^2+v^2)^(3/2))];
g(u,v)
JJ = jacobian(g(u,v))
JJM=norm(JJ,inf)


c = 1/4*NDT
psi =cell(N,1);
%Calculate psi(t)
for i=1:2
    psi{i} =transpose(B{3}*A^(2-i))*c + transpose(B{6}*A^(5-i))*c;
end

for i=3:5
    psi{i} = transpose(B{6}*A^(5-i))*c;
end

%Look at psi
%for i=1:5
 %   psi{i}
%end


%Calculate S(i)
d = ND;
```

```
S = cell(N,1);

for i=1:N
    S{i} = A^{i-1}*d;
end

%Look at values of S(i)
% for i=1:N
%   S{i}
% end

g(S{1}(1,1),S{1}(2,1))

%Calculate gamma(S(i))
gam = cell(N,1);

for i = 1:N
    gam{i} = g(S{i}(1,1),S{i}(2,1));
end


%calculate functional psi*
star = cell(N,1);
for i = 1:N
    star{i} = transpose(psi{i})*gam{i};
end

%Check sum
S= 0;
for i = 1:N
    S = abs(S+star{i});
end
S
```

```
double(S)

%calculate 2 norm for cap Psi
psiSum = 0;
for i=1:N
    psiSum = psiSum + psi{i}(1,1)^2+psi{i}(2,1)^2;
end
psiNorm = sqrt(psiSum)
double(psiNorm)
sqrt(5)*double(psiNorm)
```