CLASSES OF EXAMPLES OF *L*-TOPOLOGICAL SPACES: VARIOUS METHODS OF GENERATION

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ABSTRACT

The purpose of this thesis is to explore classes of examples of L-topological spaces born of different methods of generation. Not only the characteristic and Martin but also the Kubiak/Lowen categorical functors act as adjoint functor pairs between the categories of **TOP** and L-**TOP**; they also have a similar fibre map adjunction in common.

In addition, the Stone spectrum adjunction and its L-fuzzy counterpart are investigated, resulting in conclusions about sobriety and separation. Finally, the L-fuzzy real line and interval and their associated L-topologies are compared to previous example classes. A greater depth of L-spaces with desirable properties is found, underscoring the importance of studying poslat topological spaces.

Introduction and Preliminaries

When we break down scholarly investigation to its bare bones in mathematics, we see that much focus is placed on objects, collections of objects, and mappings between objects. With each step up from these fundamental ideas, it becomes necessary to create generalizations and classifications. Important properties evolve from investigating similar types of objects, sets and mappings, such as openness, separation, continuity, compactness, and so on. To coherently discuss these and other properties, we need to conceptualize spaces, and of course, delineate examples of spaces. The purpose of this thesis is to explore classes of examples of lattice-valued topological spaces; the examples being developed from different avenues of generation.

Topology is not the only branch of mathematics to create examples with certain properties by generating them from a more basic object or idea. For example, the integers arose from the natural numbers to facilitate subtraction. In this paper, various types of L-topological spaces that are generated or evolve in different ways are explored. The fact that no single method of generation produces all possible examples of L-topological spaces underscores the inherent richness of these spaces. We will adhere to the standardized terminology of L-topology, following Chapters 3 and 4 of [10].

The L under consideration is a complete lattice, and working with lattice-valued maps requires an understanding of properties of lattices; we refer the reader to the works of Birkhoff [1], Gierz et al [4], and Grätzer [7]. Topology requires that we investigate mappings between sets, and if those sets are lattices and we want preservation of joins and meets, we refine our language to include the ideas of semiframes, frames, semilocales, and locales as in [20], [21], and [23]. Further generalization leads to the need for understanding of category theory ([8], [15]).

Studying L-topological spaces, also known as fuzzy topology or point-set lattice-theoretic(poslat) topology, would not be possible without the advent of fuzzy sets. The first paper on fuzzy sets was authored by L.A. Zadeh in 1965 [26], fuzzy topology arose from these concepts a few years later thanks to C.L. Chang, who investigated $\mathbb{I}$-topologies[2], and J.A. Goguen, who generalized to L-subsets and L-topologies [5,6]. Significant progress has been made in this relatively young branch of topology. There are many valuable ideas of traditional topology such as continuity, compactness, completeness, separability, and so on that have been explored in a fuzzy setting.

Examples of spaces both provide a construct for verification of claims of the existence of properties, and examples also lend power to classification and representation of generalized theorems about certain kinds of spaces. Exploring ways to generate L-topological spaces can help to clarify the benefits of fuzzy topology. To begin, it is necessary to recall some fundamental definitions.

Definition. Let X be a set. The *powerset* of X, $\mathfrak{P}(X)$, is the set of all possible subsets of X.

Definition. $(X, \mathcal{T})$ is a *topological space* iff X is a set, $\mathcal{T}$ is a collection of open subsets of X ($\mathcal{T} \subset \mathfrak{P}(X)$) such that $\mathcal{T}$ is closed under arbitrary unions and finite intersections.

Definition. $f : X \to Y$ is (traditionally) *continuous* with respect to $(X, \mathcal{T})$ and $(Y, \mathcal{S})$, or

$f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous, iff

$$\forall V \in \mathcal{S}, \quad f^{\leftarrow}(V) \in \mathcal{T}$$

where $f^{\leftarrow}(V)$ is the *preimage* of f over V, that is,

$$f^{\leftarrow}(V) = \{x \in X \mid f(x) \in V\}$$

Definition. L is a [complete] *lattice* if either L is closed under binary [arbitrary] joins, or L is closed under binary [arbitrary] meets, or both.

L is a *chain* if L is totally ordered, i.e., $\forall x, y \in L$, either $x \leq y$ or $y \leq x$.

Definition. Let $\{f_\gamma | \gamma \in \Gamma\} \subset L^X$, i.e., $\forall \gamma$, $f_\gamma : X \to L$ where L is a complete lattice. We define the *join map*

$$\left(\bigvee_{\gamma \in \Gamma} f_\gamma\right) : X \to L \text{ by } \left(\bigvee_{\gamma \in \Gamma} f_\gamma\right)(x) = \bigvee_{\gamma \in \Gamma} (f_\gamma(x)) \text{ for each } x \in X$$

and the *meet map*

$$\left(\bigwedge_{\gamma \in \Gamma} f_\gamma\right) : X \to L \text{ by } \left(\bigwedge_{\gamma \in \Gamma} f_\gamma\right)(x) = \bigwedge_{\gamma \in \Gamma} (f_\gamma(x)) \text{ for each } x \in X$$

Definition. (X, τ) is an *L-topological space* iff X is a set, τ is a collection of maps from X to L ($\tau \subset L^X$) such that τ is closed under arbitrary joins and finite meets.

Definition. $f : X \to Y$ is *L-continuous* with respect to (X, τ) and (Y, σ), or

$f : (X, \tau) \to (Y, \sigma)$ is L-continuous, iff

$$\forall v \in \sigma, \quad f_L^{\leftarrow}(v) \in \tau$$

where $f_L^{\leftarrow}(v)$ is the *L-preimage* of f over v, that is,

$$f_L^{\leftarrow}(v) = v \circ f$$

Definition. A *characteristic function* $\chi_U : X \longrightarrow \{0, 1\}$ is a function that determines whether or not an element of X is a member of U.

$$\chi_U(x) = \begin{cases} 1, & \text{if } x \in U \\ 0, & \text{if } x \notin U \end{cases}$$

Definition. (Adjunction of Isotone Maps Between Posets) Let $(X, \leq), (Y, \leq)$ be posets, with $f : X \longrightarrow Y, g : X \longrightarrow Y$ isotone. Then, $f \dashv g$ (f is *left-adjoint* to g, or, g is *right-adjoint* to f) iff

$$\begin{aligned} (i)\ \forall y &\in Y,\ f \circ g(y) \leq y \\ (ii)\ \forall x &\in X,\ g \circ f(x) \geq x \end{aligned}$$

Notation. We refer to the following categories in this paper:

Category **SET** is comprised of objects that are sets, morphisms that are functions between sets, with the usual identity, composition, and associativity.

Category **TOP** has objects that are traditional topological spaces, morphisms that are continuous, and the identity, composition, and associativity of **SET**.

Category L-**TOP** is L-topological spaces, with L-continuous maps with the identity, etc. of **SET**.

Category **SFRM** is semiframes with arbitrary joins- and finite meets-preserving maps.

Category **SLOC** is the dual of **SFRM.**

Definition. $F : \mathbf{C} \rightarrow \mathbf{D}$ is a categorical functor if F maps objects of category $\mathbf{C}$ to category $\mathbf{D}$, morphisms of category $\mathbf{C}$ to category $\mathbf{D}$,and preserves the composition and identities of $\mathbf{C}$.

Definition. Let F, G be functors between categories $\mathbf{C}$ and $\mathbf{D}$, and $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{C} \leftarrow \mathbf{D}$. Then $F \dashv G$ if the following hold:

I. $\forall A \in |\mathbf{C}|$, there exists $\eta \in \mathbf{C}(A, (G \circ F)(A))$ such that $\forall B \in |\mathbf{D}|$, and $\forall f \in \mathbf{C}(A, G(B))$ there is a unique map $\overline{f} \in \mathbf{D}(F(A), B)$ such that $G(\overline{f}) \circ \eta =$ (*or* $\equiv$) f.

II. If $g \in \mathbf{C}(A_1, A_2)$, then $F(g) = (\overline{\eta_{A_2} \circ g})$, where $\eta_{A_2} : A_2 \rightarrow GF(A_2)$ is given from I.

Part I

Examples Generated By Maps Between Fibres of Topologies

1. The Characteristic and Martin Mappings and Functors

The most natural way to transform a collection of sets into a collection of lattice-valued mappings is to create a characteristic function for each set. This idea seems almost trivial in its simplicity, and its origins are best described as topological "folklore".

1.1. The Characteristic and Martin Maps on Fibres

The fibre of traditional topologies on a set X is given by:

$$\mathbb{T}_X = \{\mathcal{T} \ : (X, \mathcal{T}) \in |\mathbf{TOP}|\}$$

and the fibre of L-topologies on a et X is given by:

$$\mathbb{F}_X = \{\tau : (X, \tau) \in |L\text{-}\mathbf{TOP}|\}$$

Clearly, both $\mathbb{T}_X$ and $\mathbb{F}_X$ are partially ordered sets (in fact, complete lattices) where the order is inclusion.

Definition 1.1. *Let X be a set , $\mathbb{T}_X$ be as described above, and*

$$L^X = \{u | \ u : X \to L\}$$

We define $\forall \mathcal{T} \in \mathbb{T}_X$ the characteristic map $g_\chi : \mathbb{T}_X \to \mathfrak{P}(L^X)$ by: $\forall \mathcal{T} \in \mathbb{T}_X$,

$$g_\chi(\mathcal{T}) = \{\chi_U : U \in \mathcal{T}\}$$

When discussing characteristic functions, it is important to keep in mind what happens to the join or meet of such functions.

Lemma 1.2. *Let X be a set, and let $\{U_\gamma : \gamma \in \Gamma\} \subset \mathfrak{P}(X)$. Then, the join (meet) of the characteristic functions χ_{U_γ} is the characteristic of the union (intersection):*

$$\bigvee_{\gamma\in\Gamma} \left(\chi_{U_\gamma}\right) = \chi_{\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right)}$$

and

$$\bigwedge_{\gamma\in\Gamma} \left(\chi_{U_\gamma}\right) = \chi_{\left(\bigcap_{\gamma\in\Gamma} U_\gamma\right)}$$

Proof: To show that $\bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}) = \chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}$, let $x \in X$. Then, there are two cases to be considered.

Case 1: $x \in \bigcup_{\gamma\in\Gamma}(U_\gamma)$.

Then, $\exists \gamma_0 \in \Gamma$, $x \in U_{\gamma_0}$. Now, by how we define a characteristic function, $\chi_{U_{\gamma_0}}(x) = 1$. Then, $1 \in \left\{\chi_{U_\gamma}(x) : \gamma \in \Gamma\right\}$. But, since the range of any characteristic function is {0,1}, $\left\{\chi_{U_\gamma}(x) : \gamma \in \Gamma\right\}$ is bounded above by 1. So, 1 is the least upper bound of the set $\left\{\chi_{U_\gamma}(x) : \gamma \in \Gamma\right\}$, in other words, $1 = \bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}(x))$.

And, since $x \in \bigcup_{\gamma\in\Gamma}(U_\gamma)$, we know that $\chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}(x) = 1$, by the definition of the characteristic of $\bigcup_{\gamma\in\Gamma}(U_\gamma)$. Now we have that $\bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}(x)) = \chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}(x)$.

Case 2: $x \notin \bigcup_{\gamma\in\Gamma}(U_\gamma)$.

Then, $\forall \gamma \in \Gamma$, $x \notin U_\gamma$. So $\forall \gamma \in \Gamma$, $\chi_{U_\gamma}(x) = 0$. Then, as above, 0 is both an upper bound of $\left\{\chi_{U_\gamma}(x) : \gamma \in \Gamma\right\}$, and a member of the set, so $0 = \bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}(x))$.

And, since $x \notin \bigcup_{\gamma\in\Gamma}(U_\gamma)$, we know that $\chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}(x) = 0$, by the definition of the characteristic of $\bigcup_{\gamma\in\Gamma}(U_\gamma)$. Now we have that $\bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}(x)) = \chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}(x)$.

By the separation of cases, $\forall x \in X$, $\bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}(x)) = \chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}(x)$. So then, $\bigvee_{\gamma\in\Gamma}(\chi_{U_\gamma}) = \chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)}$.

The proof for $\bigwedge_{\gamma\in\Gamma}(\chi_{U_\gamma}) = \chi_{(\bigcap_{\gamma\in\Gamma} U_\gamma)}$ directly parallels the above proof ■.

Proposition 1.3. *Let X be a set and the characteristic map $g_X : \mathbb{T}_X \to \mathfrak{P}(L^X)$ be as defined. The following properties hold:*

(i) $g_X(\mathcal{T}) \in \mathbb{F}_X$, i.e., $g_X : \mathbb{T}_X \to \mathbb{F}_X$ is a map.

(ii) $g_X : \mathbb{T}_X \to \mathbb{F}_X$ is isotone.

Proof of (i): We want to show that $g_X(\mathcal{T})$ is an L-topology on X, so $g_X(\mathcal{T})$ must be closed under arbitrary joins and finite meets. Let $\{u_\gamma : \gamma \in \Gamma\} \subset g_X(\mathcal{T})$. Then

$\forall \gamma \in \Gamma$, $\exists U_\gamma \in \mathcal{T}$ such that $u_\gamma = \chi_{U_\gamma}$. So then $\bigvee \{u_\gamma : \gamma \in \Gamma\} = \bigvee \{\chi_\gamma : \gamma \in \Gamma\}$, which is just the characteristic of the union $\bigcup_{\gamma \in \Gamma}(U_\gamma)$ by Lemma 2. I.e.,

$$\begin{aligned} \bigvee \{u_\gamma : \gamma \in \Gamma\} &= \bigvee \{\chi_\gamma : \gamma \in \Gamma\} \\ &= \chi_{(\bigcup_{\gamma \in \Gamma} U_\gamma)} \end{aligned}$$

Since $\mathcal{T}$ is closed under arbitrary unions, $\bigcup_{\gamma \in \Gamma}(U_\gamma) \in \mathcal{T}$. Then $\bigvee \{u_\gamma : \gamma \in \Gamma\} \in g_X(\mathcal{T})$, and so $g_X(\mathcal{T})$ is closed under arbitrary joins.

Let $\{u_\delta : \delta \in \Delta,\ \Delta \text{ finite}\} \subset g_X(\mathcal{T})$. Then $\forall \delta \in \Delta$, $\exists U_\delta \in \mathcal{T}$ such that $u_\delta = \chi_{U_\delta}$. So then $\bigwedge \{u_\delta : \delta \in \Delta\} = \bigwedge \{\chi_\delta : \delta \in \Delta\}$, which is just the characteristic of the intersection $\bigcap_{\delta \in \Delta}(U_\delta)$ by Lemma 2. I.e.,

$$\begin{aligned} \bigwedge \{u_\delta : \delta \in \Delta\} &= \bigwedge \{\chi_\delta : \delta \in \Delta\} \\ &= \chi_{(\bigcap_{\delta \in \Delta} U_\delta)} \end{aligned}$$

Since $\mathcal{T}$ is closed under finite intersections, $\bigcap_{\delta \in \Delta}(U_\delta) \in \mathcal{T}$. Then $\bigwedge \{u_\delta : \delta \in \Delta\} \in g_X(\mathcal{T})$. So $g_X(\mathcal{T})$ is closed under finite meets. Therefore, $g_X(\mathcal{T}) \in \mathbb{F}_X$.

Proof of 2): Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}_X$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$. Let $u \in g_X(\mathcal{T}_1)$. Then, $\exists U \in \mathcal{T}_1$ such that $u = \chi_U$. But, $\mathcal{T}_1 \subset \mathcal{T}_2$ so $U \in \mathcal{T}_2$. Then, $\chi_U \in g_X(\mathcal{T}_2)$. So $g_X(\mathcal{T}_1) \subset g_X(\mathcal{T}_2)$ ■.

There is a corresponding isotone map from $\mathbb{F}_X$ to $\mathbb{T}_X$ which we will call the Martin map [16]. We will show that together, the characteristic map and the Martin map are adjoints.

Definition 1.4. *Let X be a set, $\mathbb{F}_X$ be as described above, and $\mathfrak{P}(X) = \{A | A \subset X\}$. We define the Martin map $m : \mathfrak{P}(\mathfrak{P}(X)) \leftarrow \mathbb{F}_X$ by: $\forall \tau \in \mathbb{F}_X$,*

$$m(\tau) = \{U \in \mathfrak{P}(X) : \chi_U \in \tau\}$$

Proposition 1.5. *Let X be a set and the Martin map $m : \mathfrak{P}(\mathfrak{P}(X)) \leftarrow \mathbb{F}_X$ be as defined. The following properties hold:*

(i) $m(\tau) \in \mathbb{T}_X$, i.e., $m : \mathbb{T}_X \leftarrow \mathbb{F}_X$ is a map.

(ii) $m : \mathbb{T}_X \leftarrow \mathbb{F}_X$ is isotone

Proof of (i): To show that $m(\tau)$ is a traditional topology on X, $m(\tau)$ must be closed under arbitrary unions and finite intersections. Let $\{U_\gamma : \gamma \in \Gamma\} \subset m(\tau)$.

For (ii), let $U \in \mathcal{T}$. Then $\chi_U \in \{\chi_V : V \in \mathcal{T}\} = g_\chi(\mathcal{T})$. So,

$$\begin{aligned} U &\in \{W \in \mathfrak{P}(X) : \chi_W \in g_\chi(\mathcal{T})\} \\ &= m\left(g_\chi(\mathcal{T})\right) \\ &= (m \circ g_\chi)(\mathcal{T}) \end{aligned}$$

Hence $\mathcal{T} \subset (m \circ g_\chi)(\mathcal{T})$ $\blacksquare$.

1.2. The Characteristic and Martin Functors on TOP and L-TOP

Now we consider the grander scheme of the categories **TOP** and L-**TOP**. Recall that with categories, we not only look at objects, but also morphims, identity, associativity, and morphism composition. So, when we have an adjoint functor relationship on categories, we benefit by having properties preserved.

Definition 1.7. *Let the characteristic functor* G_χ : **TOP** $\to L$-**TOP** *be defined as follows:*

Objects: $\forall (X, \mathcal{T}) \in |\mathbf{TOP}|$, *let* $G_\chi(X, \mathcal{T}) = (X, g_\chi(\mathcal{T}))$, *where* g_χ *is the characteristic map defined previously.*

Morphisms: $\forall f : (X, \mathcal{T}) \to (Y, \mathcal{S})$, *let* $G_\chi(f) = f$

Proposition 1.8. G_χ : **TOP** $\to L$-**TOP** *is a functor.*

Proof: For objects, let $(X, \mathcal{T}) \in |\mathbf{TOP}|$, then

$$G_\chi(X, \mathcal{T}) = (X, g_\chi(\mathcal{T})) = (X, \{\chi_U : U \in \mathcal{T}\})$$

. Now we must show that this is an L-topology on X. But by Proposition 3, $g_\chi(\mathcal{T}) \in \mathbb{F}_X$. Hence, G_χ preserves objects.

For morphisms, let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ such that f is traditionally continuous. Because we want to show that $G_\chi(f) = f$ is L-continuous, we claim that $f : (X, g_\chi(\mathcal{T})) \to (Y, g_\chi(\mathcal{S}))$ is L-continuous. Let $v \in g_\chi(\mathcal{S}) = \{\chi_V : V \in \mathcal{S}\}$ We need to have $f_L^{\leftarrow}(v) \in g_\chi(\mathcal{T})$. Keep in mind that $f_L^{\leftarrow} : L^X \leftarrow L^Y$ by

$$f_L^{\leftarrow}(w) = w \circ f, \ \forall w : Y \to L.$$

So then, $f_L^{\leftarrow}(v) = v \circ f$. It suffices to show that $v \circ f$ can be written as a characteristic function χ_U such that $U \in \mathcal{T}$. Let $x \in X$. Now,$(v \circ f)(x) = v(f(x))$, and

Then, $\forall \gamma \in \Gamma$, $\exists u_\gamma \in \tau$ such that $u_\gamma = \chi_{U_\gamma}$. But since τ is an L-topology on X, it is closed under arbitrary joins. So, $\bigvee \{u_\gamma : \gamma \in \Gamma\} \in \tau$, or,

$$\begin{aligned} \bigvee \{u_\gamma : \gamma \in \Gamma\} &= \bigvee \left\{\chi_{U_\gamma} : \gamma \in \Gamma\right\} \\ &= \chi_{(\bigcup_{\gamma \in \Gamma} U_\gamma)} \\ &\in \tau \end{aligned}$$

And, since $\chi_{(\bigcup_{\gamma\in\Gamma} U_\gamma)} \in \tau$, $\bigcup_{\gamma\in\Gamma} U_\gamma \in m(\tau)$. Hence, $m(\tau)$ is closed under arbitrary unions.

Let $\{U_\delta : \delta \in \Delta, \Delta \text{ finite}\} \subset m(\tau)$. Then $\forall \delta \in \Delta, \exists u_\delta \in \tau$ such that $u_\delta = \chi_{U_\delta}$. So then $\bigwedge \{u_\delta : \delta \in \Delta\} = \bigwedge \{\chi_{U_\delta} : \delta \in \Delta\}$, which is just the characteristic of the intersection $\bigcap_{\delta\in\Delta}(U_\delta)$. I.e.,

$$\begin{aligned} \bigwedge \{u_\delta : \delta \in \Delta\} &= \bigwedge \{\chi_{U_\delta} : \delta \in \Delta\} \\ &= \chi_{(\bigcap_{\delta\in\Delta} U_\delta)} \end{aligned}$$

And since τ is closed under finite meets, $\chi_{(\bigcap_{\delta\in\Delta} U_\delta)} \in \tau$, and so $\bigcap_{\delta\in\Delta}(U_\delta) \in m(\tau)$. So $m(\tau)$ is closed under finite intersections. Therefore, $m(\tau) \in \mathbb{T}_X$.

Proof of 2): Let $\tau_1, \tau_2 \in \mathbb{F}_X$ such that $\tau_1 \subset \tau_2$. Let $U \in m(\tau_1)$. Then, $\exists u \in \tau_1$ such that $u = \chi_U$. But $\tau_1 \subset \tau_2$, so $u \in \tau_2$. Then, $U \in m(\tau_2)$. So $m(\tau_1) \subset m(\tau_2)$ $\blacksquare$.

The codomains of both the characteristic and the Martin maps can be modified, thanks to Propositions 3 and 5: $g_\chi : \mathbb{T}_X \to \mathbb{F}_X$ and $m : \mathbb{T}_X \leftarrow \mathbb{F}_X$. Now we can convert between traditional topologies and L-topologies, and establish a relationship between these mappings.

Proposition 1.6. *$g_\chi \dashv m$ as isotone maps.*

Proof: For the characteristic map to be left-adjoint to the Martin map, we must show:

(i) $(g_\chi \circ m)(\tau) \subset \tau$ for all $\tau \in \mathbb{F}_X$, and

(ii) $(m \circ g_\chi)(\mathcal{T}) \subset \mathcal{T}$ for all $\mathcal{T}$ in $\mathbb{T}_X$.

For (i), let $u \in (g_\chi \circ m)(\tau)$. Then,

$$\begin{aligned} u &\in g_\chi(m(\tau)) \\ &= \{\chi_U : U \in m(\tau)\} \\ &= \{\chi_U : U \in \{V \in P(X) : \chi_V \in \tau\}\} \end{aligned}$$

Then by inspection, we see that $u \in \tau$. Hence, $(g_\chi \circ m)(\tau) \subset \tau$.

since $v \in g_\chi(\mathcal{S}) = \{\chi_V : V \in \mathcal{S}\}$, $\exists V \in \mathcal{S}$ so that $v = \chi_V$. Then,

$$\begin{aligned} v \circ f(x) &= v(f(x)) \\ &= \chi_V(f(x)) \\ &= \{1, \text{ if } f(x) \in V \text{ , and } 0 \text{ if } f(x) \notin V\} \\ &= \{1, \text{ if } x \in f^{\leftarrow}(V) \text{ , and } 0 \text{ if } x \notin f^{\leftarrow}(V)\} \\ &= \chi_{f^{\leftarrow}(V)}(x) \end{aligned}$$

But since f is traditionally continuous, $f^{\leftarrow}(V) \in \mathcal{T}$. So $v \circ f \in \{\chi_U : U \in \mathcal{T}\} = g_\chi(\mathcal{T})$. Hence f is L-continuous and G_χ preserves morphisms.

We must also check that composition and associativity of morphisms, as well as the identity morphism, is as defined normally in **SET**. Since morphisms are not affected by G_χ, these traits should be preserved.

For identities, recall that for $X \in |\mathbf{SET}|$, the identity function is $id_X : X \to X$ by $id_X(x) = x$ for all $x \in X$. Let $id_{(X,\mathcal{T})} : (X, \mathcal{T}) \to (X, \mathcal{T})$ be the identity function on X for any $(X, \mathcal{T}) \in |\mathbf{TOP}|$. Also, let $id_{(X,\tau)} : (X, \tau) \to (X, \tau)$ be the identity function on X for any $(X, \tau) \in |L\text{-}\mathbf{TOP}|$. We claim that $id_{(X,\tau)} = id_{(X,\mathcal{T})} = id_X$. To show $id_{(X,\mathcal{T})} = id_X$, i.e., that id_X is traditionally continuous, let $U \in \mathcal{T}$.

$$\begin{aligned} id_X^{\leftarrow}(U) &= \{y \in X : id_X(y) \in U\} \\ &= \{y \in X : y \in U\} \\ &= U \end{aligned}$$

and, $U \in \mathcal{T}$, so id_X is traditionally continuous. I.e., $id_X = id_{(X,\mathcal{T})}$. We have shown that $G_\chi(f) = f$ for $f \in \mathbf{TOP}(X, Y)$, so we should have that $id_{(X,\tau)} = G_\chi(id_{(X,\mathcal{T})}) = id_{(X,\mathcal{T})} = id_X$ is L-continuous. To check this, let $u \in \tau$.

$$\begin{aligned} (id_X)_L^{\leftarrow}(u) &= u \circ (id_X) \\ &= u, \end{aligned}$$

since the identity has this property in **SET**. Now, we have $(id_X)_L^{\leftarrow}(u) = u \in \tau$, so id_X is L-continuous. I.e., $id_X = id_{(X,\tau)}$.

In category **SET**, composition and associativity are inherited, as they will be here, so we need not check them. Then, G_χ : **TOP** $\to$$L$-**TOP** is a functor ■.

To illustrate this method of transforming a classical topological space into an L-topological space, consider the following example.

Example 1.9. *Let $(X, \mathcal{T}), (Y, \mathcal{S}) \in |\mathbf{TOP}|$, where*

$$X = \{a, b\}\,, \quad \mathcal{T} = \mathfrak{P}(X)$$
$$Y = \{1, 2, 3\}, \quad \mathcal{S} = \{\emptyset, \{3\}, \{1, 2\}, Y\}$$

and let $f \in \mathbf{TOP}((X, \mathcal{T}), (Y, \mathcal{S}))$, where $f(a) = 2, \quad f(b) = 3$.

$\mathcal{T}$ is the discrete topology on X, and $\mathcal{S}$ is closed under unions and intersections. f is continuous, since the preimage of any member of $\mathcal{S}$ is a subset of X, and hence included in the powerset of X.

Now, using the characteristic functor, we claim that $G_\chi(X, \mathcal{T}) = (X, g_\chi(\mathcal{T}))$ is an $\{0, 1\}$-topological space:

$$g_\chi(\mathcal{T}) = \left\{\chi_\emptyset = \underline{0}, \chi_{\{a\}}, \chi_{\{b\}}, \chi_X = \underline{1}\right\}$$

where $\underline{0}$ and $\underline{1}$ are the constant maps. To check that is closed under $\vee$ and $\wedge$, note that we need not check any subcollection that contains either $\underline{0}$ or $\underline{1}$:

$$\underline{0} \vee u = u, \quad \underline{0} \wedge u = \underline{0}, \quad \underline{1} \vee u = \underline{1}, \quad \underline{1} \wedge u = u$$

for any $u \in g_\chi(\mathcal{T})$ by the nature of universal upper and lower bounds. So it suffices to show that $\chi_{\{a\}} \vee \chi_{\{b\}}$ and $\chi_{\{a\}} \wedge \chi_{\{b\}}$ are elements of $g_\chi(\mathcal{T})$:

$$\left(\chi_{\{a\}} \vee \chi_{\{b\}}\right)(a) = \left(\chi_{\{a\}}(a)\right) \vee \left(\chi_{\{b\}}(a)\right) = 1 \vee 0 = 1$$
$$\left(\chi_{\{a\}} \vee \chi_{\{b\}}\right)(b) = \left(\chi_{\{a\}}(b)\right) \vee \left(\chi_{\{b\}}(b)\right) = 0 \vee 1 = 1$$

So, $\forall x \in X$, $\left(\chi_{\{a\}} \vee \chi_{\{b\}}\right)(x) = 1$. I.e., $\chi_{\{a\}} \vee \chi_{\{b\}} = \underline{1} \in g_\chi(\mathcal{T})$. Similarly, $\chi_{\{a\}} \wedge \chi_{\{b\}} = \underline{0} \in g_\chi(\mathcal{T})$. Then, $(X, g_\chi(\mathcal{T}))$ is an $\{0, 1\}$-topology on X. To show that $G_\chi(Y, \mathcal{S}) = (Y, g_\chi(\mathcal{S})) \in |\{0, 1\}\mathbf{TOP}|$, consider:

$$g_\chi(\mathcal{S}) = \left\{\chi_\emptyset = \underline{0}, \chi_{\{3\}}, \chi_{\{1,2\}}, \chi_X = \underline{1}\right\}$$

Again, we do not have to check $\vee, \wedge$ if one of the group is $\underline{0}$ or $\underline{1}$ as per above. So it suffices to show that $\chi_{\{3\}} \vee \chi_{\{2,1\}}$ and $\chi_{\{3\}} \wedge \chi_{\{2,1\}}$ belong to $g_\chi(\mathcal{T})$:

$$\left(\chi_{\{3\}} \vee \chi_{\{1,2\}}\right)(1) = \left(\chi_{\{3\}}(1)\right) \vee \left(\chi_{\{1,2\}}(1)\right) = 0 \vee 1 = 1$$
$$\left(\chi_{\{3\}} \vee \chi_{\{1,2\}}\right)(2) = \left(\chi_{\{3\}}(2)\right) \vee \left(\chi_{\{1,2\}}(2)\right) = 0 \vee 1 = 1$$
$$\left(\chi_{\{3\}} \vee \chi_{\{1,2\}}\right)(3) = \left(\chi_{\{3\}}(3)\right) \vee \left(\chi_{\{1,2\}}(3)\right) = 1 \vee 0 = 1$$

So, $\forall y \in Y$, $\left(\chi_{\{3\}} \vee \chi_{\{1,2\}}\right)(y) = 1$. I.e., $\chi_{\{3\}} \vee \chi_{\{1,2\}} = \underline{1} \in g_\chi(\mathcal{S})$. Similarly, $\chi_{\{3\}} \wedge \chi_{\{1,2\}} = \underline{0} \in g_\chi(\mathcal{S})$. Then, $(Y, g_\chi(\mathcal{S}))$ is an $\{0,1\}$-topology on Y.

The last item to consider is that $g_\chi(f) = f$ is $\{0,1\}$-continuous from $(X, g_\chi(\mathcal{T}))$ to $(Y, g_\chi(\mathcal{S}))$. I.e., that $f_L^{\leftarrow}(u) \in g_\chi(\mathcal{T})$ for all $u \in g_\chi(\mathcal{S})$.

$$f_L^{\leftarrow}(\underline{0}) = \underline{0} \circ f = \underline{0} \in g_\chi(\mathcal{T})$$
$$f_L^{\leftarrow}(\underline{1}) = \underline{1} \circ f = \underline{1} \in g_\chi(\mathcal{T})$$

To examine the remaining L-preimages, consider $x = a$ and $x = b$.

$$\begin{aligned}
f_L^{\leftarrow}(\chi_{\{3\}})(a) &= \left(\chi_{\{3\}} \circ f\right)(a) = \chi_{\{3\}}(f(a)) = \chi_{\{3\}}(2) = 0\\
f_L^{\leftarrow}(\chi_{\{3\}})(b) &= \left(\chi_{\{3\}} \circ f\right)(b) = \chi_{\{3\}}(f(b)) = \chi_{\{3\}}(3) = 1\\
f_L^{\leftarrow}(\chi_{\{1,2\}})(a) &= \left(\chi_{\{1,2\}} \circ f\right)(a) = \chi_{\{1,2\}}(f(a)) = \chi_{\{1,2\}}(2) = 1\\
f_L^{\leftarrow}(\chi_{\{1,2\}})(b) &= \left(\chi_{\{1,2\}} \circ f\right)(b) = \chi_{\{1,2\}}(f(b)) = \chi_{\{1,2\}}(3) = 0
\end{aligned}$$

And so, $\forall x \in X$, $f_L^{\leftarrow}(\chi_{\{3\}})(x) = \chi_{\{b\}}(x)$. I.e., $f_L^{\leftarrow}(\chi_{\{3\}}) = \chi_{\{b\}} \in g_\chi(\mathcal{T})$. Similarly, $f_L^{\leftarrow}(\chi_{\{1,2\}}) = \chi_{\{a\}} \in g_\chi(\mathcal{T})$. And now, we have shown that $g_\chi(f) = f$ is $\{0,1\}$-continuous from $(X, g_\chi(\mathcal{T}))$ to $(Y, g_\chi(\mathcal{S}))$.

The characteristic map has a corresponding categorical functor, and there is also a similar lifting of the Martin map[16].

Definition 1.10. *Let the Martin functor M : **TOP** $\leftarrow$$L$-**TOP** be defined as follows:*

Objects: $\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$, let $M(X, \tau) = (X, m(\tau))$, where m is the Martin map.

Morphisms: $\forall f : (X, \tau) \to (Y, \sigma)$, let $M(f) = f$.

Proposition 1.11. *M : **TOP** $\leftarrow$$L$-**TOP** is a functor.*

Proof: For objects, let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, then

$$M(X, \tau) = (X, m(\tau)) = (X, \{U \in \mathfrak{P}(X) : \chi_U \in \tau\})$$

Now we must show that this is a traditional topology on X. By Proposition 5, $m(\tau) \in \mathbb{T}_X$. Hence, M preserves objects.

For morphisms, let $f : (X, \tau) \to (Y, \sigma)$ be L-continuous. Now we want to show that $M(f) = f$ is traditionally continuous, so we claim that $f : (X, m(\tau)) \to$

$(Y, m(\sigma))$ is traditionally continuous. Let $V \in m(\sigma) = \{U \in \mathfrak{P}(X) : \chi_U \in \sigma\}$ We need to have $f^{\leftarrow}(V) \in m(\tau) = \{U \in \mathfrak{P}(X) : \chi_U \in \tau\}$. Since $V \in m(\sigma)$, $\chi_V \in \sigma$. And, f is L-continuous, so $f_L^{\leftarrow}(\chi_V) \in \tau$. Let $x \in X$.

$$\begin{aligned}
\left(f_L^{\leftarrow}(\chi_V)\right)(x) &= (\chi_V \circ f)(x) \\
&= \chi_V(f(x)) \\
&= \{1, \text{ if } f(x) \in V \text{ , and } 0 \text{ if } f(x) \notin V\} \\
&= \{1, \text{ if } x \in f^{\leftarrow}(V) \text{ , and } 0 \text{ if } x \notin f^{\leftarrow}(V)\} \\
&= \chi_{f^{\leftarrow}(V)}(x)
\end{aligned}$$

Then, $\chi_{f^{\leftarrow}(V)}(x) = f_L^{\leftarrow}(\chi_V) \in \tau$. So $f^{\leftarrow}(V) \in \{U \in P(X) : \chi_U \in \tau\} = m(\tau)$. Hence f is traditionally continuous, and M preserves morphisms.

We have identities, composition, and associativity from the proof of Proposition 8. So, $M : \mathbf{TOP} \leftarrow L\text{-}\mathbf{TOP}$ is a functor ■.

As with the characteristic and Martin maps, these characteristic and Martin functors have an adjunctive relationship.

Proposition 1.12. *$M \dashv G_\chi$ as functors, I.e.,*

I. The following, with correct quantifiers and in order, hold:

(1) $\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$,

(2) there exists $\eta_{(X,\tau)} : (X, \tau) \to (G_\chi \circ M)(X, \tau)$ such that

(3) $\forall (Y, \mathcal{S}) \in |\mathbf{TOP}|$, and

(4) $\forall f : X \to Y$ where $f : (X, \tau) \to G_\chi(Y, \mathcal{S})$ is L-continuous,

(5) there is a unique traditionally continuous map $\overline{f} : M(X, \tau) \to (Y, \mathcal{S})$ such that

(6) $G_\chi(f) \circ \eta_{(X,\tau)} = f$.

II. The following implication holds:

If $\forall f : (X, \tau) \to (Y, \sigma) \in L\text{-}\mathbf{TOP}$, then $M(f) = (\overline{\eta_{(Y,\sigma)} \circ f})$, where $\eta_{(Y,\sigma)} : (Y, \sigma) \to G_\chi M(Y, \sigma)$ is given from I.

Proof: To prove I, let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, and put $\eta_{(X,\tau)} : (X, \tau) \to (G_\chi \circ M)(X, \tau) \equiv id_{(X,\tau)} = id_X$. To check that id_X is continuous from (X, τ) to $(G_\chi \circ M)(X, \tau) = (X, (g_\chi \circ m)(\tau))$, let $v \in (g_\chi \circ m)(\tau)$. Since

$$\begin{aligned}
(g_\chi \circ m)(\tau) &= g_\chi(m(\tau)) \\
&= \{\chi_V | V \in m(\tau)\}
\end{aligned}$$

then $\exists V \in m(\tau)$ such that $v = \chi_V$. But $m(\tau) = \{U \in P(X) | \chi_U \in \tau\}$, so $\chi_V = v \in \tau$. Then $(g_\chi \circ m)(\tau) \subset \tau$, and we proved before that id_X is L-continuous from (X, τ) to (X, τ), so certainly $\forall v \in (g_\chi \circ m)(\tau) \subset \tau, (id_X)_L^{\leftarrow}(v) \in \tau$. Hence, $id_X : (X, \tau) \to (G_\chi \circ M)(X, \tau)$ is L-continuous and a valid candidate for $\eta_{(X,\tau)}$.

Now let $(Y, \mathcal{S}) \in |\mathbf{TOP}|$, and let $f : X \to Y$ where $f : (X, \tau) \to G_\chi(Y, \mathcal{S})$ is L-continuous, i.e., $f : (X, \tau) \to (Y, g_\chi(\mathcal{S}))$ is L-continuous.

We also put $\overline{f} : M(X, \tau) \to (Y, \mathcal{S}) \equiv f$ and claim that this $\overline{f}$ will make the composition in (6) work. To show that $\overline{f}$ is in fact traditionally continuous from $M(X, \tau) = (X, m(\tau))$ to $(Y, \mathcal{S})$, let $V \in \mathcal{S}$.

$$(\overline{f})^{\leftarrow}(V) = f^{\leftarrow}(V)$$

Then we need to show that $f^{\leftarrow}(V) \in m(\tau) = \{U \in P(X) | \chi_U \in \tau\}$, in other words, that $\chi_{f^{\leftarrow}(V)} \in \tau$. Now, since $f : (X, \tau) \to G_\chi(Y, \mathcal{S}) = (Y, g_\chi(\mathcal{S}))$ is L-continuous by (4), $\forall v \in g_\chi(\mathcal{S})$, $f_L^{\leftarrow}(v) = (v \circ f) \in \tau$. But since $V \in \mathcal{S}$, $\chi_V \in g_\chi(\mathcal{S})$. And so $f_L^{\leftarrow}(\chi_V) \in \tau$. We claim that $\chi_{f^{\leftarrow}(V)} = f_L^{\leftarrow}(\chi_V)$. Let $x \in X$.

$$\begin{aligned}
\chi_{f^{\leftarrow}(V)}(x) &= \{1 \text{ if } x \in f^{\leftarrow}(V),\ 0 \text{ if } x \notin f^{\leftarrow}(V)\} \\
&= \{1 \text{ if } f(x) \in V,\ 0 \text{ if } f(x) \notin V\} \\
&= \chi_V\,(f(x)) \\
&= (\chi_V \circ f)\,(x) \\
&= (f_L^{\leftarrow}(\chi_V))\,(x)
\end{aligned}$$

and since x was chosen arbitrarily, $\chi_{f^{\leftarrow}(V)} = f_L^{\leftarrow}(\chi_V)$, therefore, $\overline{f} : (X, m(\tau)) \to (Y, \mathcal{S})$ is traditionally continuous.

Then we need to show (for existence of such a map) that (6) holds with this choice of $\overline{f}$, i.e., that

$$G_\chi(\overline{f}) \circ \eta_{(X,\tau)} = f$$

Note that $G_\chi(\overline{f}) \circ \eta_{(X,\tau)} = G_\chi(f) \circ id_X$. Now we must show that $G_\chi(f) \circ id_X$ and f have the same action, so let $x \in X$.

$$\begin{aligned}
(G_\chi(f) \circ id_X)\,(x) &= G_\chi(f)\,(id_X(x)) \\
&= (G_\chi(f))\,(x) \\
&= f(x).
\end{aligned}$$

So, these functions perform on x in the same way. To check that $G_\chi(f) \circ id_X$ and f agree in continuity, recall that we originally set up f so that $f : (X, \tau) \to$

$(Y, g_\chi(\mathcal{S}))$ is L-continuous. Now, we need to have $(G_\chi(f) \circ id_X)_L^{\leftarrow}(v) \in \tau$, $\forall v \in g_\chi(\mathcal{S})$. But,

$$(G_\chi(f) \circ id_X)_L^{\leftarrow}(v) = v \circ (G_\chi(f) \circ id_X)$$

So for all $x \in X$,

$$\begin{aligned}
\left((G_\chi(f) \circ id_X)_L^{\leftarrow}(v)\right)(x) &= (v \circ (G_\chi(f) \circ id_X))(x) \\
&= v\left((G_\chi(f) \circ id_X)(x)\right) \\
&= v\left((G_\chi(f))(id_X(x))\right) \\
&= v\left((G_\chi(f))(x)\right) \\
&= v(f(x)) \\
&= (v \circ f)(x)
\end{aligned}$$

Here, we have used the definitions of composition and id_X, as well as the fact that the characteristic functor G_χ fixes morphisms, to show that $(G_\chi(f) \circ id_X)_L^{\leftarrow}(v) = v \circ f$. Since $v \in g_\chi(\mathcal{S})$, $\exists V \in \mathcal{S}$ such that $v = \chi_V$. We have already shown that

$$\chi_V \circ f = f_L^{\leftarrow}(\chi_V)$$

and f is L-continuous from (X, τ) to $(X, g_\chi(\mathcal{S}))$, so since $f_L^{\leftarrow}(v) \in \tau$ and

$$f_L^{\leftarrow}(v) = (G_\chi(f) \circ id_X)_L^{\leftarrow}(v) = \left(G_\chi(\overline{f}) \circ \eta_{(X,\tau)}\right)_L^{\leftarrow}(v)$$

then $G_\chi(\overline{f}) \circ \eta_{(X,\tau)}$ has the appropriate continuity. Hence, $G_\chi(\overline{f}) \circ \eta_{(X,\tau)} = f$ in both the action and the continuity sense, so (6) holds.

To show that this choice for $\overline{f}$ is unique, suppose $g : (X, m(\tau)) \to (Y, S)$ is L-continuous and $G_\chi(g) \circ \eta_{(X,\tau)} = f$. The action of g and $\overline{f}$ will prove to be the same: Let $x \in X$.

$$\begin{aligned}
\overline{f}(x) &= f(x) \\
&= \left(G_\chi(g) \circ \eta_{(X,\tau)}\right)(x) \\
&= G_\chi(g)\left(\eta_{(X,\tau)}(x)\right) \\
&= G_\chi(g)(id_X(x)) \\
&= (G_\chi(g))(x)
\end{aligned}$$

and since $G_\chi(g) = g$ by the definition of the characteristic functor, $g = \overline{f}$.

To prove II, let $f : (X,\tau) \to (Y,\sigma)$ be L-continuous, and put $\eta_{(Y,\sigma)} : (Y,\sigma) \to G_\chi M(Y,\sigma) \equiv id_{(Y,\sigma)} = id_Y$. This choice for $\eta_{(Y,\sigma)}$ is legitimate via the identical proof of the validity of $\eta_{(X,\tau)}$ in part I. Now, we generate a unique $\overline{id_Y \circ f}$: $M(X,\tau) \to (Y, m(\sigma))$ from Part I, and show that $M(f) = \overline{id_Y \circ f}$. Since $id_Y \circ f$: $(X,\tau) \to (Y, g_\chi \circ m(\sigma))$, in order for (6) to hold we must have that the action of $\left(G_\chi(\overline{id_Y \circ f})\right) \circ id_X$ the same as the action of $id_Y \circ f$. Let $x \in X$. Now we need to have $\left(\left(G_\chi(\overline{id_Y \circ f})\right) \circ id_X\right)(x) = (id_Y \circ f)(x)$, which is just $f(x)$.

$$\begin{aligned} \left(\left(G_\chi(\overline{id_Y \circ f})\right) \circ id_X\right)(x) &= \left(G_\chi(\overline{id_Y \circ f})\right)(id_X(x)) \\ &= \left(G_\chi(\overline{id_Y \circ f})\right)(x) \end{aligned}$$

and since $G_\chi(\overline{id_Y \circ f}) = (\overline{id_Y \circ f})$ by the definition of the characteristic functor, we must have that $\overline{id_Y \circ f} \equiv id_Y \circ f = f$. But, recall that the Martin functor M also fixes morphisms, so $M(f) = f = id_Y \circ f \equiv \overline{id_Y \circ f}$ $\blacksquare$.

The interesting thing about this adjunction is that L-spaces are more general, since every traditional space has a corresponding "characteristic space" via G_χ so $|\mathbf{TOP}| \cong |\mathbf{2}\text{-}\mathbf{TOP}| \subset |L\text{-}\mathbf{TOP}|$. On the other hand, an L-topological space will be richer than the traditional space via M, since not all mappings are required to be a characteristic map.

Of course, it is limiting to only be able to produce fuzzy topological spaces where the lattice is crisp, i.e., $L = \mathbf{2}$. We can expand our lattice to be a complete chain in the next section.

2. The Kubiak/Lowen Mappings and Functors

We will again examine the fibres $\mathbb{T}_X$ and $\mathbb{F}_X$, now defining maps which are precursors to the functors described by R. Lowen[14]. Although we will uncover another "reversing adjoint" relationship on the mappings and functors, there is a twist on our findings from the characteristic and Martin situation.

2.1. The Kubiak/Lowen Maps on Fibres

Examining lower semicontinuous, chain-valued maps to produce order-preserving adjoint fibre maps was first attempted by M.D. Weiss in [25]. These maps were based on the functors produced by R. Lowen [14] for the case of $L = \mathbb{I}$, which was later generalized to the complete lattice case by T. Kubiak [13].

Definition 2.1. *Let X be a set, and $\mathbb{T}_X$, L^X as previously described, except now we will require L to be a complete chain with universal upper bound $\top$. Define the type I Kubiak/Lowen map $\omega_L^* : \mathbb{T}_X \to \mathfrak{P}(L^X)$ by $\forall \mathcal{T} \in \mathbb{T}_X$,*

$$\omega_L^*(\mathcal{T}) = \{u : X \to L \mid \forall a \in L,\ u^{\leftarrow}(a, \top] \in \mathcal{T}\}$$

We claim that this map is from the fibre of traditional topologies on X, $\mathbb{T}_X$, to the fibre of L-topologies on X, $\mathbb{F}_X$.

Lemma 2.2. *Let X be a set, L^X as previously described, $a \in L$, and $\{u_\gamma | \gamma \in \Gamma\} \subset L^X$. Then, the preimage of the join function is the union of the preimages and the preimage of the meet function is the intersection of the preimages:*

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a, \top] = \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a, \top]\right)$$

and

$$\left(\bigwedge_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a, \top] = \bigcap_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a, \top]\right)$$

Proof: First, recall that the join function $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right) : X \to L$ is defined $\forall x \in X$,

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(x) = \bigvee_{\gamma\in\Gamma}(u_\gamma(x))$$

Let $y \in \left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a, \top]$. By definition of the backward powerset operator,

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(y) \in (a, \top]$$

So then $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(y) > a$. by how the join function is defined above,

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(y) = \bigvee_{\gamma\in\Gamma}(u_\gamma(y)) > a$$

. Since L is a chain, we know that $\exists \gamma_0 \in \Gamma$ such that $u_{\gamma_0}(y) > a$. But then,

$$u_{\gamma_0}(y) \in (a, \top], \text{ i.e., } y \in u_{\gamma_0}^{\leftarrow}(a, \top]$$

Now, if y is an element of one of the preimages, it is an element of the union of the preimages: $y \in \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right)$. Hence,

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top] \subset \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right)$$

Next let $z \in \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right)$. Then $\exists\gamma_1 \in \Gamma$ such that $z \in u_{\gamma_1}^{\leftarrow}(a,\top]$. So then $u_{\gamma_1}(z) \in (a,\top]$, or, $u_{\gamma_1}(z) > a$. But the least upper bound of a set is greater than or equal to any member of that set: $\bigvee_{\gamma\in\Gamma}(u_\gamma(z)) \geq u_{\gamma_1}(z)$. Now by transitivity, $\bigvee_{\gamma\in\Gamma}(u_\gamma(z)) > a$. Recall from the definition of the join function that $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(z) = \bigvee_{\gamma\in\Gamma}(u_\gamma(z))$, so $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(z) > a$. And so, $z \in \left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top]$. Hence,

$$\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) \subset \left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top].$$

Because we have both directions of the inclusion, we have equality of the sets:

$$\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top] = \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right).$$

Now the meet function $\left(\bigwedge_{\gamma\in\Gamma}(u_\gamma)\right) : X \to L$ is defined $\forall x \in X$,

$$\left(\bigwedge_{\gamma\in\Gamma}(u_\gamma)\right)(x) = \bigwedge_{\gamma\in\Gamma}(u_\gamma(x))$$

Let $x \in \left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(a,\top]$. Then, $\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)(x) \in (a,\top]$ by definition of the backward powerset operator. So then,

$$\begin{aligned}\left\{\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)(x) \in (a,\top]\right\} &\Leftrightarrow \left\{\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)(x) > a\right\}\\ &\Leftrightarrow \left\{\bigwedge_{\delta\in\Delta}(u_\delta(x)) > a\right\}\end{aligned}$$

Since the greatest lower bound of the set $\{u_\delta(x), \delta \in \Delta\}$ is greater than a, a must be a lower bound of this set. So, $\forall\delta \in \Delta$, $u_\delta(x) > a$. Or, $\forall\delta \in \Delta$, $u_\delta(x) \in (a,\top]$.

Then we have $\forall\delta \in \Delta$, $x \in u_\delta^{\leftarrow}(a, \top]$. Since this holds for all sets $u_\delta^{\leftarrow}(a, \top]$, x is in the intersection: $x \in \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(a, \top])$. Hence,

$$\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(a, \top] \subset \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(a, \top])$$

Now, let $y \in \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(a, \top])$. Since y is in the intersection, we know that

$$\begin{aligned} \forall\delta &\in \Delta,\ y \in u_\delta^{\leftarrow}(a, \top] \\ &\text{or,} \\ \forall\delta &\in \Delta,\ u_\delta(y) \in (a, \top] \end{aligned}$$

Since $u_\delta(y) > a$ for all $\delta \in \Delta$, a is a lower bound of the set $\{u_\delta(y), \delta \in \Delta\}$. And, any lower bound is dominated by the meet:

$$\bigwedge_{\delta\in\Delta}(u_\delta(y)) > a$$

By definition of the meet function, $(\bigwedge_{\delta\in\Delta}(u_\delta))\,(y) > a$. And so, $(\bigwedge_{\delta\in\Delta}(u_\delta))\,(y) \in (a, \top]$, which can be interpreted as the preimage of $(a,\top]$: $y \in (\bigwedge_{\delta\in\Delta}(u_\delta))^{\leftarrow}(a, \top]$. Hence,

$$\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(a, \top] \supset \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(a, \top]) .$$

Because we have both directions of the inclusion, we have equality of the sets:

$$\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(a, \top] = \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(a, \top]) \quad \blacksquare.$$

Proposition 2.3. *Let X be a set, and the Kubiak/Lowen type I map $\omega_L^* : \mathbb{T}_X \to \mathfrak{P}(L^X)$ be as defined. The following properties hold:*
(i) $\forall\mathcal{T} \in \mathbb{T}_X$, $\omega_L^*(\mathcal{T}) \in \mathbb{F}_X$, i.e., $\omega_L^* : \mathbb{T}_X \to \mathbb{F}_X$ is a map.
(ii) $\omega_L^* : \mathbb{T}_X \to \mathbb{F}_X$ is isotone.

Proof of (i): Let $\mathcal{T} \in \mathbb{T}_X$. It suffices to check that $\omega_L^*(\mathcal{T})$ is closed under arbitrary joins and finite meets. Let $\{u_\gamma : \gamma \in \Gamma\} \subset \omega_L^*(\mathcal{T})$. For arbitrary joins, we need to show:
A) $\bigvee_{\gamma\in\Gamma}(u_\gamma)$ is a mapping from X to L.

B) $\bigvee_{\gamma\in\Gamma}(u_\gamma)$ is lower semicontinuous with respect to $\mathcal{T}$, i.e.,

$$\forall a \in L, \left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top] \in \mathcal{T}.$$

For A), let $x \in X$. Then, $\forall\gamma \in \Gamma, u_\gamma \in \omega_L^*(\mathcal{T})$, so each u_γ maps from X to L. Also, $u_\gamma(x) \in L$ for all $\gamma \in \Gamma$. So certainly $\bigvee_{\gamma\in\Gamma}(u_\gamma(x)) \in L$, and then by definition of the join function, $\left(\bigvee_{\gamma\in\Gamma} u_\gamma\right)(x) \in L$. Hence *cod* $\left(\bigvee_{\gamma\in\Gamma} u_\gamma\right) = L$. And, if $x \notin X$, $u_\gamma(x)$ is not defined for any $\gamma \in \Gamma$, so then $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)(x)$ is not defined when $x \notin X$. Hence *dom* $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right) = X$. I.e., $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right) : X \to L$

For B), let $a \in L$. By Lemma 14, $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top] = \bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right)$, and we will demonstrate that $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) \in \mathcal{T}$. Since $\{u_\gamma : X \to L, \gamma \in \Gamma\} \subset \omega_L^*(\mathcal{T})$, then $\forall\gamma \in \Gamma, u_\gamma \in \omega_L^*(\mathcal{T})$. This means that $\forall\gamma \in \Gamma$ and $\forall a \in L$, $u_\gamma^{\leftarrow}(a,\top] \in \mathcal{T}$. So then because $\mathcal{T}$ is closed under arbitrary unions, we have $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) \in \mathcal{T}$. Recall that $a \in L$ was chosen arbitrarily, so $\forall a \in L$, $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) \in \mathcal{T}$, or, $\left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top] \in \mathcal{T}$. Therefore, $\bigvee_{\gamma\in\Gamma}(u_\gamma) \in \omega_L^*(\mathcal{T})$. I.e., $\omega_L^*(\mathcal{T})$ is closed under arbitrary joins.

For finite meets, let $\{u_\delta : X \to L, \delta \in \Delta, \Delta \text{ finite}\} \subset \omega_L^*(\mathcal{T})$. We want to show that $\bigwedge_{\delta\in\Delta}(u_\delta) \in \omega_L^*(\mathcal{T})$, so show:

C) $\bigwedge_{\delta\in\Delta}(u_\delta)$ is a mapping from X to L .

D) $\bigwedge_{\delta\in\Delta}(u_\delta)$ is lower semicontinuous with respect to $\mathcal{T}$. I.e.,

$$\forall b \in L, \left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(b,\top] \in \mathcal{T}.$$

For C), we use a similar argument as that in part A). Since $\forall\delta \in \Delta$, $u_\delta \in \omega_L^*(\mathcal{T})$, each u_δ maps from X to L. Let $x \in X$. For all $\delta \in \Delta$, u_δ is defined and $u_\delta(x) \in L$. So certainly $\bigwedge_{\delta\in\Delta}(u_\delta(x)) \in L$, and by definition of the meet function, $(\bigwedge_{\delta\in\Delta}(u_\delta))(x) \in L$. Hence *cod* $(\bigwedge_{\delta\in\Delta}(u_\delta)) = L$. And, if $x \notin X$, $u_\delta(x)$ is not defined for any $\delta \in \Delta$, so then $(\bigwedge_{\delta\in\Delta}(u_\delta))(x)$ is not defined when $x \notin X$. Hence *dom* $(\bigwedge_{\delta\in\Delta}(u_\delta)) = X$. I.e., $(\bigwedge_{\delta\in\Delta}(u_\delta)) : X \to L$.

For D), let $b \in L$. $(\bigwedge_{\delta\in\Delta}(u_\delta))^{\leftarrow}(b,\top] = \bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(b,\top])$ by Lemma 14, and we will show that $\bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(b,\top]) \in \mathcal{T}$. Since $\{u_\delta : X \to L, \delta \in \Delta, \Delta \text{ finite}\} \subset \omega_L^*(\mathcal{T})$, then $\forall\delta \in \Delta$, $u_\delta \in \omega_L^*(\mathcal{T})$. This means $\forall b \in L, \forall\delta \in \Delta$, $u_\delta^{\leftarrow}(b,\top] \in$

$\mathcal{T}$. But $\mathcal{T}$ is closed under finite intersections, so $\bigcap_{\delta\in\Delta}(u_\delta^{\leftarrow}(b,\top]) \in \mathcal{T}$. Then, $(\bigwedge_{\delta\in\Delta}(u_\delta))^{\leftarrow}(b,\top] \in \mathcal{T}$. Recall that $b \in L$ was chosen arbitrarily, so $\forall b \in L$,

$$\left(\bigwedge_{\delta\in\Delta}(u_\delta)\right)^{\leftarrow}(b,\top] \in \mathcal{T}$$

Therefore, $\bigwedge_{\delta\in\Delta}(u_\delta) \in \omega_L^*(\mathcal{T})$. I.e., $\omega_L^*(\mathcal{T})$ is closed under finite meets. Because $\omega_L^*(\mathcal{T})$ is closed under arbitrary joins and finite meets, $\omega_L^*(\mathcal{T})$ is an L-topology on X. I.e., $\omega_L^*(\mathcal{T}) \in \mathbb{F}_X$.

Proof of (ii): To prove that $\omega_L^* : \mathbb{T}_X \to \mathbb{F}_X$ is isotone, let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}_X$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$. Let $u \in \omega_L^*(\mathcal{T}_1)$. Then, $\forall a \in L$, $u^{\leftarrow}(a,\top] \in \mathcal{T}_1$. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, we know $\forall a \in L$, $u^{\leftarrow}(a,\top] \in \mathcal{T}_2$. So,

$$\begin{aligned} u &\in \{v : X \to L \mid \forall a \in L,\ v^{\leftarrow}(a,\top] \in \mathcal{T}_2\} \\ &= \omega_L^*(\mathcal{T}_2). \end{aligned}$$

And since u was chosen arbitrarily, $\omega_L^*(\mathcal{T}_1) \subset \omega_L^*(\mathcal{T}_2)$. Hence, ω_L^* is isotone $\blacksquare$.

Definition 2.4. *Let X be a set, and $\mathbb{F}_X$, $\mathfrak{P}(X)$ as previously described. Define the type II Kubiak/Lowen map $\iota_L^* : \mathfrak{P}(\mathfrak{P}(X)) \leftarrow \mathbb{F}_X$ by $\forall \tau \in \mathbb{F}_X$,*

$$\iota_L^*(\tau) = \{u^{\leftarrow}(a,\top] : u \in \tau,\ a \in L\}$$

We claim that this map is from the fibre of L-topologies on X, $\mathbb{F}_X$, to the fibre of traditional topologies on X, $\mathbb{T}_X$.

Proposition 2.5. *Let X be a set, and the type II Kubiak/Lowen map $\iota_L^* : \mathfrak{P}(\mathfrak{P}(X)) \leftarrow \mathbb{F}_X$ be as defined. The following properties hold:*
(i) $\forall \tau \in \mathbb{F}_X$, $\iota_L^*(\tau) \in \mathbb{T}_X$, i.e., $\iota_L^* : \mathbb{T}_X \to \mathbb{F}_X$
(ii) $\iota_L^* : \mathbb{T}_X \to \mathbb{F}_X$ is isotone.

Proof of (i): Let $\tau \in \mathbb{F}_X$. To show $\iota_L^*(\tau) \in \mathbb{T}_X$ i.e., that $\iota_L^*(\tau)$ is a traditional topology on X, we must have $\iota_L^*(\tau)$ closed under arbitrary unions and finite intersections. For arbitrary unions, let $\left\{u_\gamma^{\leftarrow}(a,1] : u_\gamma \in \tau, \gamma \in \Gamma, a \in L\right\} \subset \iota_L^*(\tau)$. We need to show that $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) \in \iota_L^*(\tau)$. In other words, show that $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right)$ is the preimage $u^{\leftarrow}(a,\top]$ of some function $u \in \tau$. By Lemma 14, $\bigcup_{\gamma\in\Gamma}\left(u_\gamma^{\leftarrow}(a,\top]\right) = \left(\bigvee_{\gamma\in\Gamma}(u_\gamma)\right)^{\leftarrow}(a,\top]$, so show that $\bigvee_{\gamma\in\Gamma}(u_\gamma) \in \tau$. Since

$\forall \gamma \in \Gamma$, $u_\gamma^{\leftarrow}(a, \top] \in \iota_L^*(\tau)$, we know that $\forall \gamma \in \Gamma$, $u_\gamma \in \tau$.The fact that τ is an L-topology, and hence closed under arbitrary joins, means that $\left(\bigvee_{\gamma \in \Gamma}(u_\gamma)\right) \in \tau$. Then, $\left(\bigvee_{\gamma \in \Gamma}(u_\gamma)\right)^{\leftarrow}(a, \top] \in \iota_L^*(\tau)$, or, $\bigcup_{\gamma \in \Gamma}\left(u_\gamma^{\leftarrow}(a, \top]\right) \in \iota_L^*(\tau)$. So, $\iota_L^*(\tau)$ is closed under arbitrary unions.

For finite intersections, let $\{u_\delta^{\leftarrow}(a, \top] : u_\delta \in \tau, \delta \in \Delta,\ \Delta \text{ finite},\ a \in L\} \subset \iota_L^*(\tau)$. We need to show that $\bigcap_{\delta \in \Delta}(u_\delta^{\leftarrow}(a, \top]) \in \iota_L^*(\tau)$. In other words, show that

$$\bigcap_{\delta \in \Delta}(u_\delta^{\leftarrow}(a, \top])$$

is the preimage $u^{\leftarrow}(a, \top]$ of some function $u \in \tau$. By Lemma 14, $\bigcap_{\delta \in \Delta}(u_\delta^{\leftarrow}(a, \top]) = (\bigwedge_{\delta \in \Delta}(u_\delta))^{\leftarrow}(a, \top]$, so it suffices to show that $\bigwedge_{\delta \in \Delta}(u_\delta) \in \tau$. Since $\forall \delta \in \Delta$, $u_\delta^{\leftarrow}(a, \top] \in \iota_L^*(\tau)$, we know that $\forall \delta \in \Delta$, $u_\delta \in \tau$. But, τ is an L-topology, and hence closed under finite meets, so $(\bigwedge_{\delta \in \Delta}(u_\delta)) \in \tau$. Now, So, $\iota_L^*(\tau)$ is closed under finite intersections.

Since $\iota_L^*(\tau)$ is closed under arbitrary unions and finite intersections, it is a traditional topology on X, i.e., $\iota_L^*(\tau) \in \mathbb{T}_X$. Therefore, $\iota_L^* : \mathbb{T}_X \to \mathbb{F}_X$.

Proof of (ii): Let $\tau_1, \tau_2 \in \mathbb{F}_X$ such that $\tau_1 \subset \tau_2$. Now let $U \in \iota_L^*(\tau_1)$. Then, $\exists u \in \tau_1, a \in L$ such that $U = u^{\leftarrow}(a, \top]$. But since $\tau_1 \subset \tau_2$, $u \in \tau_2$. And so,

$$\begin{aligned} U &\in \{v^{\leftarrow}(b, \top] \mid v \in \tau,\ b \in L\} \\ &= \iota_L^*(\tau_2) \end{aligned}$$

which means $\iota_L^*(\tau_1) \subset \iota_L^*(\tau_2)$. Then, $\iota_L^* : \mathbb{T}_X \leftarrow \mathbb{F}_X$ is isotone ■.

Proposition 2.6. *$\iota_L^* \dashv \omega_L^*$ as isotone maps on fibres.*

Proof: For this adjunction, we need to show:
(i) $\forall \tau \in \mathbb{F}_X$, $(\omega_L^* \circ \iota_L^*)(\tau) \supset \tau$
(ii) $\forall \mathcal{T} \in \mathbb{T}_X$, $(\iota_L^* \circ \omega_L^*)(\mathcal{T}) \subset \mathcal{T}$.
For (i), let $\tau \in \mathbb{F}_X$, $u \in \tau$. Then, for all $a \in L$,

$$u^{\leftarrow}(a, \top] \in \{w^{\leftarrow}(b, \top] \mid b \in L, w \in \tau\} = \iota_L^*(\tau)$$

Then, u is a map from X to L such that $u^{\leftarrow}(a, \top] \in \iota_L^*(\tau)$, $\forall a \in L$:

$$u \in \{v : X \to L \mid \forall a \in L,\ v^{\leftarrow}(a, \top] \in \iota_L^*(\tau)\}$$

But, this is just the definition of $\omega_L^*(\iota_L^*(\tau))$, so since $\omega_L^*(\iota_L^*(\tau)) = (\omega_L^* \circ \iota_L^*)(\tau)$,

$$\tau \subset (\omega_L^* \circ \iota_L^*)(\tau).$$

For (ii), let $\mathcal{T} \in \mathbb{T}_X$, $U \in (\iota_L^* \circ \omega_L^*)(\mathcal{T})$. Then,

$$U \in \iota_L^*(\omega_L^*(\mathcal{T})) = \{u^{\leftarrow}(a, \top] \mid u \in \omega_L^*(\mathcal{T}), a \in L\}$$

So $\exists u_0 \in \omega_L^*(\mathcal{T})$ and $a_0 \in L$ such that $U = u_0^{\leftarrow}(a_0, \top]$. But since $u_0 \in \omega_L^*(\mathcal{T})$ and $a_0 \in L$, $u_0^{\leftarrow}(a_0, \top] \in T$ by how we define $\omega_L^*(\mathcal{T})$. Then,

$$(\iota_L^* \circ \omega_L^*)(\mathcal{T}) \subset \mathcal{T}$$

Therefore, $\iota_L^* \dashv \omega_L^*$ ■.

When this fibre map adjoint relationship is compared to the associated relationship in section 1.1, we see that instead of the "$\mathbb{T}_X$ to $\mathbb{F}_X$" map being *left*-adjoint to the "$\mathbb{F}_X$ to $\mathbb{T}_X$" map, as in the characteristic/Martin case, we have the "$\mathbb{T}_X$ to $\mathbb{F}_X$" map being *right*-adjoint to the "$\mathbb{F}_X$ to $\mathbb{T}_X$" map here in the Kubiak/Lowen case. It stands to reason that, as functors, the Kubiak/Lowen adjunctive relationship will also be the opposite direction.

2.2. The Kubiak/Lowen Functors on TOP and L-TOP

L is a complete chain.

Definition 2.7. *Define the functor ω_L : **TOP** $\to$ L-**TOP** as follows:*
*Objects:$\forall (X, \mathcal{T}) \in |$**TOP**$|$, put $\omega_L(X, \mathcal{T}) = (X, \omega_L(\mathcal{T}))$, where*

$$\omega_L(\mathcal{T}) = \omega_L^*(\mathcal{T}) = \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\}$$

Morphisms:$\forall f : (X, \mathcal{T}) \to (Y, \mathcal{S})$, let $\omega_L(f) = f$.

Proposition 2.8. *ω_L : **TOP** $\to$ L-**TOP** is a functor.*

Proof: For objects, let $(X, \mathcal{T}) \in |$**TOP**$|$, then

$$\omega_L(X, \mathcal{T}) = (X, \omega_L(\mathcal{T})) = (X, \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\})$$

So to have $\omega_L(X, \mathcal{T}) \in |L$-**TOP**$|$, we must show that $\omega_L(\mathcal{T})$ is an L-topology on X. But by Proposition 15, $\omega_L^*(\mathcal{T}) \in \mathbb{F}_X$. And since $\omega_L(\mathcal{T}) = \omega_L^*(\mathcal{T})$, $\omega_L(\mathcal{T}) \in \mathbb{F}_X$ as well. I.e., $\omega_L(\mathcal{T})$ preserves objects.

For morphisms, let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ such that f is traditionally continuous. Because we want to show that $\omega_L(f) = f$ is L-continuous, we want $f : (X, \omega_L(\mathcal{T})) \to (Y, \omega_L(\mathcal{S}))$ to be L-continuous. Let $v \in \omega_L(\mathcal{S}) = \{w \in$

$L^Y | \forall a \in L, w^{\leftarrow}(a, \top] \in \mathcal{S}\}$ We need to have $f_L^{\leftarrow}(v) \in \omega_L(\mathcal{T})$. Keep in mind that $f_L^{\leftarrow} : L^X \leftarrow L^Y$ by $\forall w \in L^Y$,

$$f_L^{\leftarrow}(w) = w \circ f$$

So then, $f_L^{\leftarrow}(v) = v \circ f$. It suffices to show that $v \circ f$ can be written as a map $u \in L^X$ such that $\forall a \in L$, $u^{\leftarrow}(a, \top] \in \mathcal{T}$. Now, $f : X \to Y$ and $v : Y \to L$, so certainly $(v \circ f) : X \to L$, i.e., $(v \circ f) \in L^X$. To check that $f_L^{\leftarrow}(v) = (v \circ f)$ is lower semicontinuous, let $a \in L$. We want the preimage of $f_L^{\leftarrow}(v) = v \circ f$ over $(a, \top]$ to be a member of $\mathcal{T}$.

$$\begin{aligned}(f_L^{\leftarrow}(v))^{\leftarrow}(a, \top] &= (v \circ f)^{\leftarrow}(a, \top] \\ &= (f^{\leftarrow} \circ v^{\leftarrow})(a, \top] \\ &= f^{\leftarrow}(v^{\leftarrow}(a, \top])\end{aligned}$$

But, since $v \in \omega_L(\mathcal{S})$, $v^{\leftarrow}(a, \top] \in \mathcal{S}$. And f is traditionally continuous, so $f^{\leftarrow}(v^{\leftarrow}(a, \top]) \in \mathcal{T}$. We chose $a \in L$ arbitrarily, so $\forall a \in L$, $(f_L^{\leftarrow}(v))^{\leftarrow}(a, \top] \in \mathcal{T}$. So, since $v \in \omega_L(\mathcal{S})$ was arbitrary, we know

$$\forall v \in \omega_L(\mathcal{S}),\ f_L^{\leftarrow}(v) \in \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\} = \omega_L(\mathcal{T})$$

Hence $f : (X, \omega_L(\mathcal{T})) \to (Y, \omega_L(\mathcal{S}))$ is L-continuous and so ω_L preserves morphisms.

At this point, it is not necessary to check that identities, composition, and associativity are preserved, since this was accomplished in Proposition 8; again, our functor does not affect the action of morphisms. So, ω_L : **TOP** $\to$ L-**TOP** is a functor ■.

Example 2.9. *We will examine how the Kubiak/Lowen functor ω_L affects the spaces $(X, \mathcal{T}), (Y, \mathcal{S})$ and the map f introduced in the previous example. We claim that $\omega_L(\mathcal{T})$ is an L-topology on X.*

$$\begin{aligned}\omega_L(\mathcal{T}) &= \{u \in L^X | \forall l \in L, u^{\leftarrow}(l, \top] \in \mathcal{T}\} \\ &= \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \subset X\} \\ &= L^X\end{aligned}$$

which is the discrete L-topology on X.

We also claim that $\omega_L(\mathcal{S})$ is an L-topology on Y.

$$\omega_L(\mathcal{S}) = \{u \in L^Y | \forall l \in L, u^{\leftarrow}(l, \top] \in \mathcal{S}\}$$
$$= \{u \in L^Y | \forall a \in L, u^{\leftarrow}(a, \top] \notin \{\{1\}, \{2\}, \{2,3\}, \{1,3\}\}$$

It is not possible to explicitly define all the mappings allowed, but we note that one map not included in $\omega_L(\mathcal{S})$ would be:

$$v : Y \to L \text{ by}$$
$$v(1) = \bot,\ v(2) = l_1 \text{ for some lattice value, and } v(3) = \top$$

This map v cannot be included because for $l = l_2 < l_1$,

$$v^{\leftarrow}(l_2, \top] = \{y \in Y | v(y) \in (l_2, \top]\}$$
$$= \{y \in Y | v(y) > l_2\}$$
$$= \{2, 3\} \notin \mathcal{S}$$

So both the L-topological space $(Y, \omega_L(\mathcal{S}))$ and the L-continuity of f must be discussed strictly in terms of the definitions.

As before with the Martin functor, we can transform an L-topological space into a classical traditional space by defining a categorical functor.

Definition 2.10. *Define the functor ι_L : **TOP** $\leftarrow$ **L-TOP** as follows:*
Objects:$\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$, put $\iota_L(X, \tau) = (X, \iota_L(\tau))$, where

$$\iota_L(\tau) = \iota_L^*(\tau) = \{u^{\leftarrow}(a, \top] : u \in \tau,\ a \in L\}$$

Morphisms:$\forall f : (X, \tau) \to (Y, \sigma)$, let $\iota_L(f) = f$.

Proposition 2.11. *ι_L : **TOP** $\leftarrow$ **L-TOP** is a functor.*

Proof: For objects, let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, then

$$\iota_L(X, \tau) = (X, \iota_L(\tau)) = (X, \{u^{\leftarrow}(a, \top] | a \in L, u \in \tau\})$$

So to have $\iota(X, \tau) \in |\mathbf{TOP}|$, we must show that $\iota_L(\tau)$ is a traditional topology on X. But by Proposition 17, $\iota_L^*(\tau) \in \mathbb{T}_X$. And since $\iota_L(\tau) = \iota_L^*(\tau)$, $\iota_L(\tau) \in \mathbb{T}_X$ as well. I.e., ι_L preserves objects.

For morphisms, let $f : (X, \tau) \to (Y, \sigma)$ such that f is L-continuous. Because we want to show that $\iota_L(f) = f$ is traditionally continuous, we want $f : (X, \iota_L(\tau)) \to (Y, \iota_L(\sigma))$ to be L-continuous. Let $V \in \iota_L(\sigma) = \{u^{\leftarrow}(a, \top] | a \in L, u \in \sigma\}$. Then $\exists v \in \sigma$, $\exists b \in L$ such that $V = v^{\leftarrow}(b, \top]$. We need to have $f^{\leftarrow}(V) \in \iota_L(\tau)$. But,

$$\begin{aligned} f^{\leftarrow}(V) &= f^{\leftarrow}(v^{\leftarrow}(b, \top]) \\ &= (f^{\leftarrow} \circ v^{\leftarrow})(b, \top] \\ &= (v \circ f)^{\leftarrow}(b, \top] \end{aligned}$$

It is reasonable to have this composition since the domains and codomains agree: $f : X \to Y$ and $v : Y \to L$ since $v \in \sigma$. Beyond this, we know that $v \circ f = f_L^{\leftarrow}(v)$. And f is L-continuous, so $f_L^{\leftarrow}(v) \in \tau$. Then, for any $V \in \iota_L(\sigma)$, we can find $u \in \tau$ and $a \in L$ so that $f^{\leftarrow}(V)$ is the preimage of u under $(a, \top]$:

$$\begin{aligned} f^{\leftarrow}(V) &= (f_L^{\leftarrow}(v))^{\leftarrow}(b, \top] \\ &\in \{u^{\leftarrow}(a, \top] | a \in L, u \in \tau\} = \iota_L(\tau) \end{aligned}$$

And hence, $f^{\leftarrow}(V) \in \iota_L(\tau)$. So $f : (X, \iota_L(\tau)) \to (Y, \iota_L(\sigma))$ is traditionally continuous. Again, our functor does not affect the action of morphisms. So, as in Proposition 8, we inherit from SET the identities and morphism composition and associativity needed. Therefore, ι_L : **TOP** $\leftarrow$ L-**TOP** is a functor ■.

Proposition 2.12. *$\omega_L \dashv \iota_L$ as functors, I.e.,*

I. The following, with correct quantifiers and order, hold:

(1) $\forall (X, \mathcal{T}) \in |\mathbf{TOP}|$,

(2) there exists $\eta_{(X,\mathcal{T})} : (X, \mathcal{T}) \to (\iota_L \circ \omega_L)(X, \mathcal{T})$ such that

(3) $\forall (Y, \sigma) \in |L\text{-}\mathbf{TOP}|$, and

(4) $\forall f : X \to Y$ where $f : (X, \mathcal{T}) \to \iota_L(Y, \sigma)$ is traditionally continuous,

(5) there is a unique L-continuous map $\overline{f} : \omega_L(X, \mathcal{T}) \to (Y, \sigma)$ such that

(6) $\iota_L(\overline{f}) \circ \eta_{(X,\mathcal{T})} = f$.

II. The following implication holds:

If $\forall f : (X, \mathcal{T}) \to (Y, \mathcal{S}) \in \mathbf{TOP}(X, Y)$, then $\omega_L(f) = (\overline{\eta_{(Y,\mathcal{S})} \circ f})$, where $\eta_{(Y,\mathcal{S})} : (Y, \mathcal{S}) \to (\iota_L \circ \omega_L)(Y, \mathcal{S})$ is given from Part I.

Proof: To prove I, let $(X, \mathcal{T}) \in |\mathbf{TOP}|$, and put $\eta_{(X,\mathcal{T})} : (X, \mathcal{T}) \to (\iota_L \circ \omega_L)(X, \mathcal{T}) \equiv id_{(X,\mathcal{T})} = id_X$. To check that id_X is continuous from $(X, \mathcal{T})$ to $(\iota_L \circ \omega_L)(X, \mathcal{T}) = (X, (\iota_L \circ \omega_L)(\mathcal{T}))$, let $V \in (\iota_L \circ \omega_L)(\mathcal{T})$. Since

$$\begin{aligned} (\iota_L \circ \omega_L)(\mathcal{T}) &= \iota_L(\omega_L(\mathcal{T})) \\ &= \{u^{\leftarrow}(a, \top] | a \in L, u \in (\omega_L(\mathcal{T}))\} \end{aligned}$$

then $\exists v \in \omega_L(\mathcal{T}), \exists b \in L$ such that $V = v^{\leftarrow}(b, \top]$. But $\omega_L(\mathcal{T}) = \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\}$, so $V \in \mathcal{T}$. Then $(\iota_L \circ \omega_L)(\mathcal{T}) \subset \mathcal{T}$, and we proved before that id_X is continuous from $(X, \mathcal{T})$ to $(X, \mathcal{T})$, so certainly $\forall V \in (\iota_L \circ \omega_L)(\mathcal{T}) \subset \mathcal{T}$, $(id_X)^{\leftarrow}(V) \in \mathcal{T}$. Hence, $id_X : (X, \mathcal{T}) \to (\iota_L \circ \omega_L)(X, \mathcal{T})$ is continuous and a valid candidate for $\eta_{(X,\mathcal{T})}$.

Now let $(Y, \sigma) \in |L\text{-}\mathbf{TOP}|$, and let $f : X \to Y$ where $f : (X, \mathcal{T}) \to \iota_L(Y, \sigma)$ is continuous, i.e., $f : (X, \mathcal{T}) \to (Y, \iota_L(\sigma))$ is continuous.

We also put $\overline{f} : \omega_L(X, \mathcal{T}) \to (Y, \sigma) \equiv f$ and claim that this $\overline{f}$ will make the composition in (6) work. To show that $\overline{f}$ is in fact L-continuous from $\omega_L(X, \mathcal{T}) = (X, \omega_L(\mathcal{T}))$ to (Y, σ), let $v \in \sigma$.

$$(\overline{f})_L^{\leftarrow}(v) = f_L^{\leftarrow}(v) = v \circ f$$

Then we need to show that $v \circ f \in \omega_L(\mathcal{T}) = \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\}$, in other words, that $(v \circ f) : X \to L$ such that $\forall a \in L$, $(v \circ f)^{\leftarrow}(a, \top] \in \mathcal{T}$. Now, since $f : (X, \mathcal{T}) \to (Y, \iota_L(\sigma))$ is continuous by (4), $\forall V \in \iota_L(\sigma)$, $f^{\leftarrow}(V) \in \mathcal{T}$. But, $v \in \sigma$, so $\forall a \in L$, $v^{\leftarrow}(a, \top] \in \iota_L(\sigma)$. So for our v, we can say that $\forall a \in L$, $f^{\leftarrow}(v^{\leftarrow}(a, \top]) \in \mathcal{T}$.

$$\begin{aligned} f^{\leftarrow}(v^{\leftarrow}(a, \top]) &= (f^{\leftarrow} \circ v^{\leftarrow})(a, \top] \\ &= (v \circ f)^{\leftarrow}(a, \top] \end{aligned}$$

So, $\forall a \in L$, $(v \circ f)^{\leftarrow}(a, \top] \in \mathcal{T}$. And then, $(v \circ f) \in \{u \in L^X | \forall a \in L, u^{\leftarrow}(a, \top] \in \mathcal{T}\}$. Or,

$$(v \circ f) = (\overline{f})_L^{\leftarrow}(v) \in \omega_L(\mathcal{T})$$

Hence, $\overline{f}$ has the appropriate continuity.

Then we need to show (for existence of such a map) that (6) holds with this choice of $\overline{f}$, i.e., that

$$\iota_L(\overline{f}) \circ \eta_{(X,\mathcal{T})} = f$$

Note that $\iota_L(\overline{f}) \circ \eta_{(X,\mathcal{T})} = \iota_L(f) \circ id_X$. Now we must show that $\iota_L(f) \circ id_X$ and f have the same action, so let $x \in X$.

$$\begin{aligned} (\iota_L(f) \circ id_X)(x) &= \iota_L(f)(id_X(x)) \\ &= (\iota_L(f))(x) \\ &= f(x). \end{aligned}$$

So, these functions perform on x in the same way. To check that $\iota_L(f) \circ id_X$ and f agree in continuity, recall that we originally set up f so that $f : (X, \mathcal{T}) \to$

$(Y, \iota_L(\sigma))$ is continuous. Now, we need to have $(\iota_L(f) \circ id_X)^{\leftarrow}(V) \in \mathcal{T}$, $\forall V \in \iota_L(\sigma)$. But, $\exists a \in L, \exists v \in \sigma$ such that $V = v^{\leftarrow}(a, \top]$.

$$\begin{aligned}(\iota_L(f) \circ id_X)^{\leftarrow}(V) &= f^{\leftarrow}(V)\\ &= f^{\leftarrow}(v^{\leftarrow}(a, \top])\\ &= (f^{\leftarrow} \circ v^{\leftarrow})(a, \top]\\ &= (f_L^{\leftarrow}(v))^{\leftarrow}(a, \top]\end{aligned}$$

So since $f = \overline{f}$, and we originally had that $\overline{f} : \omega_L(X, \mathcal{T}) \to (Y, \sigma)$ was L-continuous, and $v \in \sigma$, then

$$(\overline{f})_L^{\leftarrow}(v) = f_L^{\leftarrow}(v) \in \omega_L(\mathcal{T})$$

in other words, $f_L^{\leftarrow}(v)$ is a map u in L^X such that $\forall a \in L$, $u^{\leftarrow}(a, \top] \in \mathcal{T}$. Hence

$$(\iota_L(f) \circ id_X)^{\leftarrow}(V) = (f_L^{\leftarrow}(v))^{\leftarrow}(a, \top] \in \mathcal{T}$$

and so our $\overline{f}$ has the appropriate continuity. And, $\iota_L(\overline{f}) \circ \eta_{(X,\mathcal{T})} = f$ in both the action and the continuity sense, so (6) holds.

To show that this choice for $\overline{f}$ is unique, suppose $g : \omega_L(X, \mathcal{T}) \to (Y, \sigma)$ continuous and makes $\iota_L(g) \circ \eta_{(X,\mathcal{T})} = f$. The action of g and $\overline{f}$ will prove to be the same: Let $x \in X$.

$$\begin{aligned}\overline{f}(x) &= f(x)\\ &= \left(\iota_L(g) \circ \eta_{(X,\mathcal{T})}\right)(x)\\ &= \iota_L(g)\left(\eta_{(X,\mathcal{T})}(x)\right)\\ &= \iota_L(g)\left(id_X(x)\right)\\ &= \left(\iota_L(g)\right)(x)\end{aligned}$$

and since $\iota_L(g) = g$ by the definition of the functor, $g = \overline{f}$.

To prove II, let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be continuous, and put $\eta_{(Y,\mathcal{S})} : (Y, \mathcal{S}) \to (\iota_L \circ \omega_L)(Y, \mathcal{S}) \equiv id_{(Y,\mathcal{S})} = id_Y$. This choice for $\eta_{(Y,\mathcal{S})}$ is legitimate via the identical proof of the validity of $\eta_{(X,\mathcal{T})}$ in part I. Now, we generate a unique $\overline{id_Y \circ f} : \omega_L(X, \mathcal{T}) \to (Y, \omega_L(\mathcal{S}))$ via Part I, and show that $\omega_L(f) = \overline{id_Y \circ f}$. Since $id_Y \circ f : (X, \mathcal{T}) \to (Y, (\iota_L \circ \omega_L)(\mathcal{S}))$, in order for the appropriate composition to work, we must have that the action of $\left(\iota_L(\overline{id_Y \circ f})\right) \circ id_X$ is the same as the action of $id_Y \circ f$.

Let $x \in X$. Now we need to have $\left(\left(\iota_L(\overline{id_Y \circ f})\right) \circ id_X\right)(x) = (id_Y \circ f)(x)$, which is just $f(x)$.

$$\begin{aligned} \left(\left(\iota_L(\overline{id_Y \circ f})\right) \circ id_X\right)(x) &= \left(\iota_L(\overline{id_Y \circ f})\right)(id_X(x)) \\ &= \left(\iota_L(\overline{id_Y \circ f})\right)(x) \end{aligned}$$

and since $\iota_L(\overline{id_Y \circ f}) = (\overline{id_Y \circ f})$ by the definition of the functor, we must have that $\overline{id_Y \circ f} \equiv id_Y \circ f = f$. But, recall that the functor ω_L also fixes morphisms, so $\omega_L(f) = f = id_Y \circ f \equiv \overline{id_Y \circ f}$ ■.

Using ω_L to generate L-topological spaces greatly increases the number of spaces we can have. But we are still restricted by the requirement that L be a complete chain. One interesting note is that neither of the two previous methods of generation accounts for all possible L-topological spaces, and they do not duplicate each other.

Part II

Examples Generated by Maps Between Semiframes as Points

3. Classical Stone Representations

Terminology and fundamentals regarding sobriety and ideas underlying the Stone representation theorems that are used in this section date back to the early 1980s work of Johnstone [12], Höhle [9], and Rodabaugh [20]. More recent work on the subject comes from [19], [24].

3.1. Introduction of points as mappings

To build the notion of points as mappings, let $(X, \mathcal{T}) \in |\mathbf{TOP}|$, $x \in X$. We now consider $\mathcal{T}_{\{x\}}$, the subspace topology on $\{x\}$. By definition, $\mathcal{T}_{\{x\}} = \{U \cap \{x\} : U \in \mathcal{T}\}$. But this particular subspace topology will be crisp: since $\mathcal{T}$ is a topology on X, $\emptyset \in \mathcal{T}$, so $\emptyset \cap \{x\} = \emptyset \in \mathcal{T}_{\{x\}}$. And, $X \in \mathcal{T}$, so $X \cap \{x\} = \{x\} \in \mathcal{T}_{\{x\}}$. For any other $U \in \mathcal{T}$, if $x \in U$, $U \cap \{x\} = \{x\}$, and if $x \notin U, U \cap \{x\} = \emptyset$. Hence, $\mathcal{T}_{\{x\}} = \{\emptyset, \{x\}\}$.

We can also note that $\mathcal{T}_{\{x\}}$ is lattice-isomorphic to $\{0,1\}$: Define $\varphi : \mathcal{T}_{\{x\}} \to \{0,1\}$ by $\varphi(\emptyset) = 0$ and $\varphi(\{x\}) = 1$.

For our x, the inclusion map $\hookrightarrow_x : \{x\} \to X$ is naturally continuous. I.e.,

$$\hookrightarrow_x : (\{x\}, \mathcal{T}_{\{x\}}) \to (X, \mathcal{T}) \text{ is traditionally continuous.}$$

Inspection of the preimage of this map will lead to our "point-making" morphism.

Note that $\hookrightarrow_x^{\leftarrow} : \mathfrak{P}(\{x\}) \leftarrow \mathfrak{P}(X)$, where, $\forall B \in \mathfrak{P}(X)$,

$$\begin{aligned} \left(\hookrightarrow_x^{\leftarrow}\right)(B) &= \{x\}, \text{ if } x \in B \\ &= \emptyset, \text{ if } x \notin B \end{aligned}$$

Definition 3.1. *The point function $p_x : \mathcal{T} \to \{0,1\}$ is defined $p_x = \varphi \circ \hookrightarrow_x^{\leftarrow}$.*

We can also see that for each $x \in X$, our point function p_x is unique: If $x = y$, then $\{x\} = \{y\}$, and so $\hookrightarrow_x = \hookrightarrow_y$, which implies that $\varphi \circ \hookrightarrow_x = \varphi \circ \hookrightarrow_y$, or, $p_x = p_y$.

Also, the action of the point function is characteristic: $\forall U \in \mathcal{T}$,

$$\begin{aligned} p_x(U) &= \left(\varphi \circ \hookrightarrow_x^{\leftarrow}\right)(U) \\ &= \varphi\left(\hookrightarrow_x^{\leftarrow}(U)\right) \\ &= \varphi\left(\{x\}, \text{ if } x \in U; \text{ or } \emptyset, \text{ if } x \notin U\right) \\ &= \varphi(\{x\}), \text{ if } x \in U; \text{ or } \varphi(\emptyset), \text{ if } x \notin U \\ &= 1, \text{ if } x \in U; \text{ or } 0, \text{ if } x \notin U \\ &= \chi_U(x) \end{aligned}$$

Definition 3.2. *Category **SFRM** consists of objects that are complete lattices, morphisms that preserve arbitrary joins and finite meets, and where composition is the same as in category **SET**.*

We will view the topology $\mathcal{T}$ as a complete lattice, where the order is containment, in order to show that these point maps are indeed semiframe morphisms.

Proposition 3.3. *$p_x : \mathcal{T} \to \{0,1\}$ is a semiframe morphism.*

Proof: First, we claim that both $\mathcal{T}$ and $\{0,1\}$ are complete lattices. $\mathcal{T}$ is clearly closed under arbitrary unions, hence is a complete join semilattice, hence a complete lattice. $\{0,1\}$ contains all joins and meets, hence is a complete lattice.

To show that p_x preserves arbitrary joins, let $\{U_\gamma : \gamma \in \Gamma\} \subset \mathcal{T}$. We want to show that

$$p_x\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) = \bigvee_{\gamma\in\Gamma} (p_x(U_\gamma))$$

$\mathcal{T}$ is closed under arbitrary unions, so $\bigcup_{\gamma\in\Gamma} U_\gamma \in \mathcal{T}$. Then, we consider two cases:

Case 1: $x \in \bigcup_{\gamma\in\Gamma} U_\gamma$

First,

$$\begin{aligned}
p_x\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) &= \varphi\circ \hookrightarrow_x^{\leftarrow} \left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) \\
&= \varphi\left(\hookrightarrow_x^{\leftarrow}\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right)\right) \\
&= \varphi(\{x\}) \\
&= 1
\end{aligned}$$

by our definitions of the point function, composition, the $\hookrightarrow_x^{\leftarrow}$ function, and the φ isomorphism.

Second, since $x \in \bigcup_{\gamma\in\Gamma} U_\gamma$, $\exists \gamma_0 \in \Gamma$ such that $x \in U_{\gamma_0}$. Then,

$$\begin{aligned}
p_x(U_{\gamma_0}) &= \varphi\circ \hookrightarrow_x^{\leftarrow} (U_{\gamma_0}) \\
&= \varphi(\hookrightarrow_x^{\leftarrow} (U_{\gamma_0})) \\
&= \varphi(\{x\}) \\
&= 1
\end{aligned}$$

Now, $1 \in \{p_x(U_\gamma) : \gamma \in \Gamma\}$. And, since the codomain of $p_x = \{0,1\}$, $1 = u.b.\,\{p_x(U_\gamma) : \gamma \in \Gamma\}$. Hence,

$$\bigvee_{\gamma\in\Gamma} p_x(U_\gamma) = 1$$

Then, by transitivity, $p_x\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) = \bigvee_{\gamma\in\Gamma} (p_x(U_\gamma))$. In other words, p_x is closed under arbitrary joins.

Case 2: $x \notin \bigcup_{\gamma\in\Gamma} U_\gamma$

First,

$$p_x\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) = \varphi\circ \hookrightarrow_x^{\leftarrow} \left(\bigcup_{\gamma\in\Gamma} U_\gamma\right)$$

$$\begin{aligned} &= \varphi\left(\hookrightarrow_x^{\leftarrow}\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right)\right) \\ &= \varphi(\emptyset) \\ &= 0 \end{aligned}$$

by our definitions of the point function, composition, the $\hookrightarrow_x^{\leftarrow}$ function, and the φ isomorphism.

Second, since $x \notin \bigcup_{\gamma\in\Gamma} U_\gamma$, we know that $\forall\gamma \in \Gamma$, $x \notin U_\gamma$. Then, $\forall\gamma \in \Gamma$,

$$\begin{aligned} p_x(U_\gamma) &= \varphi\circ \hookrightarrow_x^{\leftarrow} (U_\gamma) \\ &= \varphi(\hookrightarrow_x^{\leftarrow} (U_\gamma)) \\ &= \varphi(\emptyset) \\ &= 0 \end{aligned}$$

Now, $0 = u.b. \{p_x(U_\gamma) : \gamma \in \Gamma\} \geq \bigvee_{\gamma\in\Gamma} p_x(U_\gamma)$. And, since the codomain of $p_x = \{0, 1\}$, $\bigvee_{\gamma\in\Gamma} p_x(U_\gamma) \geq 0$. Hence,

$$\bigvee_{\gamma\in\Gamma} p_x(U_\gamma) = 0$$

And, by transitivity, $p_x\left(\bigcup_{\gamma\in\Gamma} U_\gamma\right) = \bigvee_{\gamma\in\Gamma} (p_x(U_\gamma))$. In other words, p_x is closed under arbitrary joins.

By separation of cases, p_x is closed under arbitrary joins.

Now let $\{U_\delta : \delta \in \Delta, \Delta \text{ finite}\} \subset \mathcal{T}$. We want to show that

$$p_x\left(\bigcap_{\delta\in\Delta} U_\delta\right) = \bigwedge_{\delta\in\Delta} (p_x(U_\delta))$$

First, since T is closed under finite meets, $\bigcap_{\delta\in\Delta}(U_\delta) \in T$. Again, consider the two cases.

Case 1: $x \in \bigcap_{\delta\in\Delta} U_\delta$.

Then, similarly to Case 1 of the first part of the proof, $p_x\left(\bigcap_{\delta\in\Delta} U_\delta\right) = 1$. And, $\forall\delta \in \Delta, x \in U_\delta$, which means $p_x(U_\delta) = 1$. So $\forall\delta \in \Delta$, $p_x(U_\delta) \geq 1$, hence 1 is a lower bound. Since all lower bounds of a set are dominated by its meet, $\bigwedge_{\delta\in\Delta} (p_x(U_\delta)) \geq 1$. And because the range of p_x is {0,1}, $\bigwedge_{\delta\in\Delta} (p_x(U_\delta)) \leq 1$, hence $\bigwedge_{\delta\in\Delta} (p_x(U_\delta)) = 1$. So

$$p_x\left(\bigcap_{\delta\in\Delta} U_\delta\right) = \bigwedge_{\delta\in\Delta} (p_x(U_\delta))$$

Case 2: $x \notin \bigcap_{\delta \in \Delta} U_\delta$.

Again, $p_x\left(\bigcap_{\delta\in\Delta} U_\delta\right) = 0$. And, $\forall \delta \in \Delta, x \notin U_\delta$, which means $p_x(U_\delta) = 0$. So $0 \in \{p_x(U_\delta) : \delta \in \Delta\}$ and $0 = l.b.\{p_x(U_\delta) : \delta \in \Delta\}$ which means $0 = \bigwedge \{p_x(U_\delta) : \delta \in \Delta\}$. Hence, $p_x\left(\bigcap_{\delta\in\Delta} U_\delta\right) = \bigwedge_{\delta\in\Delta}(p_x(U_\delta))$. So,

$$p_x \in \mathbf{SFRM}(\mathcal{T}, \{0,1\})$$

i.e., $p_x : \mathcal{T} \to \{0,1\}$ is a semiframe morphism ■.

The collection of point maps associated with the topological space $(X, \mathcal{T})$ is identified as a group by the topology $\mathcal{T}$, and the individual maps are each identified by a member of X. We will call this collection the "points of $\mathcal{T}$".

Definition 3.4. *Let $A \in |\mathbf{SFRM}|$. The points of A, $Pt(A)$, is the collection of semiframe mappings from A into $\{0,1\}$:*

$$\begin{aligned} Pt(A) &= \mathbf{SFRM}(A, \{0,1\}) \\ &= \left\{p : A \to \{0,1\} \mid p \text{ preserves } \bigvee, \wedge\right\} \end{aligned}$$

3.2. Sobriety, The Stone Comparison Operator Ψ, and Separation

Overlying these point functions of a topological space $(X, \mathcal{T})$ is a morphism $\Psi : X \to \mathbf{SFRM}(\mathcal{T}, \{0,1\})$ which associates each point to its point function.

Definition 3.5. *The Stone Comparison operator Ψ maps a set to the points of its topology:*

$$(X, \mathcal{T}) \in |\mathbf{TOP}|, \ \Psi : X \to Pt(\mathcal{T}) \quad \text{by } \Psi(x) = p_x$$

The question of whether or not the points of a topology is "equivalent" to the set upon which the topology acts determines the sobriety of a space.

Definition 3.6. *$(X, \mathcal{T})$ is sober iff $\Psi : X \to Pt(\mathcal{T})$ is bijective. Alternatively, $(X, \mathcal{T})$ is sober iff every irreducible closed set F in X is the closure of a unique singleton:*

$$\exists x \in X, F = \overline{\{x\}}$$

We say that F is irreducible closed if it cannot be written as a non-trivial union of nonempty closed sets, i.e., $\nexists F_1, F_2 \subset X$ such that

$$F = F_1 \cup F_2, \text{ where } F_1, F_2 \text{ are closed, } F_1, F_2 \neq \emptyset, F$$

We recall that the topological space $(X, \mathcal{T})$ may have certain "separation" properties. We will show that there is a relationship between separation and sobriety.

T_0 : For every two distinct points $x, y \in X$, we can either separate x from y, i.e., $\exists U \in \mathcal{T} : x \in U, y \notin U$, or we can separate y from x, i.e., $\exists V \in \mathcal{T} : y \in V, x \notin V$. ("One or the other can be separated off, but not necessarily both, and we have no choice over which one is being separated")

T_1 : For every two distinct points $x, y \in X$, we can choose one and separate it from the other, i.e., WLOG, $\exists U \in \mathcal{T} : x \in U, y \notin U$. ("One can be separated from the other, and we can choose which one")

T_2 : (Hausdorff) For every two distinct points $x, y \in X$, we can separate each from the other, i.e., $\exists U, V \in \mathcal{T} : x \in U,\ y \in V, U \cap V = \emptyset$. ("Any two distinct points can be 'housed off' ").

Proposition 3.7. *Let $(X, \mathcal{T}) \in |\mathbf{TOP}|, \Psi$ as defined above. The following hold:*
(i) $(X, \mathcal{T})$ is $T_0 \Leftrightarrow \Psi$ is injective
(ii) $(X, \mathcal{T})$ is Hausdorff $\Rightarrow (X, \mathcal{T})$ is sober $\Rightarrow (X, \mathcal{T})$ is T_0

Proof: For *(i)*, assume that $(X, \mathcal{T})$ is T_0, and let $x, y \in X$ such that $x \neq y$. We want to show that $\Psi(x) \neq \Psi(y)$. Since $x \neq y$ and X is T_0, $\exists U \in T$ such that, WLOG, $x \in U$ but $y \notin U$. Then,

$$\hookrightarrow_x^{\leftarrow}(U) = \{x\} \text{ and } \hookrightarrow_y^{\leftarrow}(U) = \emptyset$$

So then $\hookrightarrow_x^{\leftarrow}: \mathfrak{P}(\{x\}) \leftarrow \mathfrak{P}(X) \ \neq \hookrightarrow_y^{\leftarrow}: \mathfrak{P}(\{y\}) \leftarrow \mathfrak{P}(X)$, thanks to this U guaranteed by T_0. Now,

$$\begin{aligned} (\varphi \circ \hookrightarrow_x^{\leftarrow}) &\neq \left(\varphi \circ \hookrightarrow_y^{\leftarrow}\right) \\ p_x &\neq p_y \\ \Psi(x) &\neq \Psi(y) \end{aligned}$$

by how we define the point functions and Ψ. To verify this, we observe that

$$\begin{aligned} (\Psi(x))(U) &= p_x(U) = \varphi(\hookrightarrow_x^{\leftarrow}(U)) = \varphi(\{x\}) = 1 \\ (\Psi(y))(U) &= p_y(U) = \varphi\left(\hookrightarrow_y^{\leftarrow}(U)\right) = \varphi(\emptyset) = 0 \end{aligned}$$

since $x \neq y$. Therefore, Ψ is injective.

Now, assume that $(X, \mathcal{T})$ is injective. Let $x, y \in X$ such that $x \neq y$. We want to show that there is a set $U \in \mathcal{T}$ so that either $x \in U$ and $y \notin U$, or $x \notin U$ and

$y \in U$. Since $x \neq y$ and $(X, \mathcal{T})$ is injective, $\Psi(x) \neq \Psi(x)$. Then by how we define Ψ, $p_x \neq p_y$. So there must be an element $U \in dom(p_x) = dom(p_y) = \mathcal{T}$ such that $p_x(U) \neq p_y(U)$. I.e., $\exists U \in \mathcal{T}$ such that

$$\{p_x(U) = 1 \text{ and } p_y(U) = 0\} \text{ or } \{p_x(U) = 0 \text{ and } p_y(U) = 1\}$$

Equivalently, via the definition of a point function,

$$\left\{\varphi(\hookrightarrow_x^{\leftarrow}(U)) = 1 \text{ and } \varphi(\hookrightarrow_y^{\leftarrow}(U)) = 0\right\} \text{ or } \left\{\varphi(\hookrightarrow_x^{\leftarrow}(U)) = 0 \text{ and } \varphi(\hookrightarrow_y^{\leftarrow}(U)) = 1\right\}$$

which means, by definition of φ, that

$$\left\{\hookrightarrow_x^{\leftarrow}(U) = \{x\} \text{ and } \hookrightarrow_y^{\leftarrow}(U) = \emptyset\right\} \text{ or } \left\{\hookrightarrow_x^{\leftarrow}(U) = \emptyset \text{ and } \hookrightarrow_y^{\leftarrow}(U) = \{y\}\right\}$$

and now by definition of the preimage of an inclusion map,

$$\{x \in U \text{ and } y \notin U\} \text{ or } \{x \notin U \text{ and } y \in U\}$$

Therefore, $(X, \mathcal{T})$ is T_0.

To prove *(ii)*, first we want to show that $(X, \mathcal{T})$ is Hausdorff $\Rightarrow (X, \mathcal{T})$ is sober, so assume that $(X, \mathcal{T})$ is Hausdorff. Suppose F is an irreducible closed subset of X. Our goal is to find an $x \in X$ such that $F = \overline{\{x\}}$. Now, $A \cup B = \emptyset$ only if $A, B = \emptyset$, so $\emptyset$ is not irreducible closed, and hence $F \neq \emptyset$. So say $x \in F$. Then, by definition of closure, $F \supset \overline{\{x\}}$. To show that $F \subset \overline{\{x\}}$, and therefore $F = \overline{\{x\}}$, assume F is not a subset of $\overline{\{x\}}$. So $\exists y \in F - \overline{\{x\}}$, which means $y \neq x$. Since $(X, \mathcal{T})$ is Hausdorff, $\exists$ open sets $U, V \subset X$ such that $x \in U, y \in V, U \cap V = \emptyset$. To come to a contradiction, and hence upset our assumption that F is not a subset of $\overline{\{x\}}$, we will show that F is not irreducible closed. Put

$$\begin{aligned} F_1 &= (X - V) \cap F \\ F_2 &= \left[\overline{\left(F - \overline{\{x\}}\right)} \cup (X - U)\right] \cap F \end{aligned}$$

Neither F_1 nor F_2 are F: We know that $y \in F$, and also that $y \in V$. But then, $y \notin X - V$, so $y \notin (X - V) \cap F = F_1$. Then, F is not a subset of F_1, so $F \neq F_1$. And, we know that $x \in F$, also $x \in U$. Now, $x \notin F - \overline{\{x\}}$ and $x \notin X - U$, so $x \notin F_2$. Then, F is not a subset of F_2, so $F \neq F_2$.

Neither F_1 nor F_2 are $\emptyset$: We know that $x \in U, U \cap V = \emptyset$, so $x \notin V$. I.e., $x \in X - V$. But $x \in F$, so $x \in F_1$. Hence F_1 is nonempty. Also, $y \in V, U \cap V = \emptyset$, so $y \notin U$. I.e., $y \in X - U$, and so $y \in \left(F - \overline{\{x\}}\right) \cup (X - U)$. But $y \in F$, so $y \in F_2$. Hence F_2 is nonempty.

$F_1 \cup F_2 = F$:

$$
\begin{aligned}
F_1 \cup F_2 &= ((X - V) \cap F) \cup \left[\left((F - \overline{\{x\}}) \cap F\right) \cup ((X - U) \cap F)\right] \\
&= (F - V) \cup (F - U) \cup \left(F - \overline{\{x\}}\right) \\
&= F \cup (F - \overline{\{x\}}) \\
&= F
\end{aligned}
$$

So F is not irreducible closed, which is a contradiction, and which means that our assumption F is not a subset of $\overline{\{x\}}$ is false. Hence, $F = \overline{\{x\}}$, and so $(X, \mathcal{T})$ is sober.

Now, we want to show that $(X, \mathcal{T})$ is sober $\Rightarrow$ $(X, \mathcal{T})$ is T_0. Since $(X, \mathcal{T})$ is sober, Ψ is bijective, hence injective. So by part (i), $(X, \mathcal{T})$ is T_0 ■.

Now, we can relate sobriety with Hausdorff, T_0, and the Stone operator Ψ. It is interesting to note that the reverse implications do not hold, that is,

$$(X, \mathcal{T}) \text{ is Hausdorff} \not\Leftarrow (X, \mathcal{T}) \text{ is sober} \not\Leftarrow (X, \mathcal{T}) \text{ is } \mathrm{T}_0$$

Showing that this is in fact the case is most easily accomplished by providing counterexamples, which are explained in further detail in subsection 2.1.3.

3.3. The Stone Comparison Operator Φ and its Relationship with Ψ

Just as Ψ facilitates a reworking of the set of a topological space, there is a Stone operator Φ which redescribes the topology of a topological space. In fact, Φ can be applied to any semiframe, with the added benefit that a new topological space is generated.

Definition 3.8. *For any semiframe S, define the Stone comparison operator $\Phi : S \to \mathfrak{P}(Pt(S))$ by $\forall s \in S$,*

$$\Phi(s) = \{p \in Pt(S) \mid p(s) = 1\}$$

If Φ is injective, S is spatial.

Lemma 3.9. *For any semiframe S, $\Phi^{\rightarrow}(S)$ is a topology on $Pt(S)$.*

Proof: Let S be a semiframe, and consider the points of S,

$$Pt(S) = \mathbf{SFRM}(S, \{0,1\})$$

Also recall $\Phi : S \to \mathfrak{P}(Pt(S))$ by

$$\forall s \in S, \quad \Phi(s) = \{p \in Pt(S) | p(s) = 1\}$$

So then the image of S under Φ, $\Phi^{\rightarrow}(S)$, will be given by $\Phi^{\rightarrow}(S) = \{\Phi(s) | s \in S\} \subset \mathfrak{P}(Pt(S))$. To prove that $(Pt(S), \Phi^{\rightarrow}(S))$is a traditional topological space, we must show that $\Phi^{\rightarrow}(S)$ is closed under arbitrary unions and finite intersections.

Let $\{\Phi(s_\gamma) | \gamma \in \Gamma\} \subset \Phi^{\rightarrow}(S)$. We claim that $\bigcup_{\gamma\in\Gamma} \Phi(s_\gamma) = \Phi(\bigcup_{\gamma\in\Gamma}(s_\gamma))$. By the definition of Φ and union,

$$\begin{aligned} \bigcup_{\gamma\in\Gamma} \Phi(s_\gamma) &= \bigcup_{\gamma\in\Gamma} \{p \in Pt(S) | p(s_\gamma) = 1\} \\ &= \{p \in Pt(S) | \exists \gamma \in \Gamma, p(s_\gamma) = 1\} \end{aligned}$$

But then for every point p in this set, $p(\bigcup_{\gamma\in\Gamma}(s_\gamma)) = 1$, so

$$\begin{aligned} \bigcup_{\gamma\in\Gamma} \Phi(s_\gamma) &= \left\{p \in Pt(S) |\; p(\bigcup_{\gamma\in\Gamma} s_\gamma) = 1\right\} \\ &= \Phi(\bigcup_{\gamma\in\Gamma} s_\gamma) \end{aligned}$$

by the definition of the Stone operator Φ. Since S is a semiframe, it is closed under arbitrary unions, and so $(\bigcup_{\gamma\in\Gamma} s_\gamma) \in S$. Hence, $\bigcup_{\gamma\in\Gamma} \Phi(s_\gamma) = \Phi(\bigcup_{\gamma\in\Gamma} s_\gamma) \in \Phi^{\rightarrow}(S)$, so $\Phi^{\rightarrow}(S)$ is closed under arbitrary unions.

Let $\{\Phi(s_\delta) | \delta \in \Delta, \Delta \text{ finite}\} \subset \Phi^{\rightarrow}(S)$. We claim that $\bigcap_{\gamma\in\Delta} \Phi(s_\delta) = \Phi(\bigcap_{\delta\in\Delta}(s_\delta))$. By the definition of Φ and intersection,

$$\begin{aligned} \bigcap_{\delta\in\Delta} \Phi(s_\delta) &= \bigcap_{\delta\in\Delta} \{p \in Pt(S) | p(s_\delta) = 1\} \\ &= \{p \in Pt(S) | \forall \delta \in \Delta, p(s_\delta) = 1\} \end{aligned}$$

But then for every point p in this set, $p(\bigcap_{\delta\in\Delta}(s_\delta)) = 1$, so

$$\begin{aligned} \bigcap_{\delta\in\Delta} \Phi(s_\delta) &= \left\{p \in Pt(S) |\; p(\bigcap_{\delta\in\Delta} s_\delta) = 1\right\} \\ &= \Phi(\bigcap_{\delta\in\Delta} s_\delta) \end{aligned}$$

by the definition of the Stone operator Φ. Since S is a semiframe, it is closed under finite intersections, and so $(\bigcap_{\delta\in\Delta} s_\delta) \in S$. Hence, $\bigcap_{\delta\in\Delta}\Phi(s_\delta) = \Phi(\bigcap_{\delta\in\Delta} s_\delta) \in \Phi^{\rightarrow}(S)$, so $\Phi^{\rightarrow}(S)$ is closed under finite intersections.

Therefore, $\Phi^{\rightarrow}(S)$ is a topology on $Pt(S)$ ■.

The topological space $(Pt(S), \Phi^{\rightarrow}(S))$ is called the spectrum of S.

Proposition 3.10. $\Psi : (X, \mathcal{T}) \to (Pt(\mathcal{T}), \Phi^{\rightarrow}(\mathcal{T}))$ *is continuous and open with respect to its range.*

Proof: To show that Ψ is continuous, let $B \in \Phi^{\rightarrow}(\mathcal{T})$. Then, there is a $U \in \mathcal{T}$ such that $B = \Phi(U)$. So now,

$$\begin{aligned}\Psi^{\leftarrow}(B) &= \{x \in X \mid \Psi(x) \in \Phi(U)\}\\ &= \{x \in X \mid \Psi(x) \in \{p \in Pt(\mathcal{T}) \mid p(U) = 1\}\}\\ &= \{x \in X \mid \Psi(x)(U) = 1\}\end{aligned}$$

by the definition of Φ. If $\Psi(x)(U) = 1$, then $\chi_U(x) = 1$. Then we can redescribe our preimage as

$$\begin{aligned}\Psi^{\leftarrow}(B) &= \{x \in X \mid \chi_U(x) = 1\}\\ &= U\end{aligned}$$

And since $U \in \mathcal{T}$, Ψ is continuous.

To show that Ψ is relatively open (with respect to its range), we must show $\forall U \in \mathcal{T}$, $\Psi^{\rightarrow}(U) \in \Phi^{\rightarrow}(\mathcal{T}) \cap \Psi^{\rightarrow}(X)$. Let $U \in \mathcal{T}$. Then, $\Phi(U) \in \Phi^{\rightarrow}(\mathcal{T})$. We claim that $\Psi^{\rightarrow}(U) = \Phi(U) \cap \Psi^{\rightarrow}(X)$. Since $U \subset X$, $\Psi^{\rightarrow}(U) \subset \Psi^{\rightarrow}(X)$. So it suffices to show that $\Psi^{\rightarrow}(U) = \Phi(U)$.

$$\begin{aligned}\Psi^{\rightarrow}(U) &= \{\Psi(x) \mid x \in U\}\\ &= \{p_x \mid x \in U\}\\ &= \{p \in Pt(\mathcal{T}) \mid p(U) = 1\}\\ &= \Phi(U)\end{aligned}$$

And so Ψ is relatively open ■.

The Φ operator is needed to introduce several counterexamples that show the reverse implications of Proposition 27 do not hold.

Example 3.11. $(\{a, b\}, \{\emptyset, \{a, b\}, \{b\}\})$ *is sober but not Hausdorff.*

To show that this is sober, we will use the irreducible closed definition of sobriety. $\{a,b\} = \{a\} \cup \{b\}$. Neither $\{a\}$ nor $\{b\}$ are $\emptyset$ or $\{a,b\}$, and they are closed. So $\{a,b\}$ is irreducible closed. This is the only irreducible closed set of $\{a,b\}$, and $\{a,b\} = \overline{\{b\}}$, so this space is sober.

But, a cannot be separated from b by sets of this Sierpinski topology:

$$a \in \{a,b\}, \; b \in \{b\}, \text{ but } \{a,b\} \cap \{b\} \neq \emptyset$$

which means that this space is not Hausdorff.

Example 3.12. *$(\mathbb{R}, cofinite)$ is T_0 but not sober.*

The space $(\mathbb{R}, cofinite)$ is T_1: let $x, y \in R$ such that $x \neq y$. Then, $\{x\}$ is finite, so $R - \{x\} = (-\infty, x) \cup (x, \infty) \in cofinite$. Since $y \neq x, y \in R - \{x\}$. So we have a $U = R - \{x\} \in cofinite$ such that $y \in U$ and $x \notin U$. Also, $\mathrm{T}_1 \Rightarrow \mathrm{T}_0$, so $(\mathbb{R}, cofinite)$ is T_0.

To show that this space is not sober, we refer to an equivalent definition of sobriety: $(\mathbb{R}, cofinite)$ is sober iff every irreducible closed set F is the closure of a unique singleton. We say that F is irreducible closed if it cannot be written as a non-trivial union of nonempty closed sets, i.e.,

$$F \neq F_1 \cup F_2, \text{ where } F_1, F_2 \text{ are closed, } F_1, F_2 \neq \emptyset, F$$

So, we want to find an irreducible closed set F such that F is not the closure of a singleton set. Consider $\mathbb{R}$. $\mathbb{R}$ is irreducible closed because we cannot write it as $\mathbb{R} \cup F_2$; the real numbers cannot be written as the union of finite sets. Hence, since $\mathbb{R}$ is not the closure of a singleton set, $(\mathbb{R}, cofinite)$ is not sober.

3.4. The PT and Ω Functors

Since each topological space $(X, \mathcal{T})$ is associated with its own semiframe $\mathcal{T}$, and any semiframe can be associated with its spectrum topology, we claim that there must be some categorical relationship between **TOP** and **SFRM.** We consider **SLOC** because of the duality between semilocales and semiframes,

$$\mathbf{SLOC} \equiv \mathbf{SFRM}^{op}$$

which will be used in the definition below.

Definition 3.13. *Let $PT : \mathbf{SLOC} \to \mathbf{TOP}$ by:*

Objects: $\forall S \in |\mathbf{SLOC}|, \quad PT(S) = (Pt(S), \Phi^{\to}(S))$

Morphisms: $\forall f \in \mathbf{SLOC}(S_1, S_2), \forall p \in Pt(S_1), \quad PT(f)(p) = p \circ f^{op}$, *where f^{op} is just the dual of f.*

For $f \in \mathbf{SLOC}(S_1, S_2), \exists f^{op} \in \mathbf{SFRM}(S_2, S_1)$. We can say this because of the fact that $\forall A, B \in \mathbf{SLOC}$, the set of localic morphisms $\mathbf{SLOC}(A, B)$ is bijective with the set of semiframe morphisms in the opposite (and desired, in our case) direction $\mathbf{SFRM}(B, A)$. This bijection is expressed by $f \leftrightarrow f^{op}$ for all $f \in \mathbf{SLOC}(A, B)$.

We say this bi-level function PT takes a semiframe to its spectrum, and takes the morphism (via its opposite-direction semiframe morphism) to the appropriate continuous "point-to-point" morphism.

Proposition 3.14. $PT : \mathbf{SFRM} \to \mathbf{TOP}$ *is a functor.*

Proof: To show that PT maps semilocales to topological spaces, let $S \in \mathbf{SLOC}$. Then, $PT(S) = (Pt(S), \Phi^{\to}(S))$, which was shown to be a topological space in Lemma 29.

To show that PT respects domains and codomains of morphisms, let $f \in \mathbf{SLOC}(A, B)$. Now, we want to show that

$$PT(f : A \to B) = PT(f) : PT(A) \to PT(B)$$

Let $p \in PT(A) = (Pt(A), \Phi^{\to}(A))$. By definition, $p \in dom(PT(f))$. To check that $PT(B) = (Pt(B), \Phi^{\to}(B))$ is the codomain of $PT(f)$, we must show that $(PT(f))(p) \in PT(B)$. Recall that $f^{op} : A \leftarrow B$ because of the bijection between **SLOC** and **SFRM**, and $p : A \to \{0, 1\}$. Then,

$$\begin{aligned}(PT(f))(p) &= p \circ f^{op} \\ &= (p : A \to \{0,1\}) \circ (f^{op} : B \to A) \\ &= (p \circ f^{op}) : B \to \{0,1\}\end{aligned}$$

So our map has the right domain and codomain, but to complete the requirements to be one of the points of B, it must preserve arbitrary joins and finite meets. Let $\{b_\gamma : \gamma \in \Gamma\} \subset B$, and show that

$$((PT(f))(p)) \left(\bigvee_{\gamma \in \Gamma} b_\gamma \right) = \bigvee_{\gamma \in \Gamma} ((PT(f))(p))(b_\gamma)$$

Since $(PT(f))(p) = p \circ f^{op}$, which is just $p(f^{op}(b))$ for any $b \in B$, and since $f^{op} \in \mathbf{SFRM}(B, A)$, f^{op} preserves arbitrary joins. I.e.,

$$\begin{aligned}
((PT(f))(p))\left(\bigvee_{\gamma\in\Gamma} b_\gamma\right) &= (p \circ f^{op})\left(\bigvee_{\gamma\in\Gamma} b_\gamma\right) \\
&= p\left(f^{op}\left(\bigvee_{\gamma\in\Gamma} b_\gamma\right)\right) \\
&= p\left(\bigvee_{\gamma\in\Gamma} f^{op}(b_\gamma)\right)
\end{aligned}$$

and p is a semiframe map from A to $\{0, 1\}$, so for any subset of A, say $\{f^{op}(b_\gamma) \mid \gamma \in \Gamma\}$, we have

$$\begin{aligned}
p\left(\bigvee_{\gamma\in\Gamma} f^{op}(b_\gamma)\right) &= \bigvee_{\gamma\in\Gamma} p\left(f^{op}(b_\gamma)\right) \\
&= \bigvee_{\gamma\in\Gamma} (p \circ f^{op})(b_\gamma) \\
&= \bigvee_{\gamma\in\Gamma} \left(PT(f)(p)\right)(b_\gamma)
\end{aligned}$$

Hence, $PT(f)$ preserves arbitrary joins. The proof for finite meets directly parallels this. So $PT(f)(p)$ is a semiframe morphism from B to {0,1}, i.e., $PT(f)(p) \in PT(B)$ for all $p \in PT(A)$.

To show that PT respects identities, recall that $id_S : S \to S$ is the same as in **SET**. I.e, $\forall s \in S, id_S(s) = id(s) = s$. So now, is $PT(id_S) = id_{Pt(S)}$? To see that the action is the same, let $p \in Pt(S)$. Clearly, $id_{Pt(S)}(p) = p$. And,

$$\begin{aligned}
PT(id_S)(p) &= p \circ (id_S)^{op} \\
&= p \circ id_S \\
&= p
\end{aligned}$$

So PT respects the identity maps.

We need to check that PT maps each semilocalic morphism to a continuous morphism. Let $f \in \mathbf{SLOC}(A, B)$. We claim that $PT(f) : (Pt(A), \Phi^{\rightarrow}(A)) \to (Pt(B), \Phi^{\rightarrow}(B))$ is continuous. Let $\Phi(b) \in \Phi^{\rightarrow}(B)$, and show that

$$(PT(f))^{\leftarrow}(\Phi(b)) \in \Phi^{\rightarrow}(A).$$

by the definitions of preimage, $PT(f)$, Φ, and composition,

$$\begin{aligned}
(PT(f))^{\leftarrow}(\Phi(b)) &= \{p \in Pt(A) \mid PT(f)(p) \in \Phi(b)\} \\
&= \{p \in Pt(A) \mid p \circ f^{op} \in \Phi(b)\} \\
&= \{p \in Pt(A) \mid p \circ f^{op} \in \{q \in Pt(S_2) \mid q(b) = 1\}\} \\
&= \{p \in Pt(A) \mid (p \circ f^{op})(b) = 1\} \\
&= \{p \in Pt(A) \mid p(f^{op}(b)) = 1\} \\
&= \Phi(f^{op}(b))
\end{aligned}$$

and since $f^{op} : B \to A$, $(f^{op}(b)) \in A$. Then, $\Phi(f^{op}(b)) \in \Phi^{\to}(A)$, so

$$(PT(f))^{\leftarrow}(\Phi(b)) \in \Phi^{\to}(A),$$

and hence $PT(f) : (Pt(A), \Phi^{\to}(A)) \to (Pt(B), \Phi^{\to}(B))$ is continuous.

Also, PT preserves morphism composition in **SET**: Let $f : A \to B, g : B \to C$ be mappings in **SET**. Then, $(g \circ f) : A \to C$ is also such a map. We claim that

$$PT(g \circ f) = PT(g) \circ PT(f)$$

. Clearly, the domain $Pt(A)$ and the codomain $Pt(C)$ are the same for both of $PT(g \circ f)$ and $PT(g) \circ PT(f)$. To check the action, let $p \in Pt(A)$. We want to show $(PT(g \circ f))(p) = (PT(g) \circ PT(f))(p)$.

$$\begin{aligned}
(PT(g \circ f))(p) &= p \circ (g \circ f)^{op} \\
&= p \circ (f^{op} \circ g^{op})
\end{aligned}$$

And so,

$$\begin{aligned}
p \circ (f^{op} \circ g^{op}) &= (p \circ f^{op}) \circ g^{op} \\
&= PT(g)(p \circ f^{op}) \\
&= PT(g)(PT(f)(g)) \\
&= (PT(g) \circ PT(f))(p)
\end{aligned}$$

hence $(PT(g \circ f))(p) = (PT(g) \circ PT(f))(p)$, and PT preserves continuity in terms of the **SET**. But we have also shown that PT preserves domains and codomains, so PT preserves morphisms of **SLOC.**

Therefore, $PT : \mathbf{SLOC} \to \mathbf{TOP}$ is a functor ■.

Definition 3.15. *Let* $\Omega : \mathbf{SLOC} \leftarrow \mathbf{TOP}$ *by:*

Objects: $\forall (X, \mathcal{T}) \in |\mathbf{TOP}|, \quad \Omega(X, \mathcal{T}) = \mathcal{T}$

Morphisms: $\forall f \in \mathbf{TOP}((X, \mathcal{T}), (Y, \mathcal{S}))$,

$$\Omega(f : (X, \mathcal{T}) \to (Y, \mathcal{S})) = [f^{\leftarrow}|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}]^{op}$$

The Ω map acts on the traditionally continuous morphism $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ in an unusual way to get the "correct" semilocalic morphism. Ω takes the lattices $\mathcal{T}$ and $\mathcal{S}$, which have a natural relationship via the image and preimage operators $f^{\rightarrow}$ and $f^{\leftarrow}$, to be candidates for the domain and codomain of $\Omega(f)$. We cannot go directly to the image operator, which would have the "correct" direction, because $f^{\rightarrow}$ (when restricted to $\mathcal{T}$) is not guaranteed to be an **SLOC** morphism from $\mathcal{T}$ to $\mathcal{S}$. So instead, we look to the preimage $f^{\leftarrow}$. Since $f^{\leftarrow} : \mathfrak{P}(X) \leftarrow \mathfrak{P}(Y)$ and not $f^{\leftarrow} : \mathcal{T} \leftarrow \mathcal{S}$, we restrict the domain of $f^{\leftarrow}$ to the subcollection $\mathcal{S}$. Then, because f is traditionally continuous, we know that $\forall V \in \mathcal{S}$, $f^{\leftarrow}(V) \in \mathcal{T}$. So,

$$f^{\rightarrow}|_{\mathcal{S}} : \mathcal{T} \leftarrow \mathcal{S}$$

And, not only are $\mathcal{S}$ and $\mathcal{T}$ complete lattices, they are also semiframes. So $f^{\rightarrow}|_{\mathcal{S}} \in \mathbf{SFRM}(\mathcal{S}, \mathcal{T})$ and by the bijective relationship between $\mathbf{SFRM}(\mathcal{S}, \mathcal{T})$ and $\mathbf{SLOC}(\mathcal{T}, \mathcal{S})$, we know $(f^{\rightarrow}|_{\mathcal{S}})^{op} \in \mathbf{SLOC}(\mathcal{T}, \mathcal{S})$. Hence Ω maps morphisms of **TOP** to morphisms of **SLOC**.

Proposition 3.16. $\Omega : \mathbf{SLOC} \leftarrow \mathbf{TOP}$ *is a functor.*

Proof: Let $(X, \mathcal{T}) \in |\mathbf{TOP}|$. Then, $\Omega(X, \mathcal{T}) = \mathcal{T}$, which is a complete lattice, and hence an object of **SLOC**.

To show that Ω respects domains and codomains, let $f \in \mathbf{TOP}\left((X, \mathcal{T}), (Y, \mathcal{S})\right)$. We claim that $\Omega(f)$ maps from $\Omega(X, \mathcal{T})$ to $\Omega(Y, \mathcal{S})$, i.e., that $\Omega(f) : \mathcal{T} \to \mathcal{S}$, and that $\Omega(f) : \mathcal{T} \to \mathcal{S}$ is in fact a member of $\mathbf{SLOC}(\mathcal{T}, \mathcal{S})$. This was demonstrated immediately preceding this proposition.

Now, to show that Ω respects identities, recall that $id_{(X,\mathcal{T})} = id_X$. So,

$$\Omega(id_{(X,\mathcal{T})}) = \Omega(id_X) = [id_X^{\leftarrow}|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}]^{op}$$

But, $id_X^{\leftarrow} : \mathfrak{P}(X) \leftarrow \mathfrak{P}(X)$ is just the identity of the powerset, $id_{\mathfrak{P}(X)}$. So when this is restricted,

$$\begin{aligned} \Omega(id_X) &= [id_X^{\leftarrow}|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}]^{op} \\ &= \left[id_{\mathfrak{P}(X)}|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}\right]^{op} \\ &= [id_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}]^{op} \end{aligned}$$

and the natural bijection here is $(id_{\mathcal{T}})^{op} = id_{\mathcal{T}}$. So, $\Omega(id_{(X,\mathcal{T})}) = id_{\mathcal{T}} = id_{\Omega(X,\mathcal{T})}$, that is, Ω preserves identities.

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ and $g : (Y, \mathcal{S}) \to (Z, \mathcal{U})$ in **TOP**. Since composition preserves continuity in **TOP**, $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{U})$ is in **TOP**. We want to show that

$$\Omega(g \circ f) = \Omega(g) \circ \Omega(f)$$

Since $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{U})$, $\Omega(g \circ f) : \mathcal{T} \to \mathcal{U}$. And, $\Omega(g) : \mathcal{S} \to \mathcal{U}$ and $\Omega(f) : \mathcal{T} \to \mathcal{S}$, so $\Omega(g) \circ \Omega(f) : \mathcal{T} \to \mathcal{U}$. Hence the domains and codomains match up. Now, to check that the action is the same, let $U \in \mathcal{T}$.

$$\begin{aligned}
\Omega(g \circ f)(U) &= [(g \circ f)^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{T}]^{op}(U) \\
&= [(f^{\leftarrow} \circ g^{\leftarrow})|_{\mathcal{U}}]^{op}(U)
\end{aligned}$$

But $g^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S}$ and so $f^{\leftarrow}(g^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S}) : \mathcal{S} \to \mathcal{T}$ since both f and g are traditionally continuous.

$$\begin{aligned}
\Omega(g \circ f)(U) &= [(f^{\leftarrow} \circ g^{\leftarrow})|_{\mathcal{U}}]^{op}(U) \\
&= [f^{\leftarrow}(g^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S}) : \mathcal{S} \to \mathcal{T}]^{op}(U) \\
&= [(f^{\leftarrow}|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}) \circ (g^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S})]^{op}(U) \\
&= [(f^{\leftarrow}|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T})^{op} \circ (g^{\leftarrow}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S})^{op}](U) \\
&= [\Omega(f) \circ \Omega(g)](U)
\end{aligned}$$

Hence, Ω preserves morphism composition. So Ω is a functor $\blacksquare$.

4. Lattice-Dependent Stone Representations

In the traditional case, we consider a traditional topological space $(X, \mathcal{T})$ and define the operator $\Psi : X \to Pt(\mathcal{T})$, where $Pt(\mathcal{T}) = \mathbf{SFRM}(\mathcal{T}, \{0, 1\})$. $\Psi(x)$ is an evaluation map for x; it takes each $U \in \mathcal{T}$ and matches x with a member of $\{0, 1\}$ via $\chi_U(x)$. Suppose (X, τ) is an L-topological space. Our only requirement of L is to be a complete lattice. We will build a similar transformation.

4.1. L-Stone Operators Ψ_L and Φ_L

Definition 4.1. *Let $A \in |\mathbf{SFRM}|$. The L-points of A, $Lpt(A)$ are given by*

$$Lpt(A) = \mathbf{SFRM}(A, L)$$

Definition 4.2. *Let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$. Define the L-Stone operator $\Psi_L : X \to Lpt(\tau)$ by: $\forall x \in X, \forall p \in Lpt(\tau)$,*

$$\Psi_L(x)(p) = p(x)$$

$\Psi_L(x)$ is also an evaluation map for x; it takes each map $u \in \tau$ and matches x with a member of L via $u(x)$. Separation and a new twist on sobriety can now be defined.

Definition 4.3. *Let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$ Then, (X, τ) is T_0 iff Ψ_L is injective and (X, τ) is L-sober iff Ψ_L is bijective.*

The other Stone operator Φ also has an L-dual, which we will call Φ_L.

Definition 4.4. *Let $A \in |\mathbf{SFRM}|$. Define the L-Stone operator $\Phi_L : A \to L^{Lpt(A)}$ by: $\forall a \in A, \forall p \in Lpt(A)$,*

$$\Phi_L(a)(p) = p(a)$$

$\Phi_L(a)$ is an evaluation map for a; it takes each $p \in Lpt(A)$ and matches up a with a member of L via $p(a)$.

Lemma 4.5. *For any semiframe A, $(Lpt(A), \Phi_L^{\rightarrow}(A))$ is an L-topological space.*

Proof: Let $\{\Phi_L(a_\gamma) | \gamma \in \Gamma\} \subset \Phi_L^{\rightarrow}(A)$. We claim that

$$\bigvee_{\gamma \in \Gamma} \Phi_L(a_\gamma) = \Phi_L(\bigvee_{\gamma \in \Gamma} (a_\gamma))$$

Let $p \in Lpt(A)$.

$$\begin{aligned}
\left(\bigvee_{\gamma \in \Gamma} \Phi_L(a_\gamma)\right)(p) &= \bigvee_{\gamma \in \Gamma} (\Phi_L(a_\gamma)(p)) \\
&= \bigvee_{\gamma \in \Gamma} p(a_\gamma) \\
&= p\left(\bigvee_{\gamma \in \Gamma} a_\gamma\right) \\
&= \Phi_L\left(\bigvee_{\gamma \in \Gamma} a_\gamma\right)
\end{aligned}$$

since the "join map" is the "join of the maps", and since p is a semiframe morphism and hence preserves arbitrary joins, and by the definition of Φ_L. So, $\Phi_L^{\rightarrow}(A)$ is closed under arbitrary joins. The proof for finite meets mirrors the above $\blacksquare$.

And so, the L-spectrum of A is an L-topological space.

Proposition 4.6. *$\Psi_L : (X, \tau) \to (Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$ is L-continuous and "open" with respect to its range.*

Proof: To show that Ψ_L is L-continuous, let $\Phi_L(u) \in \Phi_L^{\rightarrow}(\tau)$. We want to show that $(\Psi_L)^{\leftarrow}(\Phi_L(u)) \in \tau$. I.e., show that this L-preimage is a map from X to L that preserves $\bigvee$ and $\wedge$.

$$(\Psi_L)^{\leftarrow}(\Phi_L(u)) = (\Phi_L(u)) \circ (\Psi_L)$$

by definition of L-preimage. Now the domain of this composition is the domain of Ψ_L, which is X. And the codomain of the composition is since $\Psi_L : X \to Lpt(\tau)$ and $\Phi_L : \tau \to L^{Lpt(\tau)}$, which means $\Phi_L(u) \in L^{Lpt(\tau)}$, or, $\Phi_L(u) : Lpt(\tau) \to L$.

$$(\Phi_L(u)) \circ (\Psi_L) : X \to Lpt(\tau) \to L$$

So we need only check that $(\Phi_L(u)) \circ (\Psi_L) \in \tau$. Claim: this has the same action as u, and hence is a member of τ. Let $x \in X$.

$$\begin{aligned} \left((\Psi_L)_L^{\rightarrow}(\Phi_L(u))\right)(x) &= (\Phi_L(u) \circ \Psi_L)(x) \\ &= \Phi_L(u)(\Psi_L(x)) \end{aligned}$$

and since $\Psi_L : X \to Lpt(\tau), \exists p \in Lpt(\tau)$ such that $\Psi_L(x) = p$. Then,

$$\begin{aligned} \left((\Psi_L)_L^{\rightarrow}(\Phi_L(u))\right)(x) &= \Phi_L(u)(p) \\ &= p(u) \\ &= \Psi_L(x)(u) \\ &= u(x) \end{aligned}$$

Now, $(\Psi_L)_L^{\rightarrow}(\Phi_L(u)) = u \in \tau$, so Ψ_L is L-continuous $\blacksquare$.

Proposition 4.7. *$\forall A \in |\mathbf{SFRM}|$, $\Phi_L^{\rightarrow}(A)$ is L-topology on $Lpt(A)$*

Definition 4.8. *A is L-spatial iff Φ_L injective* or, A is L-spatial iff Φ_L isomorphism onto $\Phi_L^{\rightarrow}(A)$

Definition 4.9. *(L-Point Functor)* LPT : **SLOC** $\rightarrow$ L-**TOP** *is defined:*

Objects: $\forall A \in |\mathbf{SLOC}|$,

$$LPT(A) = (Lpt(A), \Phi_L^{\rightarrow}(A))$$

Morphisms: $\forall f : A \rightarrow B$,

$$LPT(f) : LPT(A) \rightarrow LPT(B)$$

$\forall p \in Lpt(A)$, $LPT(f)(p) = p \circ f^{op}$

Definition 4.10. *Let* $(X, \tau) \in |L\text{-}\mathbf{TOP}|$. $\Psi_L : X \rightarrow Lpt(\tau)$ *by*

$$\begin{aligned} \Psi_L(x) &: \tau \rightarrow L \text{ by} \\ \Psi_L(x)(u) &= u(x) \text{ (evaluation map)} \end{aligned}$$

Proposition 4.11. $\Psi_L : (X, \tau) \rightarrow (Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$ *is* L*-continuous*

Definition 4.12. (X, τ) *is* L*-sober iff* Ψ_L *is bijective*

Definition 4.13. *(L-Omega Functor)* $L\Omega$: **SLOC** $\leftarrow$ L-**TOP** *is defined:*

Objects: $\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$,

$$L\Omega(X, \tau) = \tau$$

Morphisms: $\forall g : (X, \tau) \rightarrow (Y, \sigma)$,

$$L\Omega(g) = \left([g_L^{\leftarrow}]_{|\sigma} : \tau \leftarrow \sigma\right)^{op}$$

Proposition 4.14. $L\Omega \dashv LPT$

The proofs of these propositions both simplify and extend those of the classical case, and can be found in [19, 21].

Part III
The Fuzzy Real Line, $\mathbb{R}(L)$

5. Set-Up of the Fuzzy Real Line

To discuss the fuzzy real line $\mathbb{R}(L)$, sometimes called the Hutton real line [11, 3], the following requirements must be met: Let L be a deMorgan algebra. That is,

1. L is a lattice with the order $\leq$, i.e., L is closed under finite $\vee$ and finite $\wedge$.
2. L has universal upper and lower bounds $\top$ and $\bot$, respectively
3. L has $'$, an order-reversing involution, i.e., $' : L \longrightarrow L$ where $\forall x, y \in L$,

$$[(x \leq y) \Rightarrow (y' \leq x')] \text{ and } \left[x'' = (x')' = x\right]$$

We will also be discussing maps of the form $\lambda : \mathbb{R} \longrightarrow L$ such that λ is antitone, λ is bounded above by $\top$, and λ is bounded below by $\bot$. These maps will be developed to produce the members of the fuzzy real line, which can be found in [11], [3], and [13].

Definition 5.1. *Let $\lambda : \mathbb{R} \longrightarrow L$ as per above, and let $t \in \mathbb{R}$. Then, the right hand limit of λ at t, $\lambda(t^+)$, is the join of all lattice values $\lambda(s)$ where $s > t$. I.e.,*

$$\lambda(t^+) = \bigvee_{s>t} \lambda(s)$$

The left hand limit of λ at t is the meet of all $\lambda(s)$ with $s < t$:

$$\lambda(t^-) = \bigwedge_{s<t} \lambda(s)$$

Lemma 5.2. *Let λ and μ be these type of antitone maps. Then the following are equivalent:*
(i) $\forall t \in \mathbb{R}, \quad \lambda(t^+) = \mu(t^+)$
(ii) $\forall t \in \mathbb{R}, \quad \lambda(t^-) = \mu(t^-)$

Proof: Assume that *(i)* holds, and let $t \in \mathbb{R}$. To prove *(ii)*, we will show both directions of inequality. To show $\lambda(t^-) \leq \mu(t^-)$, it suffices to show that

$$\forall s_1 < t, \quad \bigwedge_{s_1<s<t} (\lambda(s)) \leq \mu(s_1)$$

Justification: $\mu(t^-) = \bigwedge_{s_1<t} (\mu(s_1)) \geq$ any lower bound of $\{\mu(s_1)|s_{1<}t\}$, so we need to show that $\lambda(t^-)$ is a lower bound.

Since $\mu(t^-) = \bigwedge_{s_1<t} (\mu(s_1)) = g.l.b\{\mu(s_1)|s_{1<}t\}$, and for any set W, $l.b.W \leq \bigwedge W$, then we need only show that $\lambda(t^-)$ is a lower bound of $\{\mu(s_1)|s_{1<}t\}$. I.e., show that $\lambda(t^-) \leq \mu(s_1)$ for all $s_1 < t$. And since

$$\begin{aligned}
\lambda(t^-) &= \bigwedge_{s<t} (\lambda(s)) \\
&= g.l.b\{\lambda(s)|s < t\} \\
&= g.l.b\{\lambda(s)|s \leq s_1 \text{ or } s_1 < s < t\}
\end{aligned}$$

for our purposes, we want $\lambda(t^-) \leq \bigwedge_{s_1<s<t} (\lambda(s))$. But $\bigwedge_{s_1<s<t} (\lambda(s))$ is equivalent to $g.l.b\{\lambda(s)|s \leq s_1 \text{ or } s_1 < s < t\}$ because λ is antitone. So now, if we demonstrate that $\bigwedge_{s_1<s<t} (\lambda(s)) \leq \mu(s_1)$, then by transitivity we get that $\lambda(t^-) \leq \mu(t^-)$.

Fix $s_1 < t$. Because μ is antitone, we have $\mu(s_1) \geq \mu(s)$ for any s such that $s_1 < s < t$. So, $\mu(s_1)$ is an upper bound for $\{\mu(s)|s_1 < s < t\}$. Hence,

$$\begin{aligned}
\mu(s_1) &\geq \bigvee_{s_1<s<t} \mu(s) \\
&= \bigvee_{s_1<s<t} \lambda(s)
\end{aligned}$$

by our initial assumption that *(i)* holds. Now, in any lattice, the join of lattice values is always greater than or equal to the meet of those values, so

$$\bigvee_{s_1<s<t} \lambda(s) \geq \bigwedge_{s_1<s<t} \lambda(s)$$

And so for this choice of s_1, by transitivity, $\bigwedge_{s_1<s<t} \lambda(s) \leq \mu(s_1)$. Since s_1 was arbitrarily selected, $\forall s_1 > t$,

$$\bigwedge_{s_1<s<t} \lambda(s) \leq \mu(s_1),$$

and by our previous argument, $\lambda(t^-) \leq \mu(t^-)$. Now we have that $(i) \Rightarrow (ii)$. The proof of $(ii) \Rightarrow (i)$ parallels this proof ■.

Since we have shown that the right hand limits of two maps always being equal implies that the left hand limits are always equal (and vice versa), we now set up an equivalence relation, which will define equivalence classes.

Definition 5.3. *Given $\lambda, \mu \in L^{\mathbb{R}}$ are antitone, we say $\lambda \equiv \mu$ iff either condition of the above Lemma holds.*

Proposition 5.4. *$\equiv$ is an equivalence relation.*

Proof: We must show that $\equiv$ is reflexive, symmetric, and transitive on the set of all such maps. Let $\lambda, \mu, \nu \in L^{\mathbb{R}}$ as before, and let $t \in \mathbb{R}$. Since λ is well-defined, we know $\forall s > t, \quad s = s \Rightarrow \lambda(s) = \lambda(s)$. But then, $\bigvee_{s>t} \lambda(s) = \bigvee_{s>t} \lambda(s)$, and so $\lambda(t^+) = \lambda(t^+)$. Because t arbitrary, $\forall t \in \mathbb{R}, \quad \lambda(t^+) = \lambda(t^+)$, which is part (i) of the previous lemma. So, $\lambda \equiv \lambda$, and hence $\equiv$ is reflexive.

Suppose $\lambda \equiv \mu$. Then, $\forall t \in \mathbb{R}, \quad \lambda(t^+) = \mu(t^+)$, or, $\forall t \in \mathbb{R}, \bigvee_{s>t} \lambda(s) = \bigvee_{s>t} \mu(s)$ and since equality is symmetric, $\bigvee_{s>t} \mu(s) = \bigvee_{s>t} \lambda(s)$. I.e,

$$\forall t \in \mathbb{R}, \ \mu(t^+) = \lambda(t^+)$$

So then $\mu \equiv \lambda$, and hence $\equiv$ is symmetric.

Suppose $\lambda \equiv \mu$ and $\mu \equiv \nu$. Then,

$$\forall t \in \mathbb{R}, \ \lambda(t^+) = \mu(t^+) \text{ and } \mu(t^+) = \nu(t^+)$$

Or, $\forall t \in \mathbb{R}, \quad \bigvee_{s>t} \lambda(s) = \bigvee_{s>t} \mu(s)$ and $\bigvee_{s>t} \mu(s) = \bigvee_{s>t} \nu(s)$. But equality is transitive, so $\forall t \in \mathbb{R}, \bigvee_{s>t} \lambda(s) = \bigvee_{s>t} \nu(s)$. I.e., $\forall t \in \mathbb{R}, \quad \lambda(t^+) = \nu(t^+)$. So, $\lambda \equiv \nu$ ■.

Proposition 5.5. *Given the equivalence class $[\lambda] = \{\lambda_\gamma | \gamma \in \Gamma, \lambda_\gamma \equiv \lambda\}$, there exist $\mu, \nu \in [\lambda]$ such that μ is left-continuous and ν is right-continuous.*

Proof: Let $[\lambda] \subset L^{\mathbb{R}}$, and define $\mu : \mathbb{R} \to L$ by

$$\forall t \in \mathbb{R}, \quad \mu(t) = \lambda(t^-)$$

We claim that μ is left continuous, that is, $\forall t \in \mathbb{R}, \mu(t^-) = \mu(t)$. Let $t \in \mathbb{R}$. Since μ must be antitone, we have

$$\mu(s) \geq \mu(t) \quad \forall s < t$$

Then, $\mu(t)$ is a lower bound for $\{\mu(s)|s<t\}$. And,

$$\begin{aligned}\mu(t) &= l.b.\{\mu(s)|s<t\} \\ &\leq \bigwedge_{s<t}\mu(s) \\ &= \mu(t^-)\end{aligned}$$

So, $\mu(t) \leq \mu(t^-)$. Now, to show that $\mu(t) \geq \mu(t^-)$, recall that our definition for $\mu(t)$ is $\lambda(t^-)$. So,

$$\mu(t) = \lambda\left(t^-\right) = \bigwedge_{s<t}\lambda(s)$$

And now we consider the left hand limit of μ at t :

$$\begin{aligned}\mu(t^-) &= \bigwedge_{s<t}\mu(s) \\ &= \bigwedge_{s<t}\lambda(s^-) \\ &= \bigwedge_{s<t}\left(\bigwedge_{z<s}\lambda(z)\right) \\ &= g.l.b.\left\{\left(\bigwedge_{z<s}\lambda(z)\right) : s<t\right\}\end{aligned}$$

So for each $s<t$,

$$\mu(t^-) \leq \bigwedge_{z<s}\lambda(z) \leq \lambda(s)$$

Since $\mu(t^-) \leq \lambda(s)$ for all such s, it is certainly a lower bound:

$$\begin{aligned}\mu\left(t^-\right) &= l.b.\{\lambda(s)|s<t\} \\ &\leq \bigwedge\{\lambda(s)|s<t\} \\ &= \bigwedge_{s<t}\lambda(s) \\ &= \lambda(t^-) \\ &= \mu(t)\end{aligned}$$

So we have shown that $\mu(t) \leq \mu\left(t^-\right)$ and $\mu(t) \geq \mu\left(t^-\right)$, and hence $\mu(t) = \mu\left(t^-\right)$, which means that μ is left continuous.

But we also need to show that $\mu \in [\lambda]$, i.e., that $\mu \equiv \lambda$, which by definition means we have to show that one of the conditions of Lemma 43 holds. Let $t \in \mathbb{R}$.

Since we proved that μ is left continuous, $\mu(t) = \mu(t^-)$. But by how we define μ, $\mu(t) = \lambda(t^-)$. Then, $\forall t \in \mathbb{R}$, $\mu(t^-) = \lambda(t^-)$. Hence, $\mu \equiv \lambda$.

Now define $\nu : \mathbb{R} \to L$ by

$$\forall t \in \mathbb{R}, \quad \nu(t) = \lambda(t^+)$$

We claim that ν is right continuous, that is, $\forall t \in \mathbb{R}$, $\nu(t^+) = \nu(t)$. Let $t \in \mathbb{R}$. Since ν antitone,

$$\nu(t) \geq \nu(s), \quad \forall s > t$$

So $\nu(t)$ is an upper bound.

$$\begin{aligned} \nu(t) &= u.b.\{\nu(s) | s > t\} \\ &\geq \bigvee_{s>t} \nu(s) \\ &= \nu(t^+) \end{aligned}$$

Now to show that $\nu(t) \leq \nu(t^+)$, recall how we define ν.

$$\begin{aligned} \nu(t) &= \lambda(t^+) \\ &= \bigvee_{s>t} \lambda(s) \end{aligned}$$

Then, the right hand limit of ν at t is

$$\begin{aligned} \nu(t^+) &= \bigvee_{s>t} \nu(s) \\ &= \bigvee_{s>t} \lambda(s^+) \\ &= \bigvee_{s>t} \left(\bigvee_{z>s} \lambda(z) \right) \end{aligned}$$

Now since λ is antitone, then for each s, we have

$$\lambda(s) \geq \lambda(z), \quad \forall \dot{z} > s$$

So $\lambda(s)$ is an upper bound:

$$\begin{aligned} \lambda(s) &= u.b.\{\lambda(z) | \dot{z} > s\} \\ &\geq l.u.b.\{\lambda(z) | \dot{z} > s\} = \bigvee_{z>s} \lambda(z) \end{aligned}$$

This is true for any $s > t$, so the least upper bound of the $\lambda(s)$'s is also greater than $\bigvee_{z>s} \lambda(z)$:

$$\bigvee_{s>t} \lambda(s) \geq \bigvee_{z>s} \lambda(z), \quad \forall s > t$$

And so, $\bigvee_{s>t} \lambda(s)$ is an upper bound of $\{\bigvee_{z>s} \lambda(z) : s > t\}$, and hence greater than the least upper bound:

$$\begin{aligned} \bigvee_{s>t} \lambda(s) &= u.b. \left\{ \bigvee_{z>s} \lambda(z) : s > t \right\} \\ &\geq \bigvee \left\{ \bigvee_{z>s} \lambda(z) : s > t \right\} \\ &= \bigvee_{s>t} \left(\bigvee_{z>s} \lambda(z) \right) \\ &= \nu(t^+) \end{aligned}$$

And since $\bigvee_{s>t} \lambda(s) = \lambda(t^+) = \nu(t)$, then

$$\nu(t) \geq \nu(t^+)$$

Hence we have both directions of the inequality, which means that ν is right-continuous.

Clearly, $\nu \in [\lambda]$. ν is right-continuous, so, $\nu(t) \geq \nu(t^+)$ for all $t \in \mathbb{R}$. And by how we define ν, $\forall t \in \mathbb{R}, \quad \nu(t) = \lambda(t^+)$. So, $\forall t \in \mathbb{R}, \quad \nu(t^+) = \lambda(t^+)$, which means that Lemma 43 is satisfied, hence, $\nu \equiv \lambda$ $\blacksquare$.

Now we can define the L-fuzzy real line $\mathbb{R}(L)$.

Definition 5.6. $\mathbb{R}(L) = \{[\lambda] \mid \lambda : R \to L$ *is as required in this part*$\}$.

6. L-Open Sets for the Fuzzy Real Line

As in the previous parts, we want to establish an L-topological space, so to build that topology, consider the following:

Definition 6.1. *For the set product* $\mathbb{R}(L) \times \mathbb{R}$, *define* $\mathcal{L}, \mathcal{R} : (\mathbb{R}(L) \times \mathbb{R}) \to L$ *by* $\forall ([\lambda], t) \in \mathbb{R}(L) \times \mathbb{R}$,

$$\mathcal{L}([\lambda], t) = \left(\lambda\left(t^-\right)\right)'$$
$$\mathcal{R}([\lambda], t) = \lambda\left(t^+\right)$$

where $'$ *is the order-reversing involution required of* L *being a deMorgan algebra.*

We will use these maps $\mathcal{L}$ and $\mathcal{R}$ as a *subbase* for an L-topology on $\mathbb{R}(L) \times \mathbb{R}$.

Definition 6.2. *Let X be a set and $\sigma \in L^X$. Then, $\tau \equiv << \sigma >>$ is the smallest L-topology containing σ. We say σ is a subbase of τ.*

So for $X = \mathbb{R}(L) \times \mathbb{R}$, we consider the L-topology generated by $\{\mathcal{L}, \mathcal{R}\}$. Denote $<< \{\mathcal{L}, \mathcal{R}\} >> \equiv \tau_{\{\mathcal{L},\mathcal{R}\}}$ and note that

$$\tau_{\{\mathcal{L},\mathcal{R}\}} = \left\{\bigvee \emptyset = \perp,\ \mathcal{L},\ \mathcal{R},\ \mathcal{L} \wedge \mathcal{R},\ \mathcal{L} \vee \mathcal{R},\ \bigwedge X = \top\right\}$$

where $\perp$ is the constant "bottom" map and $\top$ is the constant "top" map.

For such mappings, that is, mappings from $\mathbb{R}(L) \times \mathbb{R}$ into L, we can look at the projections:

If t is fixed, we create a mapping from $\mathbb{R}(L)$ to L.

If $[\lambda]$ is fixed, we create a mapping from $\mathbb{R}$ to L.

For $\mathcal{L}$ and $\mathcal{R}$, we define these projections as follows:

Definition 6.3. *For fixed $t \in \mathbb{R}$, define $\mathcal{L}_t, \mathcal{R}_t : \mathbb{R}(L) \to L$ by $\forall [\lambda] \in \mathbb{R}(L)$,*

$$\mathcal{L}_t([\lambda]) = \mathcal{L}([\lambda], t) = \left(\lambda\left(t^-\right)\right)'$$
$$\mathcal{R}_t([\lambda]) = \mathcal{R}([\lambda], t) = \lambda\left(t^+\right)$$

and for fixed $[\lambda] \in \mathbb{R}(L)$, define $\mathcal{L}_{[\lambda]}, \mathcal{R}_{[\lambda]} : \mathbb{R} \to L$ by $\forall t \in \mathbb{R}$,

$$\mathcal{L}_{[\lambda]}(t) = \mathcal{L}([\lambda], t) = \left(\lambda\left(t^-\right)\right)'$$
$$\mathcal{R}_{[\lambda]}(t) = \mathcal{R}([\lambda], t) = \lambda\left(t^+\right)$$

We had before that $\mathcal{L}$ and $\mathcal{R}$ generate an L-topology on $\mathbb{R}(L) \times \mathbb{R}$, and now we can say that these projections also generate L-topologies.

$$< \ < \{\mathcal{L}_t, \mathcal{R}_t : t \in \mathbb{R}\} >> \ \equiv \tau(L) \text{ is an } L\text{-topology on } \mathbb{R}(L)$$
$$< \ < \left\{\mathcal{L}_{[\lambda]}, \mathcal{R}_{[\lambda]} : [\lambda] \in \mathbb{R}(L)\right\} >> \ \equiv co\text{-}\tau(L) \text{ is an } L\text{-topology on } \mathbb{R}$$

Now that we have these topological spaces, we would like to know how they act. First, we will examine an embedding of the real line into the L-fuzzy real line.

Definition 6.4. *Let $r, t \in \mathbb{R}$, and put*

$$\lambda_r(t) = \begin{cases} \top, & t < r \\ \bot, & t > r \end{cases}$$

Since $[\lambda_r] \in \mathbb{R}(L)$, *put* $\varphi : \mathbb{R} \to \mathbb{R}(L)$ *by* $\varphi(r) = [\lambda_r]$

For the above mapping φ, we claim that φ is an embedding, and that the image of φ over all of $\mathbb{R}$ is the fuzzy real line $\mathbb{R}(\{0,1\})$.

Proposition 6.5. $\forall t \in \mathbb{R}$, *(i)* $\mathcal{L}_t \circ \varphi = \chi_{(-\infty,t)}$, *and* *(ii)* $\mathcal{R}_t \circ \varphi = \chi_{(t,\infty)}$

Proof: For *(i)*, let $t \in \mathbb{R}$. The characteristic map $\chi_{(-\infty,t)} : \mathbb{R} \to \{0,1\}$, but note that 0 and 1 are just the universal join and meet of the lattice $L = [0,1]$ and so $\{0,1\}$ is isomorphic to $\bigvee L$ and $\bigwedge L$, i.e., to $\{\top, \bot\}$. Also, $\varphi : \mathbb{R} \to \mathbb{R}(L)$ and $\mathcal{L}_t, : \mathbb{R}(L) \to L$, so the composition $\mathcal{L}_t \circ \varphi : \mathbb{R} \to L \supset \{\top, \bot\}$. Then, the domains and codomains of our two mappings in consideration are compatible. Now, we need only show that the action of $\mathcal{L}_t \circ \varphi$ is the same as the action of $\chi_{(-\infty,t)}$. Let $r \in \mathbb{R}$.

Case 1: $r \geq t$.

$\chi_{(-\infty,t)}(r) = 0 \cong \bot$ of L.

Now, $\varphi(r) = [\lambda_r]$, and by Proposition 46, there exists a left-continuous member of this class, call it λ_r for simplicity. Then,

$$\begin{aligned} (\mathcal{L}_t \circ \varphi)(r) &= \mathcal{L}_t(\varphi(r)) \\ &= \mathcal{L}_t([\lambda_r]) \\ &= \mathcal{L}([\lambda_r], t) \\ &= \left(\lambda_r\left(t^-\right)\right)' \\ &= (\lambda_r(t))' \end{aligned}$$

Since λ_r is left continuous. And by how we define λ_r for $t \leq r$,

$$(\mathcal{L}_t \circ \varphi)(r) = (\lambda_r(t))' = (\top)' = \bot$$

So in this case, since r was chosen arbitrarily, $\mathcal{L}_t \circ \varphi = \chi_{(-\infty,t)}$ for all $r \geq t$.

Case 2: $r < t$.

$\chi_{(-\infty,t)}(r) = 1 \cong \top$ of L. Again, we choose the left continous member of the class:

$$(\mathcal{L}_t \circ \varphi)(r) = (\lambda_r(t))' = (\bot)' = \top$$

Since $r < t$. And, in this case, since r was chosen arbitrarily, $\mathcal{L}_t \circ \varphi = \chi_{(-\infty,t)}$ for all $r < t$.

By separation of cases, $\forall t \in \mathbb{R}, \quad (i)\ \mathcal{L}_t \circ \varphi = \chi_{(-\infty,t)}$.

For *(ii)*, let $t \in \mathbb{R}$. The characteristic map $\chi_{(t,\infty)} : \mathbb{R} \to \{0,1\}$, which is isomorphic to $\bigvee L$ and $\bigwedge L$, i.e., to $\{\top, \perp\}$. Also, $\varphi : \mathbb{R} \to \mathbb{R}(L)$ and $\mathcal{R}_t, : \mathbb{R}(L) \to L$, so the composition $\mathcal{R}_t \circ \varphi : \mathbb{R} \to L \supset \{\top, \perp\}$. Then, the domains and codomains are compatible, and to check that the action of the mappings is the same, let $r \in \mathbb{R}$.

Case 1: $r < t$.

$\chi_{(t,\infty)}(r) = 0 \cong \perp$ of L.

Again, we need to consider, for $\varphi(r) = [\lambda_r]$, which member of the class to use in evaluation. Since $\mathcal{R}_t$ is related to the idea of right hand limits, we will choose the right continuous member of $[\lambda_r]$, call it λ_r. Then,

$$\begin{aligned} (\mathcal{R}_t \circ \varphi)(r) &= \mathcal{R}_t(\varphi(r)) \\ &= \mathcal{R}_t([\lambda_r]) \\ &= \mathcal{R}([\lambda_r], t) \\ &= \lambda_r\left(t^+\right) \\ &= \lambda_r(t) \end{aligned}$$

since λ_r is right continuous. And by how we define λ_r for $r < t$,

$$(\mathcal{R}_t \circ \varphi)(r) = \lambda_r(t) = \perp$$

So, $\forall r < t$, $\chi_{(t,\infty)}(r) = (\mathcal{R}_t \circ \varphi)(r)$, i.e., $\chi_{(t,\infty)} = \mathcal{R}_t \circ \varphi$.

Case 2: $r \geq t$.

$\chi_{(t,\infty)}(r) = 1 \cong \top$ of L. Now,

$$\begin{aligned} (\mathcal{R}_t \circ \varphi)(r) &= \mathcal{R}([\lambda_r], t) \\ &= \lambda_r\left(t^+\right) \\ &= \lambda_r(t) \end{aligned}$$

as before, and recall for $r \geq t$, $\lambda_r(t) = \top$. So $\chi_{(t,\infty)} = \mathcal{R}_t \circ \varphi$ by separation of cases ■.

It is also reasonable to assume that the other projection will have a relationship to the characteristic map.

Proposition 6.6. $\forall r \in \mathbb{R}, \forall [\lambda_r] \in \mathbb{R}(L)$, *(i)* $\mathcal{L}_{[\lambda_r]} = \chi_{(r,\infty)}$, *and (ii)* $\mathcal{R}_{[\lambda_r]} = \chi_{(-\infty,r)}$

Proof: Let $r \in \mathbb{R}$, $[\lambda_r] \in \mathbb{R}(L)$. Note that this class of mappings in the L-fuzzy real line is "defined" by a fixed $r \in \mathbb{R}$. For *(i)*, we will rename the class so that λ_r is the left continuous member of the class. Now, $\forall x \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{L}_{[\lambda_r]}(x) &= \mathcal{L}([\lambda_r], x) \\
&= \left(\lambda_r(x^-)\right)' \\
&= (\lambda_r(x))' \\
&= \left(\begin{cases} \top, & x \leq r \\ \bot, & x > r \end{cases}\right)' \\
&= \begin{cases} \top', & x \leq r \\ \bot', & x > r \end{cases} \\
&= \begin{cases} \bot, & x \leq r \\ \top, & x > r \end{cases} \\
&= \chi_{(r,\infty)}(x)
\end{aligned}$$

and since this is true for each x, $\mathcal{L}_{[\lambda_r]} = \chi_{(r,\infty)}$.

For *(ii)*, we require that the class be named by its right continuous representative, i.e., λ_r is right continuous. So, $\forall x \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{R}_{[\lambda_r]}(x) &= \mathcal{R}([\lambda_r], x) \\
&= \lambda_r(x^+) \\
&= \lambda_r(x) \\
&= (\{\top, x \leq r; \quad \bot, x > r\}) \\
&= \chi_{(r,\infty)}(x)
\end{aligned}$$

so since this holds for each x, $\mathcal{R}_{[\lambda_r]} = \chi_{(-\infty,r)}$ $\blacksquare$.

Hutton created the $\mathbb{I}$-precursor of the L-fuzzy real line, $\mathbb{I}(L)$, circa 1975 [11]. We just substitute $\mathbb{I}$ for $\mathbb{R}$ in our above definitions. We can also have a natural L-topology on the L-fuzzy interval by the same $\mathcal{L}_t$ and $\mathcal{R}_t$ as before, only with the domains $\mathbb{I}$ and codomains $\mathbb{I}(L)$. There have been some interesting results about the fuzzy real line, one of which is the following: If L is a frame, then, $\mathbb{R}(L)$ is sober iff L is a complete Boolean algebra. This is referred to as Meßner's Theorem [17], and is valuable because it links the idea of the fuzzy real line to sobriety.

There are many avenues of exploration available to us when we consider L-fuzzy extensions of existing topological axioms, theorems, classifications, results, and open questions. In some cases, a poslat approach may mean a far simpler proof, as in sobreity vs. L-sobriety. By looking at examples of L-topological spaces, we benefit, by virtue of increased generality together with the richness of examples.

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