# ANOTHER CHARACTERIZATION OF NILPOTENT GROUPS

by

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# ABSTRACT

This paper presents an original proof of the equivalence of nilpotentcy and sylow embeddedness for all finite groups whose order is divisible by at least three distinct primes.

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I wish to express my gratitude to my thesis advisor, Dr. Neil Flowers, for his guidance, his patience, and his paper [1], which inspired me to write this thesis. I wish to thank Dr. Stephen E. Rodabaugh for his corrections to a draft of this paper. Ultimately, I would like to express my appreciation to my family for their love and support.

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# INTRODUCTION

Let $G$ be a group and $H \leq G$. In 1964, two group theorists, Kegel and Wielandt, wondered if $P \cap H \in \mathrm{Syl}_p(H)$ for all $P \in \mathrm{Syl}_p(G)$, would it always be the case that $H \trianglelefteq \trianglelefteq G$ ? Unfortunately, they were unable to answer this question affirmatively or find a counterexample. This problem became known as the Kegel-Wielandt Conjecture.

In 1991, some thirty years later, a group theorist named Peter Kleidman [4] finally proved that the Kegel-Wielandt Conjecture was true. In his intricate, seventy-page proof, he employed the classification of finite simple groups.

In 1996, not knowing of any of these events, a group theorist named Neil Flowers asked a question very similar to the Kegel-Wielandt Conjecture. His question was: If $G$ is a group such that $P \cap H \in \mathrm{Syl}_p(H)$ for each subgroup $H$ of $G$ with $|\pi(H)| \geq 2$, for all $P \in \mathrm{Syl}_p(G)$, and for all primes $p$ that divide $|G|$, then what can we say about $G$? Flowers claimed that the group would be nilpotent, and in fact this strange condition was equivalent to nilpotency.

The object of this paper is to provide a proof of the above equivalency, via mathematical induction, different from that given by Flowers [1] . Our main result is the following:

**Theorem** Let $G$ be a group such that $|\pi(G)| \geq 3$. Then $G$ is nilpotent if and only if for every $H \leq G$ with $|\pi(H)| \geq 2$, every prime $p$, and every $P \in \mathrm{Syl}_p(G)$,

$$P \cap H \in \mathrm{Syl}_p(H).$$

# CHAPTER 1

## Preliminaries

In this section we give the background concepts and notations that will be used in our proof. Some results in this section are proved, but they all are well known and can be found in [2] or any intermediate text in finite group theory.

**Definition 1.1** $G$ is a $p$-group for some prime $p$, if $|G| = p^n$ for some $n \in \mathbf{Z}^+ \cup \{0\}$.

The identity group is a $p$-group for each $p$.

**Definition 1.2** Let $G$ be a group , $p$ be a prime, and $p^n$ be the largest integer power of $p$ dividing $|G|$. Then a subgroup $P$ of $G$ is called a Sylow $p$-subgroup if $|P| = p^n$ and we define $|G|_p = p^n$. We denote the set of all Sylow $p$-subgroups of $G$ by $\mathrm{Syl}_p(G) = \{P \leq G |\, |P| = p^n\}$.

**Definition 1.3** Let $G$ be a group and $p$ be a prime. Then $\mathrm{O}_p(G) = \bigcap_{P \in \mathrm{Syl}_p(G)} P$.

**Lemma 1.4** Let $G$ be a group and $p$ be a prime, then $\mathrm{O}_p(G) \trianglelefteq G$.

Proof: Let $x \in \mathrm{O}_p(G)$, $g \in G$, and $P \in \mathrm{Syl}_p(G)$. Then $|P^{g^{-1}}| = |P|$, and so $P^{g^{-1}} \in \mathrm{Syl}_p(G)$. Since $x \in \mathrm{O}_p(G)$, $x \in P^{g^{-1}}$, which implies $x^g \in P$. Therefore, since $P$ and $g$ were chosen arbitrarily, we have $x^g \in \mathrm{O}_p(G)$ and so $\mathrm{O}_p(G) \trianglelefteq G$. ■

**Theorem 1.5** (Sylow's Theorem) Let $G$ be a group and $p$ be a prime . Then

(i) $\mathrm{Syl}_p(G) \neq \varnothing$.

(ii) Every $p$-subgroup of $G$ lies in a Sylow $p$-subgroup of $G$.

(iii) $G$ acts transitively on $\mathrm{Syl}_p(G)$ by conjugation.

(iv) $|\mathrm{Syl}_p(G)| = 1 \pmod{p}$.

(v) $|\mathrm{Syl}_p(G)| = |G| \,/\, |\mathrm{N}_G(P)|$ for any $P \in \mathrm{Syl}_p(G)$.

**Definition 1.6** For each $a \in S$, we define the stabilizer of $a$ in $G$ by

$$\mathrm{Stab}_G(a) = \{g \in G \mid ag = a\}.$$

Now for the next three lemmas, let $G$ be a group, $S$ be a set, and suppose $G$ acts on $S$.

**Lemma 1.7** Let $a \in S$. Then $\mathrm{Stab}_G(a) \leq G$.

**Lemma 1.8** Suppose $G$ acts transitively on $S$. Then $|S| = |G|/|\mathrm{Stab}_G(a)|$ for any $a \in S$.

**Lemma 1.9** If $G$ is a $p$-group, and $p$ does not divide $|S|$, then $C_S(G) \neq \varnothing$.

**Lemma 1.10** Let $G$ be a $p$-group and $H < G$. Then $H < N_G(H)$.

Proof: Let $S = \{H^g \mid g \in G \text{ and } H^g \neq H\}$. If $H \trianglelefteq G$, then $G = N_G(H)$ and so $H < N_G(H)$. Therefore, we may assume $H$ is not normal in $G$. Then since $G$ acts transitively on $S \cup \{H\}$ by conjugation we get,

$$|S \cup \{H\}| = \frac{|G|}{|\mathrm{Stab}_G(H)|} = \frac{|G|}{|N_G(H)|}$$

and so

$$|S| = \frac{|G|}{|N_G(H)|} - 1$$

Therefore, since $G$ is a $p$-group, $p$ divides $|G|/|N_G(H)|$. But since $p$ does not divide 1, we conclude $p$ does not divide $|S|$. Now $H$ is a $p$-group, and $H$ acts on $S$ by conjugation. Thus, by Lemma 1.9, there exists $H^g \in C_S(H)$. But then $H \leq N_G(H^g)$ and so $H^{g^{-1}} \leq N_G(H)$. Since $H^g \neq H$, we have $H < N_G(H)$. ■

**Lemma 1.11** Let $G$ be a group, $a \in G$, $H \leq G$. Let the centralizer of $a$ in $G$, the centralizer of $H$ in $G$, and the normalizer of $H$ in $G$ be defined respectively by:

(i) $\mathrm{C}_G(a) = \{g \in G \mid ag = ga\}$.

(ii) $\mathrm{C}_G(H) = \{g \in G \mid gh = hg \text{ for each } h \in H\} = \bigcap_{h \in H} C_G(h)$.

(iii) $\mathrm{N}_G(H) = \{g \in G \mid H^g = H\}$.

Then $\mathrm{N}_G(H)$, $\mathrm{C}_G(a)$, and $\mathrm{C}_G(H)$ are subgroups of $G$. Also, $\mathrm{C}_G(H) \trianglelefteq \mathrm{N}_G(H)$ and $H \trianglelefteq \mathrm{N}_G(H)$.

**Theorem 1.12** (Cauchy's Theorem) If $G$ is a group and $p$ is a prime such that $p \mid |G|$, then there exists $1 \neq x \in G$ such that $x^p = 1$.

**Lemma 1.13** Let $G$ be a group, $N \trianglelefteq G$, $H \leq G$, $L \leq G/N$ and $\gamma : G \rightarrow G/N$ be the natural map defined by $(g)\gamma = gN$. Then,

(i) $(H)\gamma = HN/N$.

(ii) $(HN/N)\gamma^{-1} = HN$.

(iii) $L = K/N$ for some $N \leq K \leq G$.

**Theorem 1.14** (First Isomorphism Theorem) Let $G$ and $G'$ be groups, and $\phi : G \to G'$ be a homomorphism with $\mathrm{Ker}\phi = K$. Then

$$G/K \cong (G)\phi$$

**Theorem 1.15** (Second Isomorphism Theorem) Let $G$ be a group and $H, K$ be subgroups of $G$. If $K \trianglelefteq G$, then $H \cap K \trianglelefteq H$ and

$$\frac{HK}{K} \cong \frac{H}{H \cap K}$$

**Lemma 1.16** Let $G$ be a group, $P \in \mathrm{Syl}_p(G)$, $N \trianglelefteq G$. Then,

$$\frac{PN}{N} \in Syl_p(\frac{G}{N}).$$

**Lemma 1.17** (Frattini Argument) Let $G$ be a group, $N \trianglelefteq G$, and $P \in \mathrm{Syl}_p(N)$. Then $G = \mathrm{N}_G(P)N$.

Proof: Let $g \in G$. Then $P \in \mathrm{Syl}_p(N)$ implies $P^g \in \mathrm{Syl}_p(N^g) = \mathrm{Syl}_p(N)$ because $N$ is normal. By Sylow's Theorem, there exists $n \in N$ such that $P^{gn} = P$. Hence, $gn \in \mathrm{N}_G(P)$ and therefore $g \in \mathrm{N}_G(P)N$. Thus, $G = \mathrm{N}_G(P)N$. ■

**Lemma 1.18** (Frattini Argument 2) Let $G$ be a group and $S$ be a set, and suppose $G$ acts on $S$. Suppose further that $H \leq G$ and $H$ acts transitively on $S$. Then $G = \mathrm{Stab}_G(s)H$ for every $s \in S$.

Proof: Let $g \in G$. Then, since $H$ acts transitively on $S$, $S = sH$ for every $s \in S$. But since $sg \in S$, there exists $h \in H$ such that $gs = hs$. Then $h^{-1}gs = s$ and so $h^{-1}g \in \mathrm{Stab}_G(s)$. Hence $g \in \mathrm{Stab}_G(s)H$ and therefore $G = \mathrm{Stab}_G(s)H$. ■

**Definition 1.19** A group $G$ is solvable if there exists a series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq G_3 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

such that $G_i/G_{i+1}$ is abelian for all $0 \leq i \leq n-1$.

Examples of solvable groups include abelian groups, $S_3$, and $p$-groups. Some of the properties of solvable groups are the following:

(i) If $G$ is solvable and $H \leq G$, then $H$ is solvable.

(ii) If $G$ is solvable and $N \trianglelefteq G$, then $G/N$ is solvable.

(iii) If $G$ is a group and $N \trianglelefteq G$ such that $G/N$ and $N$ are solvable, then $G$ is solvable.

**Definition 1.20** Let $G$ be a group , $H \leq G$, and $\sigma$ a finite set of primes. Then

(i) $\pi(G) = \{p \in \mathbf{Z}^+ \mid p$ divides $G$ and $p$ is a prime$\}$.

(ii) $\sigma' = \{p \in \mathbf{Z}^+ \mid p$ is a prime and $p \notin \sigma\}$.

(iii) $G$ is called a $\sigma$-group if $\pi(G) \subseteq \sigma$.

(iv) $H$ is a Hall $\sigma$-subgroup of $G$ if $H$ is a $\sigma$-group and $\pi(G/H) \subseteq \sigma'$.

(v) $\text{Hall}_\sigma(G) = \{H \leq G \mid H$ is a Hall $\sigma$-subgroup of G$\}$.

**Definition 1.21** Let $G$ be a group, $H \leq G$ such that $|\pi(H)| \geq 2$, and $p \in \pi(G)$. Then $H$ is a sylow $p$-embedded subgroup of $G$ if $P \cap H \in \text{Syl}_p(H)$ for each $P \in \text{Syl}_p(G)$. We denote the set of all sylow $p$-embedded subgroups of $G$ by $\text{Sylem}_p(G)$, and we define

$$\text{Sylem}(G) = \bigcap_{p \in \pi(G)} \text{Sylem}_p(G).$$

**Definition 1.22** Let $G$ be a group. Then

$$\mathrm{Subgp}^2(G) = \{H \leq G \mid |\pi(H)| \geq 2\}.$$

**Definition 1.23** A group $G$ is sylow embedded if $\mathrm{Subgp}^2(G) \subseteq \mathrm{Sylem}(G)$.

**Definition 1.24** A group $G$ is nilpotent if for each $P \in \mathrm{Syl}_p(G)$, $P \trianglelefteq G$.

The above definition is equivalent to saying $|\mathrm{Syl}_p(G)| = 1$, for each $p \in \pi(G)$. Examples of nilpotent groups include abelian groups, $p$-groups, and $D_4 \times \mathbf{Z}_3$.

Some of the properties of nilpotent groups are:

(i) If $G$ is nilpotent and $H \leq G$, then $H$ is nilpotent.

(ii) If $G$ is nilpotent and $N \trianglelefteq G$, then $G/N$ is nilpotent.

**Theorem 1.25** Let $G$ be a nilpotent group. Then $G$ is solvable.

Proof: We are going to use induction on $|G|$. Since $G$ is nilpotent, we know that for $P \in \mathrm{Syl}_p(G)$, $P \trianglelefteq G$. If $G$ is a $p$-group for some prime $p$, then $G$ is solvable. Without loss, we may assume $|\pi(G)| \geq 2$. Now, since $P$ is a $p$-group, $P$ is solvable, $G/P$ is solvable by induction, and so by 1.19 (iii), $G$ is solvable.■

**Theorem 1.26** (Hall's Theorem) Let $G$ be solvable and $\pi \subseteq \pi(G)$. Then $\mathrm{Hall}_\pi(G) \neq \varnothing$ and $G$ acts transitively on $\mathrm{Hall}_\pi(G)$ by conjugation.

**Theorem 1.27** (Burnside's Theorem) Let $G$ be a group and $P \in \mathrm{Syl}_p(G)$ such that $\mathrm{N}_G(P) = \mathrm{C}_G(P)$. Then there exists $K \trianglelefteq G$ such that $G = PK$ and $P \cap K = 1$.

**Theorem 1.28** (Frobenius' Theorem) Let $G$ be a group and $H \leq G$ such that $H = \mathrm{N}_G(H)$ and $H \cap H^x = 1$ for each $x \in G \backslash H$. Then there exists $K \trianglelefteq G$ such that $G = HK$ and $H \cap K = 1$.

# CHAPTER 2

## Proof of the Main Theorem

As stated in the introduction, our goal is to show the equivalence of sylow embeddness and nilpotency for finite groups, whose order is divisible by at least three distinct primes. First, we will consider simpler cases and then build our proof using induction for the general conclusion of our theorem.

**Lemma 2.1** Let $G$ be a group and $N \trianglelefteq G$. Then $N \in \text{Sylem}(G)$.

Proof: Let $P \in \text{Syl}_p(G)$. Then $P \cap N \leq N$ is a $p$-group and so by Sylow's Theorem, $P \cap N \leq P_0$ for some $P_0 \in \text{Syl}_p(N)$. Again, by Sylow's Theorem, there exists $g \in G$ such that $P_0 \leq P^g$. Now since $N \trianglelefteq G$, we have

$$P \cap N \leq P_0 \leq P^g \cap N = (P \cap N)^g$$

Thus, since $|P \cap N| = |(P \cap N)^g|$, we have $P \cap N = P_0$. Therefore, $P \cap N \in \text{Syl}_p(N)$ and so $N \in \text{Sylem}_p(G)$. Since $p$ was chosen arbitrarily $N \in \text{Sylem}(G)$. ■

**Lemma 2.2** Let $G$ be a group and $N \trianglelefteq\trianglelefteq G$. Then $N \in \text{Sylem}(G)$.

Proof: Let $N$ be subnormal in $G$. Then there exists a series $N_1, N_2, \ldots, N_k$ such that $N = N_k \trianglelefteq \ldots \trianglelefteq N_i \trianglelefteq \ldots \trianglelefteq N_1 \trianglelefteq G$. Use induction on $k$. If $k = 1$, we have $N \trianglelefteq G$. So, by Lemma 2.1, we are done. Now suppose the lemma holds for all subnormal subgroups with subnormal length $l$ where $l \leq k-1$. Let $P \in \mathrm{Syl}_p(G)$. Then by induction $N_l \in \mathrm{Sylem}(G)$, and so $P \cap N_l \in \mathrm{Syl}_p(N_l)$. Since, $N_{l+1} \trianglelefteq N_l$, again by using Lemma 2.1 we have,

$$P \cap N_l \cap N_{l+1} \in \mathrm{Syl}_p(N_{l+1})$$

But since $N_{l+1} \leq N_l$, $P \cap N_{l+1} = P \cap N_l \cap N_{l+1}$ which consequently implies $P \cap N_{l+1} \in \mathrm{Syl}_p(N_{l+1})$. Hence, $N_{l+1} \in \mathrm{Sylem}(G)$ and therefore we have $N \in \mathrm{Sylem}(G)$ by induction. ■

**Lemma 2.3** If $G$ is abelian, then $G$ is a sylow embedded group.

Proof: Since every subgroup in $G$ is normal, by Lemma 2.1, $G$ is a sylow embedded group.■

At this point we can prove one direction of our main theorem or "the easy half".

**Theorem 2.1 (part I):** Let $G$ be a nilpotent group, then $G$ is a sylow embedded group.

Proof: Let $G$ be a nilpotent group such that $H \leq G$, $|\pi(H)| \geq 2$, and $P \in \mathrm{Syl}_p(G)$. All we need to show is that $P \cap H \in \mathrm{Syl}_p(H)$. Since $P$ is a $p$-group $P \in \mathrm{Syl}_p(G)$ and $P \cap H \leq P$ implies $P \cap H$ is a $p$-subgroup of $H$. By Sylow's Theorem, there exists $P_0 \in \mathrm{Syl}_p(H)$ such that $P \cap H \leq P_0$. Since $P \trianglelefteq G$, by Sylow's Theorem $P_0 \leq P$. Therefore,

$$P \cap H \leq P_0 \leq P \cap H$$

Hence, $P \cap H = P_0$, which means $P \cap H \in \text{Syl}_p(H)$. ■

In the next two Lemmas we find out that Sylow embeddness is a fairly rexilant property.

**Lemma 2.4** If $G$ is a sylow embedded group and $H \leq G$, then $H$ is a Sylow embedded group.

Proof: Let $P \in \text{Syl}_p(H)$ and $K \leq H$ with $|\pi(K)| \geq 2$. Then there exists $P_0 \in \text{Syl}_p(G)$ such that $P \leq P_0$. Now $P_0 \cap H$ is a $p$-subgroup of $H$, and so there exists $h \in H$ such that $P_0 \cap H \leq P^h$. Thus, $P \leq P_0 \cap H \leq P^h$ and $P = P_0 \cap H$. Since $G$ is a sylow embedded group, $P_0 \cap K \in \text{Syl}_p(K)$. But then,

$$P \cap K = P_0 \cap H \cap K = P_0 \cap K \in \text{Syl}_p(K)$$

Since, $P$ was chosen arbitrarily as a Sylow $p$-subgroup of $H$, we can conclude that $H$ is a sylow embedded group. ■

**Lemma 2.5** If $G$ is a sylow embedded group and $N \trianglelefteq G$, then $G/N$ is a sylow embedded group.

Proof: Let $\overline{G} = G/N$, $\overline{P_0} \in \text{Syl}_p(\overline{G})$ and $\overline{H}$ is a subgroup of $\overline{G}$ such that the order of $\overline{H}$ is divisible by two primes. Then $H, P_0 \leq G$, where $H$ and $P_0$ are the preimages of $\overline{P_0}$ and $\overline{H}$ in $G$. Let $P \in \text{Syl}_p(P_0)$. Then by Lemma 1.16, $\overline{P} \in \text{Syl}_p(\overline{P_0})$. Also, we have

$$\frac{|G|}{|P|}=\frac{|G|}{|P_0|}\frac{|P_0|}{|P|}=\frac{|\overline{G}|}{|\overline{P_0}|}\frac{|P_0|}{|P|}.$$

Hence, $|G|/|P|$ is not divisible by $p$ because $\overline{P_0}$ and $P$ are Sylow $p$-subgroups of $\overline{G}$ and $P_0$ respectively. Therefore, $P \in \mathrm{Syl}_p(G)$. Now since $G$ is sylow embedded, we get $P \cap H \in \mathrm{Syl}_p(H)$. Thus, by Lemma 1.16, $\overline{P \cap H} \in \mathrm{Syl}_p(\overline{H})$. But,

$$\overline{P \cap H}=\overline{P} \cap \overline{H}=\overline{P_0} \cap \overline{H}$$

therefore, $\overline{P_0} \cap \overline{H} \in \mathrm{Syl}_p(\overline{H})$ and $\overline{G}$ is a sylow embedded group. ∎

**Lemma 2.6** Let $G$ be a sylow embedded group and $H \in \mathrm{Hall}_\pi(G)$ where $\pi \subseteq \pi(G)$. Then,

$$G = \mathrm{N}_G(P)H \quad \text{for any } P \in \mathrm{Syl}_p(H)$$

Proof: First, we want to show that $\mathrm{Syl}_p(H) = \mathrm{Syl}_p(G)$. It is enough to show $\mathrm{Syl}_p(G) \subseteq \mathrm{Syl}_p(H)$ because $H$ is a Hall subgroup. Let $P \in \mathrm{Syl}_p(G)$. Since $G$ is sylow embedded $P \cap H \in \mathrm{Syl}_p(H)$. But then,

$$|G|_p = |P| \geq |P \cap H| = |H|_p = |G|_p$$

Therefore, $|P| = |P \cap H|$ and so $P = P \cap H$. Thus, $P \leq H$ and so $P \in \mathrm{Syl}_p(H)$. Now $G$ acts on $\mathrm{Syl}_p(H)$ by conjugation, and by Sylow's Theorem, $H$ acts transitively on $\mathrm{Syl}_p(H)$ by conjugation. Hence, by using the Frattini Argument, Lemma 1.17, we get $G = \mathrm{Stab}_G(P)H = \mathrm{N}_G(P)H$ where $P$ is any sylow subgroup of $H$. ∎

**Lemma 2.7** If $G$ is a sylow embedded group and $|G| = pqr$, where $p < q < r$ are primes, then $G$ is nilpotent.

Proof: Without loss, by Sylow's Theorem, $|\mathrm{Syl}_p(G)|$ equals $1, q, r,$ or $qr$. Also, the possibilities for the $|\mathrm{Syl}_q(G)|$ and $|\mathrm{Syl}_r(G)|$ are $1, r, pr$ and $1, pq$, respectively. We claim that $G$ has a normal sylow subgroup. If not, then

$$|G| = pqr \geq (p-1)q + r(q-1) + pq(r-1) + 1$$

But then, we get

$$0 \geq q(r-1) - (r-1) = (r-1)(q-1) > 0$$

which is a contradiction. So, we may assume that without loss, $P \trianglelefteq G$ where $P \in \mathrm{Syl}_p(G)$. Let $R \in \mathrm{Syl}_r(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then $PQ$, $PR \in \mathrm{Hall}(G)$ and so by Lemma 2.6, $G = \mathrm{N}_G(Q)P = \mathrm{N}_G(R)P$. Therefore, there exists $x, y \in G$ such that $R^x \leq \mathrm{N}_G(Q)$ and $Q^y \leq \mathrm{N}_G(R)$. But then $R^xQ$ and $Q^yR \in \mathrm{Hall}(G)$. So again by Lemma 2.6, $G = \mathrm{N}_G(Q)R^x = \mathrm{N}_G(Q)$ and $G = \mathrm{N}_G(R)Q^y = \mathrm{N}_G(R)$. Thus, $R \trianglelefteq G$, $Q \trianglelefteq G$, and we can conclude that $G$ is nilpotent.■

This section includes results about general groups, which are not necessarily sylow embedded groups. Next, we are going to examine groups with square free order. The proof of our condition for such groups requires the following four lemmas.

**Lemma 2.8** Let $G$ be a group and $H, K \leq G$ such that

$$gcd(\frac{|G|}{|H|}, \frac{|G|}{|K|}) = 1,$$

then $G = HK$.

Proof: Notice that

$$\frac{|G|}{|K|}\frac{|K|}{|H \cap K|} = \frac{|G|}{|H \cap K|} = \frac{|G|}{|H|}\frac{|H|}{|H \cap K|}$$

So, both $|G|/|H|$ and $|G|/|K|$ divide $|G|/|H \cap K|$, and therefore,

$$\frac{|G|}{|H|}\frac{|G|}{|K|} \text{ divides } \frac{|G|}{|H \cap K|}, \text{ since } gcd(\frac{|G|}{|H|}, \frac{|G|}{|K|})=1$$

Hence,

$$\frac{|G|}{|H|}\frac{|G|}{|K|} \leq \frac{|G|}{|H \cap K|}.$$

It follows that $|G| \leq |HK|$ and so $G = HK$. ■

**Lemma 2.9** Let $G$ be a group, $H_1, H_2, H_3 \leq G$ such that $H_i$ are nilpotent for $i = 1,2,3$ and

$$gcd(\frac{|G|}{|H_i|}, \frac{|G|}{|H_j|}) = 1 \text{ for each } i \neq j.$$

Then $G$ is nilpotent.

Proof: Let $P \in \mathrm{Syl}_p(G)$. Then there exists $i,j$ such that $p \nmid |G|/|H_i|$ and $p \nmid |G|/|H_j|$. Thus, there exists $x,y$ such that $P^x \leq H_i$ and $P^y \leq H_j$. Then

$P^x \in \mathrm{Syl}_p(H_i)$ and $P^y \in \mathrm{Syl}_p(H_j)$. But since $H_i, H_j$ are nilpotent $P^x \trianglelefteq H_i$ and $P^y \trianglelefteq H_j$. It follows that $P \trianglelefteq H_i^{x^{-1}}$ and $P \trianglelefteq H_j^{y^{-1}}$. But

$$gcd(\frac{|G|}{|H_i^{x^{-1}}|}, \frac{|G|}{|H_j^{y^{-1}}|}) = gcd(\frac{|G|}{|H_i|}, \frac{|G|}{|H_j|}) = 1$$

and so by Lemma 2.8, $G = H_i^{x^{-1}} H_j^{y^{-1}}$. Hence, $P \trianglelefteq H_i^{x^{-1}} H_j^{y^{-1}} = G$, and $G$ is nilpotent. ■

**Lemma 2.10** Let $G$ be a group and $p = \min.\pi(G)$, and suppose the Sylow $p$-subgroups of $G$ are cyclic. Then $G = PK$, where $P \in \mathrm{Syl}_p(G)$, $K \trianglelefteq G$ and $P \cap K = 1$.

Proof: First let $p = \min.\pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Then $P$ is cyclic by assumption. Let $\mathrm{N}_G(P)$ act on $P$ by conjugation. This action induces a homomorphism from $\mathrm{N}_G(P)$ into $\mathrm{Aut}(P)$ with kernel $\mathrm{C}_G(P)$. Thus, by the First Isomorphism Theorem 1.14, $\mathrm{Aut}(P)$ contains a subgroup isomorphic to $\mathrm{N}_G(P)/\mathrm{C}_G(P)$. Therefore,

$$|\mathrm{N}_G(P)/\mathrm{C}_G(P)| \text{ divides } |\mathrm{Aut}(P)| = p^n - p^{n-1} = p(p-1).$$

Considering the above, we claim that $|\mathrm{N}_G(P)/\mathrm{C}_G(P)| = 1$. Suppose $|\mathrm{N}_G(P)/\mathrm{C}_G(P)| > 1$. Then there exists a prime $q$, such that $q$ divides $|\mathrm{N}_G(P)/\mathrm{C}_G(P)|$. Since $P$ is cyclic, $P$ is abelian. Hence, $P \leq \mathrm{C}_G(P)$. But then $p$ does not divide $|\mathrm{N}_G(P)/\mathrm{C}_G(P)|$ since $P \in \mathrm{Syl}_p(G)$. Thus $q \neq p$. Therefore, $q$ divides $|\mathrm{N}_G(P)|$ which implies that $q$ divides $|G|$. Furthermore, $q$ divides $|\mathrm{Aut}(P)|$ and so $q$ divides $(p-1)$. We have $q < p$ and $q$ divides $|G|$ which contradicts the minimality of $p$. Thus, $|\mathrm{N}_G(P)/\mathrm{C}_G(P)| = 1$ and so $\mathrm{N}_G(P) = \mathrm{C}_G(P)$. Now by Burnside's Theorem 1.27, we have $G = PK$, where $K$ is normal and $P \cap K = 1$. ■

**Lemma 2.11** If $G$ is a group such that $|G| = p_1p_2p_3...p_n$, where $p_i$ are distinct primes and $n$ is a positive integer, then $G$ is solvable.

Proof: To show that $G$ is solvable, we are going to use induction on $|G|$. When $|G| = p_1$, since $G$ is a $p$-group, $G$ is solvable. Assume that every group with square free order, whose order is less than $|G|$, is solvable. Then we want to show that $G$ is solvable. Let $P \in \mathrm{Syl}_p(G)$ and $p = \min.\pi(G)$. Then $P$ is cyclic, and so by Lemma 2.10, there exists $K \trianglelefteq G$ such that $G = PK$ and $P \cap K = 1$. Since

$$\frac{G}{K} = \frac{PK}{K} \cong \frac{P}{P \cap K} = \frac{P}{1} \cong P,$$

then $G/K$ is a $p$-group which also means it is solvable. Also $|K| < |G|$ and $K$ has square free order. Thus $K$ is solvable by our inductive hypothesis. Hence, by properties of solvable groups, 1.19 (iii), we can conclude that $G$ is solvable. ■

**Lemma 2.12** Let $G$ be a sylow embedded group such that $|G| = p_1p_2p_3...p_n$, where $p_i$ is a prime for all $1 \leq i \leq n$ and $n \geq 3$. Then $G$ is nilpotent.

Proof: We use induction on $|G|$. We may assume $|\pi(G)| \geq 4$. Since $|G|$ is square free, $G$ is solvable by Lemma 2.11. Hence, $\mathrm{Hall}_\pi(G) \neq \varnothing$ for each $\pi \subseteq \pi(G)$ by Hall's Theorem. Let $H_i \in \mathrm{Hall}_{p_i'}(G)$ for $i = 1,2,3$. Then $H_i$'s are square free order and sylow embedded groups by Lemma 2.4. So, by induction $H_i$ is nilpotent for each $i = 1,2,3$. But

$$gcd(\frac{|G|}{|H_i|}, \frac{|G|}{|H_j|}) = gcd(p_i,p_j) = 1 \text{ for } i \neq j.$$

Therefore, by Lemma 2.9, $G$ is nilpotent.■

**Lemma 2.13** Let $G$ be a sylow embedded group and $P$ be a $p$-subgroup of $G$ such that $\mathrm{N}_G(P)$ is not a $p$-group. Then $P \leq \mathrm{O}_p(G)$.

Proof: Since $\mathrm{N}_G(P)$ is not a $p$-group, there exists $H \leq \mathrm{N}_G(P)$ where $H$ is a $p'$-group. Then $K = PH \leq G$ and $P \trianglelefteq K$. If $R \in \mathrm{Syl}_p(G)$, then since $G$ is sylow embedded, $R \cap K \in \mathrm{Syl}_p(K)$. Therefore, $P = R \cap K$ and so $P \leq R$. Since $R$ was chosen arbitrarily we have $P \leq \mathrm{O}_p(G)$.■

**Lemma 2.14** If $G$ is a sylow embedded group such that $|G| = p^a q^b r^c$ for distinct primes $p, q$, and $r$, then $G$ is nilpotent.

Proof: The proof is by induction on $|G|$. We claim that it is enough to show there exists $H \in \mathrm{Hall}_\pi(G)$ for some $\pi \subseteq \pi(G)$ such that $|\pi| = 2$. Without loss, we may assume $H = PQ$ where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$. Let $R \in \mathrm{Syl}_r(G)$. Then by Lemma 2.6 we have $G = \mathrm{N}_G(P)PQ = \mathrm{N}_G(P)Q$ and $G = \mathrm{N}_G(Q)QP = \mathrm{N}_G(Q)P$. Therefore, by Sylow's Theorem, there exists $x,y \in G$ such that $R^x \subseteq \mathrm{N}_G(P)$ and $R^y \subseteq \mathrm{N}_G(Q)$. But then $R^xP$ and $R^yQ$ are Hall subgroups of $G$ and by the same Lemma we get:

$$G = \mathrm{N}_G(P)R^x = \mathrm{N}_G(Q)R^y$$

Hence, $G = \mathrm{N}_G(P)R^x = \mathrm{N}_G(P)$ and so $P \trianglelefteq G$. Using the same argument, we get $Q \trianglelefteq G$. Now $PR$ and $QR$ are Hall subgroups of $G$. Again by Lemma 2.6, we have

$G = \mathrm{N}_G(R)P = \mathrm{N}_G(R)Q$. Thus, since $P, Q \trianglelefteq G$, $Q \leq \mathrm{N}_G(R)$ and $P \leq \mathrm{N}_G(R)$ by the first and the second equalities, respectively. Furthermore, since $PQ \leq G$, we get $R \trianglelefteq RPQ = G$. Therefore, $G$ is nilpotent.

We may assume $G$ has no normal Sylow subgroups and $G$ is nonsolvable. Let $P, Q \in \mathrm{Syl}_p(G)$ such that $|P \cap Q|$ is maximal. Suppose $|P \cap Q| \neq 1$. Then $Q \cap P < P$ and $Q \cap P < Q$. If $\lambda_p(\mathrm{N}_G(P \cap Q)) = 1$, let $R \in \mathrm{Syl}_p(N_G(P \cap Q))$ and $R \leq S \in \mathrm{Syl}_p(G)$. Now since $P$ is a $p$-group, $P$ is nilpotent. Therefore, by Lemma 1.10, we get

$$P \cap Q < \mathrm{N}_P(P \cap Q) \leq P \cap R \leq P \cap S$$

and so $P = S$ by the maximality of $|P \cap Q|$. Similarly, we get $Q = S$. Therefore, $P = Q$ which is a contradiction. Thus, $\lambda_p(\mathrm{N}_G(P \cap Q)) > 1$ and $\mathrm{N}_G(P \cap Q)$ is not a $p$-group. Hence, by Lemma 2.13, $1 \neq P \cap Q \leq \mathrm{O}_p(G) \trianglelefteq G$. Let $\overline{G} = G/\mathrm{O}_p(G)$. Then by Lemma 2.5, $\overline{G}$ is sylow embedded. Also, $|\pi(\overline{G})| = 3$. Therefore, by induction $G/\mathrm{O}_p(G)$ is nilpotent. Since $\mathrm{O}_p(G)$ is nilpotent, $G$ is solvable, by properties of solvable groups 1.19 (iii), we get a contradiction.

Thus, $|P \cap Q| = 1$ and so $P \cap P^g = 1$ for each $g \in G \backslash \mathrm{N}_G(P)$. Since $P$ is not normal in $G$, by Lemma 2.13, $\mathrm{N}_G(P)$ is a $p$-group. But since $P \leq N_G(P)$ and $P \in \mathrm{Syl}_p(G)$, we get $P = \mathrm{N}_G(P)$. By Frobenius' Theorem, there exists $K \trianglelefteq G$ such that $G = PK$ and $P \cap K = 1$. Now,

$$|K| = \frac{|K|}{1} = \frac{|K|}{|P \cap K|} = \frac{|PK|}{|P|} = \frac{|G|}{|P|} = |G|_{\{q,r\}} = q^b r^c$$

Therefore, $K \in \mathrm{Hall}_{\{q,r\}}(G)$, and so $G$ is nilpotent.■

**Lemma 2.15** Let $G$ be a solvable sylow embedded group such that $|\pi(G)| \geq 3$. Then $G$ is nilpotent.

Proof: The proof is by induction on $|G|$. By Lemma 2.14, we may assume $|\pi(G)| \geq 4$. Let $|G| = \prod_{i=1}^{n} p_i^{e_i}$, where $n \geq 4$, $p_i$ is a prime, and $e_i$ is a positive integer for all $1 \leq i \leq n$. Since $G$ is solvable, by Hall's Theorem, there exist $H_i \in \mathrm{Hall}_{p_i'}(G)$ for $i = 1,2,3$. $H_i$ is a solvable, sylow embedded group for each $i$. Moreover, $|\pi(G)| \geq 4$ implies $|\pi(H_i)| \geq 3$ for each $i$. Therefore, by induction each $H_i$ is nilpotent. Since,

$$gcd(\frac{|G|}{|H_i|}, \frac{|G|}{|H_j|}) = gcd(p_i^{e_i}, p_j^{e_j}) = 1$$

By Lemma 2.9, $G$ is nilpotent.■

**Theorem 2.1** Let $G$ be group with $|\pi(G)| \geq 3$. Then $G$ is nilpotent if and only if $G$ is a sylow embedded group.

Proof: Earlier we gave the proof of one direction of this theorem. Now let $G$ be a sylow embedded group with $|\pi(G)| \geq 3$, and we use induction on $|G|$ to prove that $G$ is nilpotent. From Lemma 2.15, it suffices to show that $G$ is solvable, and we may assume that $|\pi(G)| \geq 4$. Let $P, Q \in \mathrm{Syl}_p(G)$ such that $|P \cap Q|$ is maximal. If $|P \cap Q| = 1$, by Frobenius' Theorem and the argument used in Lemma 2.14, there exists $K \trianglelefteq G$ with $G = PK$ and $P \cap K = 1$. Then $K \in \mathrm{Hall}_{p'}(G)$ and so $|\pi(K)| \geq 3$. Therefore, since $K$ is sylow embedded, $K$ is nilpotent by induction.

Moreover, by the Second Isomorphism Theorem 1.15, we have:

$$\frac{G}{K} = \frac{PK}{K} \cong \frac{P}{P \cap K}$$

and so $G/K$ is nilpotent. Hence, $G$ is solvable.

Now, suppose $|P \cap Q| \neq 1$. As before, the maximality of $|P \cap Q|$ implies $\lambda_p(\mathrm{N}_G(P \cap Q)) > 1$. Thus, $\mathrm{N}_G(P \cap Q)$ is not a $p$-group, and therefore,

$$1 \neq P \cap Q \leq \mathrm{O}_p(G) \trianglelefteq G$$

$\overline{G} = G/\mathrm{O}_p(G)$ which is a sylow embedded group, and $|\pi(\overline{G})| \geq 3$. Therefore, $\overline{G}$ is nilpotent (and solvable) by induction. Since $O_p(G)$ is a $p$-group, and therefore solvable, by properties of solvable groups 1.19 (iii), $G$ is solvable.■

# CHAPTER 3

## Counterexamples

We want to emphasize that embeddness implies nilpotency and vice versa is not always true for a group $G$ where $|\pi(G)| = 2$. Let's illustrate this by two counterexamples:

**Example 3.1** $A_4$ is a sylow embedded group. $|A_4| = 12 = 2^2 3$. So, if we have a subgroup $H$ such that $|\pi(H)| \geq 2$ then $|H| = 6$. However, $A_4$ does not have a subgroup $H$ of order 6. If it did, then by Cauchy's Theorem 1.12, there exists $h \in H$ such that $|h| = 3$. Since $|A_4|/|H| = 2$, then $H \trianglelefteq A_4$.

Without loss, let $h = (123)$. Then

$$(123)^{(12)(34)} = (214) \in H$$
$$(214)^{(23)(14)} = (341) \in H$$

Also, the squares of the above elements $(123), (214), (341)$ are in $H$. Hence, $|H| \geq 7$, including the identity, which is a contradiction. However, $A_4$ is not nilpotent because

$$< (123) >, < (234) >, < (124) >, < (134) > \; \in \mathrm{Syl}_3(A_4) \text{ and } |\mathrm{Syl}_3(A_4)| > 1.$$

**Example 3.2** $D_6$ is a sylow embedded group. Let $H \leq D_6$ such that $|\pi(H)| \geq 2$.

Then $|H| = 6$. Hence, $|D_6|/|H| = 2$ and so $H \trianglelefteq D_6$. Since $H \trianglelefteq D_6$, by Lemma 2.1, $H \in \text{Sylem}(D_6)$. Therefore $D_6$ is a sylow embedded group.

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$$

$$\delta 3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$$

So, $\delta_1^2 = \delta_2^2 = \delta_3^2 = 1 = \mu_1^2 = \mu_2^2 = \mu_3^2$. By Sylow's Theorem, $|\text{Syl}_2(D_6)| = 1$ or 3. We know that $|P| = 4$, where $P \in \text{Syl}_2(D_6)$, and there are six elements of order 2. Therefore, $|\text{Syl}_2(D_6)| = 3 \neq 1$ and so $D$ is not nilpotent.

# OPEN QUESTIONS

Now that we have proven the equivalence of nilpotency with sylow embeddness for certain finite groups, other questions related to this proof have arisen and still remain unanswered. These open questions include the following:

1 - Are the concepts of nilpotency and sylow embeddedness equivalent for infinite groups with a sylow theory?

2 - For a given prime $p$ what kinds of groups act transitively on the set $\text{Sylem}_p(G)$ by conjugation?

3 - If G is a group such that $\text{Sylem}_p(G) \neq \varnothing$ for all primes $p$, can the structure of $G$ be determined?

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