Investigating Normality in Lattice Valued Topological Spaces

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Sciences

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

May, 2022

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Abstract

Separation axioms are a useful property in general topology. Important among them is the concept of normality. Normality can be used to guarantee the existence of various continuous functions. Three theorems towards that goal are Urysohn's Lemma, the Katetov-Tong Insertion Lemma and Tietze Extension Theorem. Much work has been done on extending these theorems into the realm of lattice valued topological spaces. This paper compiles much of the existent work on the topic, clearly and concisely elaborating on the proofs of these theorems in the current literature.

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Introduction

A fundamental concept within analysis is the notion of continuity. Continuous functions are nice to work with, and continuity is necessary for properties such as differentiability. Intuitively, a function is continuous if, when one brings two inputs "close", the corresponding outputs can be made arbitrarily "close".

In order to address this concept of continuity, metric spaces become useful. A metric space is a set, together with a function which expresses a "distance" between two points. Equipped with such a function, "close" can be defined rigorously and continuity can be studied in depth.

A great insight of topology is the knowledge that in order to discuss continuity, one does not need a defined notion of distance, just a notion of open and closed sets. Metric spaces generate topologies, and some topologies generate metric spaces. Given sets X and Y, equipped with metrics d, p respectively, we can generate topological spaces, (X, τ_d) and (Y, τ_p). Notably, a function

$$f : (X, d) \to (Y, p)$$

viewed as a function between metric spaces is continuous in the metric sense if and only if the corresponding function

$$f : (X, \tau_d) \to (Y, \tau_p)$$

between topological spaces is continuous in the topological sense.

As it was laid out in [12], general topological spaces hold no guarantees as to how many open sets are available. In the trivial topology, the only open sets are the empty set and the space itself, as such there are very few continuous functions. In the discrete topology, every set is open, and so every function is continuous. Neither of these are useful to work with, instead, useful spaces are spaces with a richness of open sets, and yet not an overabundance of them.

Normality is a property which guarantees a richness of open sets within a space sufficient to obtain a richness of real-valued continuous functions on the space [12]. As such it is a useful property to understand.

There are a plethora of theorems showing the existence of various types of continuous functions on normal topological spaces. Three famous results in the study of normal spaces are Urysohn's Lemma, the Tietze Extension Theorem and the Katetov-Tong Insertion Lemma.

A useful generalization of the concept of a topological space is that of a lattice-valued topological space. Here instead of open sets, we have open L-valued subsets.

The natural questions one might ask next become, "Can one define a notion of normality in lattice valued topological spaces?" and "Do the above theorems have counterparts in lattice valued topological spaces?"

This has been an active area of research over the last fifty years which has led to many useful results [6] [4] [2]. This paper reviews many of the important results which have been achieved in that time, adding clarifications and expansions of the proofs where useful.

Mathematical Preliminaries

2.1 Topological Spaces

Definition 2.1.1. Suppose X is a set. A subset $\tau \subset \wp(X)$ is a topology if:

- τ is closed under arbitrary unions;
- τ is closed under finite intersections;
- $\varnothing \in \tau$ and $X \in \tau$.

Definition 2.1.2. Suppose X and Y are sets, and $f : X \to Y$. This gives rise to two functions, $f^{\to} : \wp(x) \to \wp(Y)$ and $f^{\leftarrow} : \wp(Y) \to \wp(Y)$, given by

$$f^{\to}(A) = \{y \in Y : x \in X, f(x) = y\},$$

$$f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}$$

Definition 2.1.3. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces and $f : X \to Y$. Then f is continuous if

$$\forall\, V \in \tau_Y, f^{\leftarrow}(V) \in \tau_X$$

Definition 2.1.4. Suppose (X, τ) is a topological space. Then X is normal if and only if whenever A and B are disjoint closed sets in X, $U, V \in \tau$ with $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.

These motivate our ideas as we move into lattice valued topological spaces.

2.2 Lattice Valued Topological Spaces

Definition 2.2.1 [4]. Let $(L, \leq, ')$ be a complete lattice with an order reversing involution. Suppose X is a set. Then an L-valued subset of X is a map $A : X \to L$.

Definition 2.2.2 [1] [4]. We define the usual set theory operations and relations as

$$(\bigcup_{\lambda \in \Lambda} A_\lambda)(x) = \bigvee_{\lambda \in \Lambda} A_\lambda(x)$$

$$(\bigcap_{\lambda \in \Lambda} A_\lambda)(x) = \bigwedge_{\lambda \in \Lambda} A_\lambda(x)$$

$$A'(x) = A(x)'$$

$$A \subset B \Leftrightarrow A(x) \leq B(x), \forall x \in X.$$

Definition 2.2.3 [1] [10]. An L-topological space is a pair (X, τ) where X is a set and τ is a collection of L-valued sets closed under arbitrary union and finite intersection. An L-valued set is called open if it is in τ and called closed if its complement, constructed from the order reversing involution, $'$, is in τ. We call τ an L-topology.

Definition 2.2.4 [10] [15]. Suppose X and Y are sets and $f : X \to Y$ is a function. Then we can construct $f_L^{\rightarrow} : L^X \to L^Y$ by

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : f(x) = y\}$$

and $f_L^{\leftarrow} : L^Y \to L^X$ by

$$f_L^{\leftarrow}(B) = B \circ f.$$

Note 2.2.1 [10] [11]. We have the adjunction relationship we would expect based on the notation of these functions. Namely,

$$f_L^{\rightarrow} \dashv f_L^{\leftarrow}.$$

Definition 2.2.5[1] [13]. Let (X, τ_X) and (Y, τ_Y) be L-topological spaces and let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. Then f is continuous if whenever $A \in \tau_Y$, we have $f^{\leftarrow}(A) \in \tau_X$.

Note 2.2.2 [11]. It can be shown that a function is continuous if and only if the preimage of a subbasic open set in Y is open in X.

Definition 2.2.6 [4]. an L-topological space is normal if and only if for every closed K and open U with $K \subset U$ there exists a V with

$$K \subset V^\circ \subset \overline{V} \subset U.$$

In order to discuss Urysohn's Lemma in the lattice valued case, we will need to define the L-valued unit interval.

2.3 The L-valued unit interval

Definition 2.3.1 [4]. Consider the set F of antitone maps $\lambda : \mathbb{R} \to L$, satisfying

- $\lambda(t) = 1, \forall t < 0$
- $\lambda(t) = 0, \forall t > 1.$

Define $\lambda(t-) = \bigwedge_{s<t}\{\lambda(s)\}$ and $\lambda(t+) = \bigvee_{s>t}\{\lambda(s)\}$.

We define an equivalence class of functions in F by $\mu \in [\lambda]$ if and only if, $\forall t \in \mathbb{R}, \lambda(t-) = \mu(t-), \lambda(t+) = \mu(t+)$.

Definition 2.3.2 [4]. We define the L-valued unit interval to be the set of equivalence classes of F.

Definition 2.3.3 [4]. We define an L-valued topology on $[0,1](L)$ by taking $\{L_t, R_t : t \in \mathbb{R}\}$ as a subbase, where

$$L_t([\lambda]) = \lambda(t-)', R_t([\lambda]) = \lambda(t+).$$

Note 2.3.1. This is well defined by construction.

Note 2.3.2 [4]. We obtain a partial ordering on $[0,1](L)$ by taking $[\mu] \leq [\lambda]$ if and only if,

$$\forall t \in \mathbb{R}, \mu(t-) \leq \lambda(t-), \mu(t+) \leq \lambda(t+).$$

Lemma 2.3.1. In the case $L = 2$, $[0,1](L)$ and its L-topology reduce to the family of step functions with jumps occurring in $[0,1]$ together with the characteristic functions of the open sets of the usual topology on $[0, 1]$ evaluated at those jumps.

Proof. Let $[\lambda] \in [0,1](L)$.

Then each $\mu \in [\lambda]$ is a step function, which jumps from $\top$ to $\perp$ at some $s \in [0, 1]$.

$$R_t([\lambda]) = \lambda(t+) = \begin{cases} \top & \text{if } s > t \\ \perp & \text{if } s \leq t \end{cases}$$

$$= \chi_{(t,1]}(s).$$

Now we can view $\chi_{(t,1]}(s)$ as the set $(t, 1]$ via the natural correspondence.

By the same process, we get $L_t([\lambda])' = \chi_{[0,t)}(s)$ which we may view as $[0, t)$.

Now we have $\{[0, t), (t, 1] : t \in [0, 1]\}$, which forms a subbase for the usual topology on $[0, 1]$.

□

Note 2.3.3 [2]. We can define $\mathbb{R}(L)$ in an appropriate way.

Definition 2.3.4. [6] Let $(X.\tau)$ be an L-topological space and $f : X \to \mathbb{R}(L)$. Then f is called lower semicontinuous if $f^{\leftarrow}(R_t)$ is open for every $t \in \mathbb{R}$. Likewise a function is called upper semicontinuous if $f^{\leftarrow}(L_t)$ is open for every $t \in \mathbb{R}$.

Urysohn's Lemma

3.1 Importance of Urysohn's Lemma

As described in the introduction, a main goal of the study of separation properties in topological spaces is to ensure a sufficient amount of continuous functions on a space, without containing an unnecessary excess of continuous functions [12]. Especially valuable are functions from a topological space to itself, and functions into the real numbers or the unit interval.

Urysohn's Lemma is a fundamental theorem toward our goal of constructing continuous functions. It allows one to build functions into the unit interval which separate closed sets. The statement of the theorem is given below.

Theorem 3.1.1 (Urysohn's Lemma) [12]. A topological space (X, τ) is normal if and only if whenever A and B are disjoint closed sets in X, there exists a continuous $f : X \to [0, 1]$ with $f^{\to}(A) \subset \{0\}$ and $f^{\to}(B) \subset \{1\}$.

Important to this theorem is that it is bidirectional. If a space is normal, one can build a continuous function into the unit interval which separates closed sets. But also, if one can always build such a Urysohn function, then the space is normal.

3.2 Proof of Urysohn's Lemma

Below we will consider the proof of Urysohn's Lemma. The proof below is an expanded version of that given in [14].

Proof. $(\Rightarrow)$ Suppose (X, τ) is normal, A, B are disjoint closed sets in X.

By normality, there exists $U_{\frac{1}{2}} \in \tau$ with $A \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \cap B = \varnothing$.

Now A and $X \setminus U_{\frac{1}{2}}$ are disjoint closed sets. So are $\overline{U_{\frac{1}{2}}}$ and B.
So there exists open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ so that

$$A \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}}$$

With $\overline{U_{\frac{3}{4}}} \cap B = \varnothing$.

We will now use induction to continue this process.
Suppose $n \in \mathbb{N}$ and that we have defined sets $U_{\frac{k}{2^n}}$ for all
$k = 1, 2, ...2^n - 1$ such that

$$A \subset U_{\frac{1}{2^n}} \subset \overline{U_{\frac{1}{2^n}}} \subset U_{\frac{2}{2^n}} \subset ...\overline{U_{\frac{2^n-2}{2^n}}} \subset U_{\frac{2^n-1}{2^n}}.$$

And $\overline{U_{\frac{2^n-1}{2^n}}} \cap B = \varnothing$.

Then by normality we can insert open sets between each of these sets, extending to n+1.

So by induction we achieve a collection of open sets indexed by the dyadic rationals, subject to the properties that:

- $A \subset U_r$ and $B \cap \overline{U_r} = \varnothing$ for every dyadic rational r.
- $U_r \subset U_s, \forall r < s$.

Note, we will denote the set of dyadics as D. We now define a function $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} 1 & \forall x, x \notin U_r, \forall r \in D \\ inf\{r : x \in U_r\} & otherwise \end{cases}$$

Clearly $f^{\to}(A) \subset \{0\}$ and $f^{\to}(B) \subset \{1\}$.
Claim: f is continuous.
Case 1: $f(x) = 1$.

Let U be a basic open set with $f(x) \in U$. Then $U = (r, 1]$ for some $r \in (0, 1)$. Then $X \setminus \overline{U_r} \in \tau$

and $x \in X \diagdown \overline{U_r}$ and $f^{\rightarrow}(X \diagdown \overline{U_r}) \subset (r, 1]$.

The cases of $f(x) = 0$ and $f(x) \in (0, 1)$ follow similarly. So f is continuous.

($\Leftarrow$) Suppose whenever A and B are disjoint closed set in X and $f : X \rightarrow [0, 1]$ there is a continuous function with $f^{\rightarrow}(A) \subset \{0\}$ and $f^{\rightarrow}(B) \subset \{1\}$.

Let A and B be disjoint closed sets and $f : X \rightarrow [0, 1]$ be defined from the hypothesis. Then $f^{\leftarrow}([0, \frac{1}{2}))$ and $f^{\leftarrow}((\frac{1}{2}, 1])$ are disjoint open sets which separate A and B.

□

3.3 Generalization into Lattice Valued Topology

In his 1975 paper, Hutton proposed a generalization of Urysohn's Lemma for lattice valued topological spaces. A slightly modified version of his statement is given below.

Theorem 3.3.1 Hutton's Lattice Valued Urysohn's Lemma [4]. Let (X, τ) be an L-topological space. Then (X, τ) is normal if and only if (X, τ) has the "Hutton Property", i.e. for every closed set K and open set U with $K \subset U$ there exists a continuous function $f : X \rightarrow [0, 1](L)$ such that for every $x \in X,\ K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x)$

This statement is initially confusing. It does not, at surface level appear particularly related to Urysohn's lemma, which builds a continuous function to separate closed sets. However, in the case of $L = 2$, the Hutton Property reduces to Urysohn's Lemma.

Consider the case of $L = 2$. We have, $\forall x \in X$,

$$K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x).$$

Let $x \in X$. Now $K(x) \leq f(x)(1-)$. So $\chi_K(x) \leq f(x)(1-) = [\lambda_{r_x}](1-)$. If $x \in K$, $\top \leq f(x)(1-) = [\lambda_{r_x}](1-) = \top$. So $\forall x \in K$, $f(x) \in [\lambda_1]$ which we can view as 1. So $K \subset f^{\leftarrow}(\{1\})$.

Similarly we have $f(x)(0+) \leq U(x)$. So $f(x)(0+) \leq \chi_U(x)$. If $x \notin U, f(x)(0+) \leq \bot$. So for $x \in U', f(x) \in [\lambda_0]$. Or, $U' \subset f^{\leftarrow}(\{0\})$.

So in the case of $L = 2$, the statement of the Hutton Property can be read as:

Property 3.3.1 (Hutton Property). For every open set U, closed set K, with $K \subset U$, there exists a continuous function $f : X \to [0, 1]$, with

$$K \subset f^{\leftarrow}(\{1\}), U' \subset f^{\leftarrow}(\{0\}).$$

Lemma 3.3.1. The Hutton Property and Urysohn Property are equivalent when $L = 2$.

Proof. ($\Rightarrow$) Suppose a topological space (X, τ) has the Hutton Property.

Let A and B be disjoint closed sets. Choose $U \in \tau$ by $U = X \diagdown B$. Then $\exists f : X \to [0, 1]$ with $A \subset f^{\leftarrow}(\{1\})$ and $U' \subset f^{\leftarrow}(\{0\})$.

Now $U' = (X \diagdown B)' = B$ so

$$f^{\rightarrow}(B) = f^{\rightarrow}(U') \subset f^{\rightarrow}(f^{\leftarrow}(\{0\})) \subset \{0\}.$$

And likewise,

$$f^{\rightarrow}(A) \subset f^{\rightarrow}(f^{\leftarrow}(\{1\})) \subset \{1\}.$$

($\Leftarrow$) Suppose the space has Urysohn Property.

Let K be a closed set and U be open such that $K \subset U$. Choose $A = X \diagdown U$. Then K and A are disjoint. So there exists a Urysohn function with $f^{\rightarrow}(A) \subset \{0\}$ and $f^{\rightarrow}(K) \subset \{1\}$.

Then

$$U' = A \subset f^{\leftarrow}(f^{\rightarrow}(A)) \subset f^{\leftarrow}(\{0\})$$

and

$$K \subset f^{\leftarrow}(f^{\rightarrow}(K)) \subset f^{\leftarrow}(\{1\})$$

as desired.

□

Proof of Theorem 3.3.1.

With the above preparations, the proof of Theorem 3.3.1 is now given. Sufficiency elaborates upon the proof given in [4], while necessity, at the recommendation of [5], adapts the approach of Lemma 3.1 in [8].

Proof. ($\Leftarrow$) Let $x \in X$. By hypothesis we have,

$$K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x).$$

Since, $\forall t \in (0,1)$ it is true that $f(x)(1-) \leq f(x)(t+) \leq f(x)(t-) \leq f(x)(0+)$, we have

$$K(x) \leq f(x)(t+) \leq f(x)(t-) \leq U(x).$$

.

By definition, $f^{\leftarrow}(L'_t)(x) = f(x)(t-)$ and $f^{\leftarrow}(R_t)(x) = f(x)(t+)$ and by the continuity of f we have $f^{\leftarrow}(L'_t)$ is closed and $f^{\leftarrow}(R_t)$ is open. So

$$K \subset f^{\leftarrow}(R_t) \subset f^{\leftarrow}(L'_t) \subset U.$$

By the properties of the interior of a set we have

$$K \subset f^{\leftarrow}(R_t) \subset f^{\leftarrow}(L'_t)^{\circ} \subset f^{\leftarrow}(L'_t) \subset U.$$

So if we let $V = f^{\leftarrow}(L'_t)$ we have

$$K \subset V^{\circ} \subset \overline{V} \subset U.$$

So X is normal.

($\Rightarrow$) Construct by normality, a sequence of sets $\{V_r : r \in (0,1) \text{ and } r \in [0,1] \cap \mathbb{Q}\}$ with the properties

- $K \subset V_r^{\circ} \subset \overline{V_r} \subset U$
- If $r < s$ then $\overline{V_r} \subset V_s^{\circ}$

And now define $f : X \to [0,1](L)$ by

$$f(x)(t) = \bigwedge_{r<t} V_r(x).$$

Then by construction, for every $x \in X$

$$K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x).$$

It remains to show that f is continuous. We can do this by showing that the preimages of subbasic open sets remain open [13]. Note, since the preimage operator preserves complements, showing that the preimages of the complements of basic open sets are closed acheives our goal. Now

$$f_L^{\leftarrow}(R_t') = \bigwedge_{r>t} V_r = \bigwedge_{r>t} \overline{V_r}$$

which is the intersection of closed sets and thus closed [8]. Also,

$$f_L^{\leftarrow}(L_t) = \bigvee_{r<t} V_r = \bigvee_{r<t} V_r^{\circ}$$

is open so f is continuous as desired [8].

□

The Katetov-Tong Insertion Lemma

4.1 Importance of the Katetov-Tong Insertion Lemma

Continuing our construction of continuous functions we have the Katetov-Tong Insertion Lemma. It allows us to build a continuous function that sits between an upper semicontinuous and a lower semicontinuous function.

Theorem 4.1.1 Katetov-Tong Insertion Lemma [3]. A topological space (X, τ) is normal if and only if whenever $g : X \to \mathbb{R}$ is upper semicontinuous and $h : X \to \mathbb{R}$ is lower semicontinuous with $g \leq h$ then there exists a continuous function $f : X \to \mathbb{R}$ with $g \leq f \leq h$.

Besides furthering our goal of developing a rich theory of continuous functions [12], the Katetov-Tong Insertion Lemma has an alternate use. With the Katetov-Tong Insertion Lemma, one can prove Tietze Extension Theorem as a quick consequence. We will see this becomes very useful as we generalize into lattice-valued topological spaces. The traditional methods of proving Tietze Extension Theorem have not yielded much, yet using Katetov-Tong works in the generalized setting [6].

4.2 Proof of Katetov-Tong

The Katetov-Tong Insertion Lemma was originally proven independently by both Miroslav Katětov and Hing Tong. Multiple people have since given different proofs for the theorem. This paper analyzes and expands upon a version of the proof from [3].

Proof. $(\Rightarrow)$ Suppose (X, τ) is normal and g, h are given as above.
$\forall t \in \mathbb{Q}$, define

$$H(t) = \{x \in X : h(x) \leq t\}$$

and

$$G(t) = \{x \in X : g(x) < t\}.$$

Finally consider

$$P = \{(r, s) : r, s \in \mathbb{Q}, r < s\}.$$

This is infinite and countable so we can index it as a sequence $\{(r_n, s_n)\}_{n \in \mathbb{N}}$. Let $P_n = \{(r_k, s_k) : k \leq n\}$.

Subclaim 1: For any $r < s$, $H(r)$ is closed, $G(s)$ is open and $H(r) \subset G(s)$.

$H(r) = h^{\leftarrow}((-\infty, r])$ and h is lower semicontinuous, so $H(r)$ is closed.
$G(s) = g^{\leftarrow}((-\infty, s)) = g^{\leftarrow}(\mathbb{R} \setminus [s, \infty)) = X \setminus g^{\leftarrow}([s, \infty))$ and
g is upper semicontinuous so $g^{\leftarrow}([s, \infty))$ is closed. So $G(s)$ is open.
Now $G(r) \subset G(s)$ since $r < s$ and
$H(r) \subset G(r)$ since $g < h$.
Therefore, $H(r) \subset G(s)$ as desired.

Now we will use induction to construct a sequence of closed subsets, $D(r_n, s_n)$ in X with the properties

- $H(r) \subset D(r, s) \subset G(s), \forall r < s$
- $r < u, s < t \implies D(r, s) \subset D(u, t)^{\circ}$.

Base Case: n = 1.
By normality, we may construct a closed set $D(r_1, s_1)$ such that

$$H(r_1) \subset D(r_1, s_1)^{\circ} \subset D(r_1, s_1) \subset G(s_1).$$

Since we have only one object in our collection, the properties hold.

Inductive Step (Strong): Suppose $\forall k < n, \exists D(r_k, s_k)$ with the desired properties.
Let $J = \{j \in \mathbb{N} : j < n, r_j < r_n, s_j < s_n\}$ and
$K = \{k \in \mathbb{N} : k < n, r_k > r_n, s_k > s_n\}$.

Then

$$H(r_n) \cup (\bigcup_{j \in J} D(r_j, s_j))$$

is closed and

$$G(s_n) \cap (\bigcap_{k \in K} D(r_k, s_k)^\circ)$$

is open.

Also, we have

$$H(r_n) \cup (\bigcup_{j \in J} D(r_j, s_j)) \subset G(s_n) \cap (\bigcap_{k \in K} D(r_k, s_k)^\circ).$$

By normality, we may construct $D(r_n, s_n)$ to be a closed set such that

$$H(r_n) \cup (\bigcup_{j \in J} D(r_j, s_j)) \subset D(r_n, s_n)^\circ \subset D(r_n, s_n) \subset G(s_n) \cap (\bigcap_{k \in K} D(r_k, s_k)^\circ).$$

So we have that

$$H(r_n) \subset D(r_n, s_n)^\circ \subset D(r_n, s_n) \subset G(s_n).$$

Also, let $(r_k, s_k) \in P_n$ with $r_k < r_n, s_k < s_n$.
Then $(r_k, s_k) \in J$ and $\bigcup_{j \in J} D(r_j, s_j) \subset D(r_n, s_n)$ so $D(r_k, s_k) \subset D(r_n, s_n)$.

Now let $(r_i, s_i) \in P_n$ with $r_n < r_i, s_n < s_i$. Then $(r_i, s_i) \in K$.
So

$$D(r_n, s_n) \subset \bigcup_{k \in K} D(r_k, s_k)^\circ.$$

And therefore $D(r_n, s_n) \subset D(r_i, s_i)$ and the induction holds.

We now have our desired family of sets. Now for each rational number t, let

$$F(t) = \bigcap_{s>t} D(t,s)$$

Now each $D(t,s)$ is closed so $F(t)$ is closed. Also, $\forall s > t$ we have $H(t) \subset F(t)$ and $F(t) \subset G(s)$. Now we also have that

$$\bigcup_{t\in\mathbb{Q}} F(t) = X, \bigcap_{t\in\mathbb{Q}} F(t) = \varnothing, F(r) \subset F(s)^\circ, \forall r < s.$$

We will now define $f : X \to \mathbb{R}$ by

$$f(x) = inf\{t : X \in F(t)\}.$$

Subclaim 2: f is well defined.

The reals are a conditionally complete lattice, so every set with a lower bound has a unique infimum.

Note that $\{t : x \in F(t)\}$ is bounded below by $g(x) - 1$.

So f is well defined.

Subclaim 3: f is continuous.

Let $f(x) \in B(y,\epsilon)$ for some $y \in \mathbb{R}, \epsilon > 0, x \in X$. Then $x \in F(y+\epsilon)^\circ \diagdown F(y-\epsilon)$ and we have

$$f^{\rightarrow}(F(y+\epsilon)^\circ \diagdown F(y-\epsilon)) \subset B(y,\epsilon)$$

So f is continuous at all $x \in X$.

Subclaim 4: $g \leq f$.

Let $x \in X$ and $f(x) = y$. Then $x \in F(s), \forall s > y$. So $x \in G(s), \forall s > y$. So $g(x) \leq y = f(x)$.

Subclaim 5: $f \leq h$

Let $x \in X$ with $h(x) = z$. Then $x \in H(r) \subset F(r), \forall r \geq z$. So $f(x) \geq z$ as desired.

And so f is as desired.

$\square$

4.3 Generalizations into Lattice Valued Topology

Theorem 4.3.1 Kubiak's L-valued Katetov-Tong Insertion Lemma. Let $(L, \leq, ')$ be a complete, completely distributive lattice with an order reversing involution. An L-topological space (X, τ) is normal if and only if whenever $g, h : X \to \mathbb{R}(L)$, g is upper semicontinuous, h is lower semicontinuous and $g \leq h$, then there exists a continuous function $f : X \to \mathbb{R}(L)$ with $g \leq f \leq h$ and f continuous.

The Lattice Valued Extension of the Katetov-Tong Insertion Lemma was proven in 1987 by Tomasz Kubiak [6]. This was done as an intermediate step to his true goal, a fuzzification of the Tietze Extension Theorem. However, gaining even this result requires sophisicated mathematical machinery. Note that this theorem requires the underlying lattice to be completely distributive. For a careful description of complete distributivity see [7]. Below we will go through the proof of the theorem as given in [6], expanding where useful. Before tackling the theorem above, we will first consider some intermediate lemmas.

Lemma 4.3.1. Let (X, τ) be a normal L-topological space, $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ be countable families of elements in L^X. If there exists an $A, B \in L^X$ with $\overline{A_i} \leq \overline{A} \leq B_j^{\circ}$ and $\overline{A_i} \leq B^{\circ} \leq B_j^{\circ}, \forall i, j \in \mathbb{N}$, then there exists a $U \in L^X$ with

$$\overline{A_i} \leq U^{\circ} \leq \overline{U} \leq B_j^{\circ}, \forall i, j \in \mathbb{N}.$$

Proof. Subclaim: $\forall n \in \mathbb{N}, n \geq 2, \exists \{U_i, V_j : 1 \leq i, j \leq n\} \subset L^X$ with

- $\overline{A_i} \leq U_i^{\circ}$
- $V_j \leq B_j$
- $\overline{A} \leq V_j^{\circ}$
- $\overline{U_i} \leq B^{\circ}$
- $\overline{U_i} \leq V_j$.

We will prove this by induction. Let P_k be the statement that the above is true for $n = k$. P_2 follows immediately from normality.

Inductive Step: Suppose for some $n \geq 2$ we have defined $\{U_i, V_j : 1 \leq i, j \leq n-1\}$ such that P_n holds.

Since $\overline{A_n} \leq \overline{A} \leq V_j^\circ$ (for $j < n$), and $\overline{A_n} \leq B_j^\circ$, by normality there exists $U_n \in L^X$ with

$$\overline{A_n} \leq U_n^\circ \leq \overline{U_n}(\bigwedge_{j<n}(V_j \wedge B))^\circ.$$

Likewise, there exists a $V_n \in L^X$ with

$$\bigvee_{i \leq n}(\overline{U_n} \wedge \overline{A}) \leq V_n^\circ \leq \overline{V_n} \leq B_n^\circ.$$

and therefore P_{n+1} holds. And so the subclaim holds.

Now set

$$U = \bigwedge_{i=1}^{\infty} U_i.$$

Then $A_i \leq U_i^\circ \leq U^\circ, \forall i \in \mathbb{N}$.

Since $\overline{U_i} \leq V_j^\circ, \forall i, j \in \mathbb{N}$, we have that $U_i \leq V_j, \forall i, j \in \mathbb{N}$. So $\forall j \in \mathbb{N}, V_j$ is an upper bound for $\{U_i\}$.

Therefore $U \leq V_j, \forall j \in \mathbb{N}$. So

$$\overline{U} \leq \overline{V_j} \leq B_j^\circ, \forall j \in \mathbb{N}$$

and finally we have

$$\overline{A_i} \leq U^\circ \leq \overline{U} \leq B_j^\circ, \forall i, j \in \mathbb{N}.$$

□

Lemma 4.3.2. Let (X, τ) be a normal L-topological space. If $\{H_r\}_{r\in\mathbb{Q}}$ is a monotone increasing (isotone) collection of closed L-valued subset of X and $\{G_r\}_{r\in\mathbb{Q}}$ is a monotone increasing (isotone) collection of open L-valued subsets of X such that $H_r \leq G_s$ whenever $r < s$, we have

$$\exists\{F_r\}_{r\in\mathbb{Q}} \subset L^X$$

with the property $H_r \leq F_s^\circ, \overline{F} \leq G_s$ and $\overline{F_r} \leq F_s^\circ$ whenever $r < s$.

Proof. First we will order the the rationals $\{r_n\}_{n\in\mathbb{N}}$ without repetitions. Again, we will use inductions for this proof.

Let $S(n)$ be the statement that for $1 \leq i, j \leq n-1$ we have defined $F_{r_i} \in L^X$ with

- $r < r_i \implies H_r \leq F_{r_i}^\circ$
- $r_i < r \implies \overline{F_{r_i}} \leq G_r$
- $r_i < r_j \implies \overline{F_{r_i}} \leq F_{r_j}^\circ$.

Now note that $\{H_r : r < r_1\}$ and $\{G_t : t > r_1\}$ together with H_{r_1} and G_{r_1} satisfy that

- The families are countable families of elements of L^X
- $\overline{H_r} \leq \overline{H_{r_1}} \leq G_t^\circ$
- $\overline{H_r} \leq G_{r_1}^\circ \leq G_t^\circ$

$\forall r < r_1, t > r_1$.

This note follows from the facts that $H_r = \overline{H_r}$, $G_t \leq G_t^\circ$, $H_{r_1} = \overline{H_{r_1}}$, $G_{t_1} \leq G_{t_1}^\circ$, $H_r \leq H_{r_1}$ since the sequence of sets is monotone and $G_{r_1} \leq G_t$ since the sequence is monotone.

Now by our note and by Lemma 4.3.2 $\exists U_1 \in L^X$ with $H_r \leq U_1^\circ, \forall r < r_1$ and $\overline{U_1} \leq G_t, \forall t > r_1$. Set $F_{r_1} = U_1$ and we have $S(2)$.

Inductive Step: Suppose $F_{r_i} \in L^X$ are defined for $i < n, n \in \mathbb{N}$ such that they satisfy $S(n)$.

Define $A = \bigvee\{F_{r_i} : i < n, r_i < r_n\} \vee H_{r_n}$ and $B = \bigwedge\{F_{r_i} : j < n, r_j > r_n\} \wedge G_{r_n}$.

Now note that whenever $r_i < r_n < r_j$ with $i, j < n$ we have that

- $\overline{F_{r_i}} \leq \overline{A} \leq F_{r_j}^{\circ}$
- $\overline{F_{r_1}} \leq B^{\circ} \leq F_j^{\circ}$.

Also, whenever $r < r_n < t$, we have $H_r \leq \overline{A} \leq G_t$ and $H_r \leq B^{\circ} \leq G_t$. Therefore the collections $\{F_{r_i} : i < n, r_i < r_n\} \cup \{H_r : r < r_n\}$ and $\{F_{r_i} : j < n, r_j > r_n\} \cup \{G_r : r > r_n\}$ together with A and B fulfill the hypothesis of Lemma 4.3.1 so $\exists U_n \in L^X$ with

- $r < r_n \implies H_r \leq U_n$
- $r_i < r_n \implies \overline{F_{r_i}} \leq U_n^{\circ}$
- $r_n < r \implies \overline{U_n} \leq G_r$
- $r_n < r_j \implies \overline{U_n} \leq F_{r_j}^{\circ}$

when $1 \leq i, j \leq n-1$. Set $F_{r_n} = U_n$. Then $\{F_{r_i} : 1 \leq i \leq n\}$ satisfies $S(n+1)$. And so the lemma holds.

□

With these lemmas, we are now equipped to handle Kubiak's L-valued Katetov-Tong Insertion Lemma. The proof of this theorem is an expanded version of the one found in [6].

Theorem 4.3.1 Kubiak's L-valued Katetov-Tong Insertion Lemma.

Proof. ($\Leftarrow$) Suppose (X, τ) is an L-topological space. Suppose whenever $g, h : X \to \mathbb{R}(L)$ with g upper semicontinuous and h lower semicontinuous and $g \leq h$ then $\exists f : X \to \mathbb{R}(L)$ with f continuous and $g \leq f \leq h$.

Let $U \in \tau$ and K be a closed L-valued subset with $K \leq U$. Define $g, h : X \to \mathbb{R}(L)$ by $\forall x \in X$

$$g(x)(t) = \begin{cases} 1 & t < 0 \\ K(x) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

$$h(x)(t) = \begin{cases} 1 & t < 0 \\ U(x) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

Note that these functions are upper/lower semicontinuous functions respectively. Also note that since $K(x) \leq U(x), \forall x \in X$ we have $g \leq h$. Now let $t \in (0,1)$. Then

$$K = R_t(g) = g_L^{\leftarrow}(R_t) \leq f_L^{\leftarrow}(R_t) \leq f_L^{\leftarrow}(L_t') \leq h_L^{\leftarrow}(L_t') = L_t(h)' = U$$

And thus (X, τ) is normal.

($\Rightarrow$) Suppose (X, τ) is normal. Also suppose $g, h : X \to \mathbb{R}(L)$ with $g \leq h$, g upper semicontinuous and h lower semicontinuous. Define $H, G : \mathbb{Q} \to L^X$ by $\forall r \in \mathbb{Q}$

$$H(r) = H_r = h_L^{\leftarrow}(R_r'), G(r) = G_r = g_L^{\leftarrow}(L_r)$$

Now if $r < s$ with $r, s \in \mathbb{Q}$ then $R_r' \subset R_s', L_r \subset L_s$ so

$$h_L^{\leftarrow}(R_r') \subset h_L^{\leftarrow}(R_s'), g_L^{\leftarrow}(L_r) \subset g_L^{\leftarrow}(L_s)$$

and so we have

- H and G are isotone
- $G_r \in \tau$ and G_r is closed $\forall r \in \mathbb{Q}$
- $H_r \leq G_s$ whenever $r < s$

so by Lemma 4.3.2 $\exists \{F_r\}_{r \in \mathbb{Q}} \subset L^X$ with $H_r \leq F_s^\circ, \overline{F_r} \leq F_s^\circ, \overline{F_r} \leq G_s$ whenever $r < s$ $(r, s \in \mathbb{Q})$.

Now for each $t \in \mathbb{R}$ define

$$V_t = \bigwedge_{r<t} F_r'.$$

Note that this family is clearly antitone. Also whenever $s < t$ we have $\overline{V_t} \leq V_s^\circ$.

Now suppose that $q, t \in \mathbb{R}$ and $r, s \in \mathbb{Q}$ with $q < r < s < t$. Then

$$V_q' \leq \overline{F_r} \leq F_s^\circ \leq V_s'$$

and thus $\overline{V_t} \leq V_q^\circ$. So we have

$$\bigvee_{t\in\mathbb{R}} V_t = \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} F'_r$$

$$\geq \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} G'_r$$

$$= \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} g_L^{\leftarrow}(L'_r)$$

$$= \bigvee_{t\in\mathbb{R}} g_L^{\leftarrow}(L'_t)$$

$$= g_L^{\leftarrow}(\bigvee_{t\in\mathbb{R}} L'_t)$$

$$(\bigvee_{t\in\mathbb{R}} L'_t) \circ g = 1.$$

And likewise we have

$$\bigwedge_{t\in\mathbb{R}} V_t = 0.$$

Now define $f : X \leftarrow \mathbb{R}(L)$ by

$$f(x)(t) = V_t(x), \forall x \in X, t \in \mathbb{R}$$

First we must check that f is continuous. It suffices to note that f is subbassic continuous. First note that

$$\bigvee_{s>t} V_s = \bigvee_{s>t} V_s^{\circ}, \bigwedge_{s<t} V_s = \bigwedge_{s<t} \overline{V_s}.$$

Therefore

$$f_L^{\leftarrow}(R_t) = \bigvee_{s>t} V_s = \bigvee_{s>t} V_s^{\circ} \in \tau$$

$$f_L^{\leftarrow}(L'_t) = (\bigwedge_{s<t} V_s) = \bigwedge_{s<t} \overline{V_s}$$

which is closed so f is continuous.

It remains to show that $g \leq f \leq h$. This can be done by showing that $\forall t \in \mathbb{R}$

$$g_L^{\leftarrow}(L'_t) \leq f_L^{\leftarrow}(L'_t) \leq h_L^{\leftarrow}(L'_t),$$

$$g_L^{\leftarrow}(R_t) \leq f_L^{\leftarrow}(R_t) \leq h_L^{\leftarrow}(R_t)$$

$$g_L^{\leftarrow}(L'_t) = \bigwedge_{s<t} g_L^{\leftarrow}(L'_t)$$

$$= \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} g_L^{\leftarrow}(L'_r)$$

$$= \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} G'_r$$

$$\leq \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} F'_r$$

$$= \bigwedge_{s<t} V_s = f_L^{\leftarrow}(L'_t).$$

Also

$$f_L^{\leftarrow}(L'_t) = \bigwedge_{s<t} V_s$$

$$= \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} F'_r$$

$$\leq \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} H'_r$$

$$= \bigwedge_{s<t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} h_L^{\leftarrow}(R_r)$$

$$= \bigwedge_{s<t} h_L^{\leftarrow}(L'_s) = h_L^{\leftarrow}(L'_t).$$

Now for the right subbassic sets

$$g_L^{\leftarrow}(R_t) = \bigvee_{s>t} g_L^{\leftarrow}(R_s)$$

$$= \bigvee_{s>t} \bigvee_{\substack{r>s \\ r\in\mathbb{Q}}} g_L^{\leftarrow}(L'_r)$$

$$= \bigvee_{s>t} \bigvee_{\substack{r>s \\ r\in\mathbb{Q}}} G'_r$$

$$\leq \bigwedge_{s>t} \bigvee_{\substack{r<s \\ r\in\mathbb{Q}}} F'_r$$

$$= \bigwedge_{s>t} V_s = f_L^{\leftarrow}(R_t).$$

And

$$
\begin{aligned}
f_L^{\leftarrow}(R_t) &= \bigvee_{s>t} V_s \\
&= \bigvee_{s>t} \bigwedge_{\substack{r<s \\ r\in\mathbb{Q}}} F'_r \\
&\leq \bigvee_{s>t} \bigvee_{\substack{r>s \\ r\in\mathbb{Q}}} H'_r \\
&= \bigvee_{s>t} \bigvee_{\substack{r>s \\ r\in\mathbb{Q}}} h_L^{\leftarrow}(R_r) \\
&= \bigvee_{s>t} h_L^{\leftarrow}(R_s) = h_L^{\leftarrow}(R_r).
\end{aligned}
$$

And thus the theorem holds.

□

Tietze Extension Theorem

5.1 Importance of Tietze Extension Theorem

A key concept within topology is the idea of hereditary properties and subspace topologies. If a space has a given property, does its subspaces? Likewise, one can try to work backwards. If a subspace of a topological space has a property, does the original?

One way to approach this problem is to construct morphisms from both a space and its subspace into a third space. Given a topological space X and a subspace A, it is useful to start with a continuous function from A into the reals, and build a continuous function from X into the reals which agrees at all points in the A. Tietze Extension Theorem allows us to do that under certain circumstances.

Theorem 5.1.1 Tietze Extension Theorem [12]. A Topological space (X, τ) is normal if and only whenever A is a closed subset of X and $f : A \to \mathbb{R}$ is continuous then there exists a continuous map $F : X \to \mathbb{R}$ with $F|_A = f$.

5.2 Proof of Tietze Extension Theorem

We will now consider the proof of the Tietze Extension Theorem. The proof below is an expanded version of that given in [14]. The author has expanded it in various areas for the purpose of clarity.

Proof. ($\Rightarrow$) Suppose (X, τ) is a normal space, $A \subset X$ is closed, $f : A \to [-1, 1]$ is continuous.
Let $A_1 = \{x \in A : f(x) \geq \frac{1}{3}\}$ and $B_1 = \{x \in A : f(x) \leq \frac{-1}{3}\}$.

Now A_1 and B_1 are disjoint closed sets in A, and therefore in X.

So there exists a continuous

$$f_1 : X \to [\frac{-1}{3}, \frac{1}{3}]$$

such that $f_1^{\rightarrow}(A_1) \subset \{\frac{1}{3}\}$ and $f_1^{\rightarrow}(B_1) \subset \{-\frac{1}{3}\}$

Now, by looking at cases, we can see that $\forall x \in A, |f(x) - f_1(x)| < \frac{2}{3}$.

So $f - f_1$ is a mapping from $A \to [-\frac{2}{3}, \frac{2}{3}]$. Let $g_1 = f - f_1$ and continue this process, dividing $[-\frac{2}{3}, \frac{2}{3}]$ into thirds at $-\frac{2}{9}$ and $\frac{2}{9}$.

Let $A_2 = \{x \in A : g_1(x) \geq \frac{2}{9}\}$ and $B_2 = \{x \in A : g_1(x) \leq -\frac{2}{9}\}$.

Then there is a Urysohn function $f_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ with $f_2^{\rightarrow}(A_2) = \{\frac{2}{9}\}$ and $f_2^{\rightarrow}(B_2) = \{-\frac{2}{9}\}$

Now $|(f - f_1) - f_2| \leq (\frac{2}{3})^2$ on A.

Continuing this process, we obtain a sequence of continuous functions on A with the property

$$|f - \sum_{i=1}^{n} f_i| \leq (\frac{2}{3})^n$$

Define $F : X \to \mathbb{R}$ by

$$F(x) = \sum_{i=1}^{\infty} f_i(x), \forall x \in X$$

Now $F(x) = f(x), \forall x \in A$.

Claim: F is continuous.

Let $x \in X$ and let $\epsilon > 0$. Choose $N > 0$ such that

$$\sum_{n=N+1}^{\infty} (\frac{2}{3})^n < \frac{\epsilon}{2}$$

For $i = 1, 2, ...N$, f_i is continuous. So for $i = 1, 2, ...N$ pick an open U_i containing x such that

$$y \in U_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{2N}$$

.

Then $U = U_1 \cap U_2 \cap ... \cap U_N$ is open in X and,

$$y \in U \implies |F(x) - F(y)| \leq (\sum_{i=1}^{N} |f_i(x) - f_i(y)|) + (\sum_{i=N+1}^{\infty} |\frac{2}{3}|^i) < N\frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon.$$

So F is continuous as desired.

We now have a continuous map to [-1,1] with the desired properties. Now note that (-1,1) is homeomorphic to $\mathbb{R}$.

Now consider a continuous map $f : A \to (-1, 1)$. We can regard it as a mapping from A to [-1,1].

Therefore, by our above work, we can find an extension

$$F' : X \to [-1, 1]$$

Let $A_0 = \{x \in X : |F'(x)| = 1\}$.

Then A and A_0 are disjoint closed sets in X. So there is a Urysohn function

$$g : X \to [0, 1]$$

with the property that $g^{\to}(A_0) \subset \{0\}$ and $g^{\to}(A) \subset \{1\}$.

Define $F : X \to (-1, 1)$ by

$$\forall x \in X, F(x) = g(x)F'(x)$$

Then F is the composition of continuous functions, so F is continuous. And if $x \in A$

$$F(x) = g(x)F'(x) = 1F'(x) = f(x)$$

So F is the desired extension function.

($\Leftarrow$)

Suppose the extension property holds. Let A and B be closed disjoint sets in X.

Then $A \cup B$ is a closed set in X. Now $f : A \cup B \to [0, 1]$ by $f^{\to}(A) \subset \{0\}$ and $f^{\to}(B) \subset \{1\}$ is continuous on $A \cup B$.

Then the extension of f onto X will be a Urysohn function. So X is normal.

□

5.3 Generalizations into Lattice Valued Topology

The generalization of Tietze Extension Theorem into lattice valued topological spaces came from Kubiak in the same paper as his version of the Katetov-Tong Insertion Lemma [6]. It requires a minor lemma which can be found in [9].

Lemma 5.3.1 [9]. If (X, τ) is a normal L-topological space and A is a closed crisp subset of X, then (A, τ_A) is normal.

With this lemma, we are equipped to handle Kubiak's proof of the lattice valued Tietze Extension Theorem. The proof below comes from Kubiak's 1983 paper.

Kubiak's Lattice Valued Tietze Extension Theorem [6]. Let (X, τ) be a normal L-topological and let A be a closed crisp set and $f : (A, \tau_A) \to [0, 1](L)$ be continuous. Then there exists a continuous function $F : (X, \tau) \to [0, 1](L)$ such that $F_{\big|A} = f$ [6].

Proof. Suppose (X, τ) is a normal L valued fuzzy topological space and A is a closed crisp subset of X and $f : (A, \tau_A) \to [0, 1](L)$ is continuous.

Let $[\lambda_i]$ be the element of $[0, 1](L)$ determined by the function $\lambda_i : \mathbb{R} \to L$ where

$$\lambda_i(t) = 1, \forall t < i$$

$$\lambda_i(t) = 0, \forall t \geq i.$$

Then define two functions $h, g : X \to [0, 1](L)$ by

$$g(x) = f(x), \forall x \in A$$

$$g(x) = [\lambda_0], \forall x \notin A$$

$$h(x) = f(x), \forall x \in A$$

$$h(x) = [\lambda_1], \forall x \notin A.$$

We will first show that g is upper semicontinuous. Let $t > 0$. Then

$$g_L^{\leftarrow}(L_t)(x) = f_L^{\leftarrow}(L_t)(x), x \in A$$

$$g_L^{\leftarrow}(L_t)(x) = 1, x \notin A.$$

Now $f_L^{\leftarrow}(L_t)$ is open in (A, τ_A). So therefore $f_L^{\leftarrow}(L_t)$ is of the form $U_t \big| A$ with $U_t \in \tau$. So we have

$$g_L^{\leftarrow}(L_t) = U_t \vee A'$$

which is open in (X, τ) so g is upper semicontinuous. By a similar process we see

$$h_L^{\leftarrow}(R_t) = V_t \vee A', t < 1$$

$$h_L^{\leftarrow}(R_t) = 0, t \geq 1$$

with $V_t \in \tau$ such that $f_L^{\leftarrow}(R_t) = V_t\big|_A$. And thus h is lower semicontinuous with $g \leq h$. Therefore, by Kubiaks L-valued Katetov-Tong Insertion Lemma there is a continuous function $F : (X, \tau) \to [0,1](L)$ with $g(x) \leq F(x) \leq h(x), \forall x \in X$. So $\forall x \in A$ we get $f(x) \leq F(x) \leq f(x)$ and thus the theorem holds.

$\square$

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