GRACEFUL AND HARMONIOUS LABELING OF THE DISJOINT UNION OF CERTAIN CYCLES AND PATHS

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

December, 2001

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ABSTRACT

A vertex *labeling* of a graph G is an assignment f of labels to the vertices of G. Each edge xy is assigned a label depending on the vertex labels $f(x)$ and $f(y)$. One of the best known type of labeling methods is called graceful labeling. A function f is called a *graceful* labeling of a graph G with m edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, m\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Over the past few decades, several different graphs have been gracefully labeled.

Another popular labeling method is harmonious labeling. A function f is called a *harmonious* labeling of a graph G with m edges if it is an injection from the vertices of G to the group of integers modulo m such that when each edge xy is assigned the label $(f(x) + f(y)) \pmod{m}$, the resulting edge labels are distinct. If the graph G is a tree or has components that are trees, then exactly one label may be used on two vertices. In this case f would not be an injection. In this thesis, we present a technique for labeling the disjoint union of the path and some cycles. We use both labeling methods, graceful and harmonious.

AKNOWLEDGEMENTS

I wish to express my gratitude to my thesis advisor, Dr. Kenneth J. Roblee, whose patience and guidance led me to believe in my own abilities. To my mother, without whose constant support none of my work is possible. Finally, I dedicate this thesis in loving memory of my father.

Chapter 1

History and Background

Labeling graphs was popularized in 1967 by Alex Rosa [6]. Rosa called a function f a $\beta - valuation$ of a graph G with m edges if f is an injection from the vertices of a graph G to the set $\{0, 1, \ldots, m\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [2] called this particular method of labeling graceful, and this is the popular term used today. We call a graph "graceful" if such an f exists. Alex Rosa, a design theorist, introduced β-valuations (i.e. graceful labelings) as well as many other labelings as tools for decomposing the complete graph into isomorphic subgraphs. First, these β-valuations were used as a means of attacking Ringel's [5] conjecture that K_{2n+1} can be decomposed into $(2n+1)$-many subgraphs that are all isomorphic to a given tree with n edges. Erdos said most graphs are not graceful, although most graphs with some"regularity of structure" are graceful. Sheppard [7] proved there are exactly $m!$ gracefully labeled graphs with m

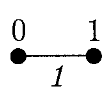
edges. It has also been proven that every graph is a subgraph of a graceful graph. Rosa [6] has given three reasons why a graph fails to be graceful. First of all, a graph may not be graceful if the graph has "too many vertices" and "not enough edges". A simple example would be K_2+K_1. The reason this fails to be graceful is because there are three vertices and only two possible labels, 0 and 1. Second, the graph may have "too many edges". For example, P_5 can be gracefully labeled, but adding an edge between the endpoints of this graph creates a non-graceful graph, namely C_5. Also, a graph that has the "wrong parity" cannot be labeled gracefully. For example, Rosa has proved that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4), then the graph can not be labeled gracefully. Specifically, C_{4n+1} and C_{4n+2} are not graceful.

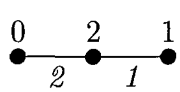
Harmonious graphs came about in the study by Graham and Sloane [3] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with m edges to be *harmonious* if there is an injection from the vertices of G to the group of integers modulo m such that when each edge xy is assigned to the label $(f(x)+f(y)) \pmod{m}$, the resulting edge labels are distinct. If G is a tree or has a component that is a tree, then exactly one label may be used on two vertices and the labeling function is not an injection. Graham and Sloane also proved that if a harmonious graph has an even number m of edges and the degree of every vertex is divisible by 2^k, then m is divisible by 2^{k+1}. This condition has been generalized by Liu and Zhang [4]: If a harmonious graph has m edges and the degree sequence $d_1, d_2, \ldots, d_p$, then $\gcd(d_1, d_2, \ldots, d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

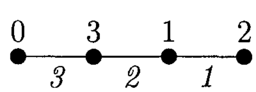
Since the time labeling graphs was introduced, particularly graceful labelings, people have been trying to prove that certain types of graphs can be labeled gracefully. Some have been successful and some have not. For example, in 1964 Ringel and Kotzig [5] made the following conjecture concerning labelings of trees.

Graceful Tree Conjecture *Every tree has a graceful labeling.*

Properly named a conjecture, the Graceful Tree Conjecture has not yet been proven for all trees. However, many families of trees have been gracefully labeled and there have been over 300 papers with varying and scattering methods to prove that a certain class of graphs is or is not graceful. One family of trees is known as the path. It has been proven that paths can always be labeled gracefully. Here are a few small examples: (Note that the number in italics is the induced edge labeling for each graph.)

P_2


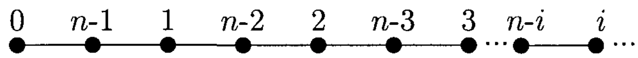
P_3


P_4


We can label the path, denoted P_n, as follows:

n odd:

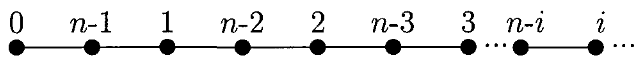
P_n



where $1 \le i \le \frac{n-1}{2}$

n even:

P_n



where $1 \le i \le \frac{n}{2}$

Another tree that has been proven to be graceful is the n-star which is the complete bipartite graph $K_{1,n}$. To gracefully label this graph, first label the center with 0. Then label the remaining vertices $1, \ldots, n$ in succession.

We can also ask if certain cyclic graphs are graceful. Cycles have been a major focus of attention. Rosa [6] proved that C_n (the n-cycle where n is the number of vertices) is graceful if and only if $n \equiv 0, 3 \mod 4$. Graham and Sloane [3] proved that C_n is harmonious if and only if $n \equiv 1, 3 \mod 4$. Here, we investigate the disjoint union of certain cycles and paths.

Chapter 2

Graceful Labeling of the Four or Six Cycle Union the Path

Labeling different types of graphs gracefully has been the focus of many papers. From the last section, we see that the family of trees known as paths are graceful. Here, we ask the question: Is the disjoint union of the path P_n and the cycle C_s, denoted C_s+P_n, graceful? Cycles in general, on the other hand, have not been proven graceful. In this thesis, an algorithm has been formulated to label $C_4 + P_n$ and $C_6 + P_n$.

For the 4-cycle, label a vertex 0 and label its non-neighbor 1. Label the other two vertices on the cycle $n+3$ and $n+1$. The vertex labels on the cycle will give the four highest edge labels. To label the path, start at an endpoint and label it $n+2$, and its neighbor 3. This will give the next highest possible edge label. The objective is to try and label the vertices so that the edge labels decrease in succession. However, this is not always possible. If not possible, then use the vertex label that will give the next highest edge label. At the next step, we will be able to get the edge label we skipped. By continuing this process of trying to get the largest induced edge label without repeating a vertex label and still having all the edge labels distinct, $C_4 + P_n$ is seen to be graceful. For example, we will label $C_4 + P_3$ to illustrate this algorithm. To start, label a vertex on the cycle 0 and its non-neighbor 1. The other two vertices on the cycle are labeled 4 and 6. To label the path start at an endpoint and label the vertex 5 since $n + 2$, where $n = 3$ is 5. Now, label its neighbor 3. This gives us an induced edge label of 2 which is the next highest possible edge label and the cycle gave us induced edge labels of 6, 5, 4, and 3. Since 1 is the only possible induced edge label left, the vertex that is a neighbor of 3 in the path must be labeled 2.

During the proof we will use letters A, B, C, and D to denote a specific part of the piecewise vertex-labeling function f, to be defined in the proof. We will also be using these letters to denote the edges used in the proof. We will use, for example, $C - A$ to denote the edge that is computed when we subtract formula A from formula C in the function f.

Theorem 1 $C_4 + P_n$, when n is not congruent to 2 modulo 3, is graceful.

Proof. To show that $C_4 + P_n$ is graceful, we present a vertex labeling function g of $C_4 + P_n$ with m edges that is one-to-one from the vertices of $C_4 + P_n$ to the set $\{0, 1, \ldots, m\}$ such that, when each edge is assigned the label $|g(i) - g(i + 1)|$, the resulting edge labels are distinct. We note that m in this case is $n + 3$.

We begin by labeling the 4-cycle as follows: label two of the non-adjacent vertices $n+3$ and $n+1$, and label the other vertices 0 and 1. Note that this results in induced edge labels of $n+3$, $n+2$, $n+1$, and n.

Next, we present a labeling function for the path P_n. Define i to be the i^{th} vertex starting at, say, the right endpoint of the path, and so $1 \leq i \leq n$. Let

$$f(i) = \begin{cases} 3 & \text{if } i{=}2 \\ (i-2) - 6\lfloor \frac{i-4}{12} \rfloor & \text{if } i \equiv 4, 6, 8 \pmod{12} \\ (i-5) - 6\lfloor \frac{i-10}{12} \rfloor & \text{if } i \equiv 0, 2, 10 \pmod{12} \text{ (except } i{=}2) \\ n - ((i-3) - 6\lfloor \frac{i+1}{12} \rfloor) & \text{if } i \equiv 1, 3, 5 \pmod{12} \\ n - ((i-6) - 6\lfloor \frac{i-7}{12} \rfloor) & \text{if } i \equiv 7, 9, 11 \pmod{12} \end{cases} \tag{2.1}$$

We first show that f is one-to-one. Actually, there are some cases where it is not one-to-one. We identify these cases and handle them separately in the lemma following the proof. We have several cases to check. In particular we have to show (1) that if $i \neq j$ and one computes $f(i)$ and $f(j)$ using the same formula in the definition of f above, then $f(i) \neq f(j)$ (note that there are four cases here) and (2) that if we use different formulas to compute $f(i)$ and $f(j)$, then we always have $f(i) \neq f(j)$ (note that there are six cases to check here). To prove (1), we note that formulas A and B are increasing functions, and that formulas C and D are decreasing functions. Thus it is clear that $f(i) \neq f(j)$ provided $i \neq j$ and if $f(i)$ and $f(j)$ use the same formula.

To prove (2), we first compare formulas A and B. Note that if $f(i)$ is computed by formula A and $f(j)$ is computed using formula B, then $f(i)$ is even while $f(j)$ is odd. Thus, they are not equal. Similarly, if we next compare formulas C and D, formula C will yield n - (an even number) while formula B yields n - (an odd number), thus they will not equal. Now to the more involved cases.

Case 1. We compare formulas A and C. Noting, as above, that formula A yields even outputs, and formula C yields n - (even) outputs, then it must be true that n is even. Note that the assumption $n \equiv 0, 1 \pmod 3$ implies that $n \equiv 0, 1, 3, 4, 6, 7, 9, 10 \pmod{12}$. The last two observations imply that $n \equiv 0, 4, 6, 10 \pmod{12}$. Thus, we have four subcases to check:

Subcase (i): $n = 12k$, for some integer $k > 0$. We prove that a formula A label is always less than a formula C label. Observe that the the maximum value of a formula A output is $(12k - 4 - 2) - 6\lfloor \frac{12k-4-4}{12} \rfloor = 6k$ (note that we used $i = 12k - 4$ because $i \equiv 4, 6, 8 \pmod{12}$ and this would be the largest value of i accepted as input). The minimum value of formula C is $12k - [(12k - 7 - 3) - 6\lfloor \frac{12k-7+1}{12} \rfloor] = 6k + 4$. Thus the minimum value of C exceeds the maximum value of A. However, by the nature of the graph we are considering, the labeling must be onto. Since there is a gap of four between the maximum of A and the minimum of C and since these are the only even label-producing formulas, some odd label must have been repeated. We resolve this problem in the lemma following the proof.

Subcase (ii): $n = 12k + 4$. Using the same argument as in (i), we see that the maximum value of A is $6k + 2$ while the minimum value of C is $6k + 4$ (Note that since there is only a gap of two, the problem mentioned in (i) does not occur).

Subcase (iii): $n = 12k + 6$. We have the maximum value of A is $6k + 4$ while the minimum value of C is $6k + 4$. We consider this later in a separate lemma.

Subcase (iv): $n = 12k + 1$. We have the maximum value of A is $6k + 6$ while the minimum value of C is $6k + 8$.

Case 2. Next, we compare formulas A and D. As reasoned above in the case 1, here we must have n odd in order for formula D to output an even number. Thus, we consider $n \equiv 1, 3, 7, 9 \pmod{12}$, and reason as in the four subcases above.

Subcase (i): $n = 12k+1$. The maximum value of A is $6k$ and the minimum value of D is $6k + 2$.

Subcase (ii): $n = 12k+3$. The maximum value of A is $6k$ and the minimum value of D is $6k + 4$. Again, see the lemma.

Subcase iii: $n = 12k + 7$. The maximum value of A is $6k + 4$ and the minimum value of D is $6k + 6$.

Subcase (iv): $n = 12k + 9$. The maximum value of A is $6k + 6$ and the minimum value of D is $6k + 6$. We consider this case also in the lemma below.

Case 3. We next compare formulas B and C. Here we must have n odd for the parities to be the same. So $n \equiv 1, 3, 7, 9 \pmod{12}$.

Subcase (i): $n = 12k + 1$. The maximum value of formula B is $6k + 1$ and the minimum value of formula C is $6k+3$.

Subcase (ii): $n = 12k+3$. The maximum value of formula B is $6k+3$ and the minimum value of formula C is $6k+3$. As earlier, we treat this separately.

Subcase (iii): $n = 12k+7$. The maximum value of formula B is $6k+3$ and the minimum value of formula C is $6k+5$.

Subcase (iv): $n = 12k+9$. The maximum value of formula B is $6k+3$ and the minimum value of formula C is $6k+7$. See the lemma.

Case 4. Finally, we compare formulas B and D, noting that n must be one of $0, 4, 6, 10 \pmod{12}$. Again, we consider all subcases.

Subcase (i): $n = 12k$. The maximum value of formula B is $6k+1$ and the minimum value of formula D is $6k+1$. This case will also be reconsidered in the lemma after the proof.

Subcase (ii): $n = 12k+4$. The maximum value of formula B is $6k+3$ and the minimum value of formula D is $6k+5$.

Subcase (iii): $n = 12k+6$. The maximum value of formula B is $6k+3$ and the minimum value of formula D is $6k+7$. See the lemma.

Subcase (iv): $n = 12k+10$. The maximum value of formula B is $6k+5$ and the minimum value of formula D is $6k+7$.

Now, we show $|f(i) - f(i+1)| \neq |f(j) - f(j+1)|$. Again, there are many cases to check, so we use Roman numerals to distinguish them from the cases above.

Case I: We compare the $C-A$ edge and the $D-A$ edge. We show $|f(i)-f(i+1)| \neq$

$|f(j) - f(j+1)|$, where $i \neq j$. The $C - A$ edge is $|n - ((i-3) - 6\lfloor \frac{i+1}{12} \rfloor)) - ((i+1-2) - 6\lfloor \frac{i+1-4}{12} \rfloor)|$. The $D - A$ edge is $|n - ((j-6) - 6\lfloor \frac{j-7}{12} \rfloor)) - ((j+1-2) - 6\lfloor \frac{j+1-4}{12} \rfloor)|$.

Subcase (i): Both i-floors and the j-floors are the same. For this subcase, we are assuming that both i-floors and the j-floors are the same. Since the floor functions are assumed to be the same, it does not matter which one is used in the simplification. After simplifying the $C - A$ edge we get $|n - 2i + 4 + 12\lfloor \frac{i+1}{12} \rfloor|$. After simplifying the $D - A$ edge we get $|n - 2j + 7 + 12\lfloor \frac{j-7}{12} \rfloor|$.

Now, $12\lfloor \frac{i+1}{12} \rfloor$ and $12\lfloor \frac{j-7}{12} \rfloor$ must differ by a multiple of 12. (Since when the floor functions are rounded down, they differ by an integer.) If we set the two induced edge labelings equal, then $-2i = -2j + 3 + 12m$ for some integer m. But this is impossible since the left hand side is even and the right hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $C - A$ edge and the $D - A$ edge which does not change the parity. We get the same result as in subcase (i).

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $C - A$ edge only. This does not change the parity since 6 is even. So, we get the same situation as in subcase (ii).

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $D - A$ edge only. This does not change the parity since 6 is even. This is the same as in subcase (i).

Hence, the $C - A$ edge and the $D - A$ edge are never equal.

Case II: We compare the $C-B$ edge and the $D-B$ edge. The $C-B$ edge is $|n-((i-3)-6\lfloor\frac{i+1}{12}\rfloor)-((i+1-5)-6\lfloor\frac{i+1-10}{12}\rfloor)|$. The $D-B$ edge is $|n-((j-6)-6\lfloor\frac{j-7}{12}\rfloor)-((j+1-5)-6\lfloor\frac{j+1-10}{12}\rfloor)|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $C-B$ edge we get $|n-2i+7+12\lfloor\frac{i+1}{12}\rfloor|$. After simplifying the $D-B$ edge we get $|n-2j+10+12\lfloor\frac{j-7}{12}\rfloor|$.

Now, $12\lfloor\frac{i+1}{12}\rfloor$ and $12\lfloor\frac{j-7}{12}\rfloor$ must differ by a multiple of 12. Since the floor functions differ by an integer, $-2i=-2j+3+12m$ for some integer m. But this is impossible since the left hand side is even and the right hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $C-B$ edge and the $D-B$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $C-B$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $D-B$ edge only. This does not change the parity since 6 is even. Hence, the $C-B$ edge and the $D-B$ edge are never equal.

Case III: We now compare the $A-C$ edge and the $B-C$ edge. The $A-C$ edge is $|(i-2)-6\lfloor\frac{i-4}{12}\rfloor))-(n-(i+1-3\lfloor\frac{i+1+1}{12}\rfloor)|$. The $B-C$ edge is $|((j-5)-6\lfloor\frac{j-10}{12}\rfloor)-(n-((j+1-3)-6\lfloor\frac{j+1+1}{12}\rfloor))|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $A - C$ edge we get $|2i - n - 4 - 12\lfloor \frac{i-4}{12} \rfloor|$. After simplifying the $B - C$ edge we get $|2j - n - 7 - 12\lfloor \frac{j-10}{12} \rfloor|$.

Now, $12\lfloor \frac{i-4}{12} \rfloor$ and $12\lfloor \frac{j-10}{12} \rfloor$ must differ by a multiple of 12. (Since floor functions differ by an integer.) Thus, $2i = 2j - 3 + 12m$ for some integer m. But this is impossible since the right hand side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $A - C$ edge and the $B - C$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $A - C$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $B - C$ edge only. This does not change the parity since 6 is even. Hence, the $A - C$ edge and the $B - C$ edge are never equal.

Case IV: We compare the $A - D$ edge and the $B - D$ edge. The $A - D$ edge is $|(i-2) - 6\lfloor \frac{i-4}{12} \rfloor) - (n - ((i+1-6) - 6\lfloor \frac{i+1-7}{12} \rfloor))|$. The $B - D$ edge is $|(j-5) - 6\lfloor \frac{j-10}{12} \rfloor) - (n - ((j+1-6) - 6\lfloor \frac{j+1-7}{12} \rfloor))|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $A - D$ edge we get $|2i - n - 7 + 12\lfloor \frac{i-4}{12} \rfloor|$. After simplifying the $B - D$ edge we get $|2j - n - 10 + 12\lfloor \frac{j-10}{12} \rfloor|$.

Now, $12\lfloor\frac{i-4}{12}\rfloor$ and $12\lfloor\frac{j-10}{12}\rfloor$ must differ by a multiple of 12. Thus, $2i = 2j-3+12m$ for some integer m. But this is impossible since the right hands side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $A-D$ edge and the $B-D$ edge which does not change the parity. We get the same result as in Subcase (i).

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $A-D$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $B-D$ edge only. This does not change the parity since 6 is even. Hence, the $A-D$ edge and the $B-D$ edge are never equal.

Case V: We compare the $D-A$ edge and the $C-B$ edge. The $D-A$ edge is $|n-((i-6)-6\lfloor\frac{i-7}{12}\rfloor))-((i+1-2)-6\lfloor\frac{i+1-4}{12}\rfloor)|$. The $C-B$ edge is $|n-((j-3)-6\lfloor\frac{j+1}{12}\rfloor))-((j+1-5)-6\lfloor\frac{j+1-10}{12}\rfloor)|$. In this case, all possible values for i make the floor functions in the $D-A$ edge round down to the same value. Also, all possible values for j make the floor functions round down to different values. So the only case to consider is when the i-floors are the same and the j-floors are different. Here we subtract 6 from the $C-B$ edge only. After simplifying the $D-A$ edge we get $|n-2i+7+12\lfloor\frac{i-7}{12}\rfloor|$. After simplifying the $C-B$ edge we get $|n-2j+1+12\lfloor\frac{j+1}{12}\rfloor|$.

Now, $12\lfloor\frac{i-7}{12}\rfloor$ and $12\lfloor\frac{j+1}{12}\rfloor$ must differ by a multiple of 12. (Since when the floor functions are rounded down, they differ by at most 1) Thus, $-2i = -2j - 6 + 12m$ for some integer m.

But, the only values for i are $i \equiv 7, 9, 11 \pmod{12}$ and the only values for j are $j \equiv 1, 3, 5 \pmod{12}$. So, $-2i \equiv -2j \mod 12$. Since any multiple of 12 is zero modulo 12, then $-2i$ and $-2j + 6$ are the same mod 12. This is impossible since $2i \equiv 2, 6, 10 \pmod{12}$ and $2j \equiv 2, 6, 10 \pmod{12}$ and the only possible differences are $4, 8 \pmod{12}$. Hence, the $D - A$ edge and the $C - B$ edge are never equal.

Case VI: We compare the $A - C$ edge and the $B - D$ edge. The $A - C$ edge is $|(i-2) - 6\lfloor\frac{i-4}{12}\rfloor - (n - ((i+1-3) - 6\lfloor\frac{i+1+1}{12}\rfloor))|$. The $B - D$ edge is $|(j-5) - 6\lfloor\frac{j-10}{12}\rfloor) - (n - ((j+1-6) - 6\lfloor\frac{j+1-7}{12}\rfloor))|$. In this case, all possible values for i make the floor functions in the $A - C$ edge round down to the same value. Also, all possible values for j make the floor functions round down to the same values. So the only case to consider is when the i-floors are the same and the j-floors are the same. After simplifying the $A - C$ edge we get $|2i - n - 4 + 12\lfloor\frac{i-4}{12}\rfloor|$. After simplifying the $B - D$ edge we get $|2j - n - 10 + 12\lfloor\frac{j-10}{12}\rfloor|$.

Now, $12\lfloor\frac{i-4}{12}\rfloor$ and $12\lfloor\frac{j-10}{12}\rfloor$ must differ by a multiple of 12. Thus, $2i = 2j - 6 + 12m$ for some integer m.

But, the only values for i are $i \equiv 4, 6, 8 \mod 12$ and the only values for j are $j \equiv 0, 2, 10 \mod 12$. So, $2i \equiv 2j \mod 12$. Since any multiple of 12 is zero modulo 12, then we have $2i$ and $2j - 6$ modulo 12. This is impossible since $2i \equiv 0, 4, 8 \mod 12$ and $2j \equiv 0, 4, 8 \mod 12$ and the only possible differences are $4, 8 \mod 12$.

Hence, the $A - C$ edge and the $B - D$ edge are never equal, and so $C_4 + P_n$, when n is not congruent to 2 modulo 3, is graceful.

□

Lemma For the subcases that were postponed in the proof above, the function f can be modified to give a graceful labeling.

Proof. We consider the subcases that we postponed above; recall that these were the subcases where the maximum value of one labeling formula was equal to the minimum value of some other labeling function. As usual, we do each case separately, with each one actually being a subcase of its respective case.

Case 1. First we postponed the subcase $n = 12k + 6$ comparing the A and C labels. There, we found that the maximum of the A labels is $6k + 4$, which is also the minimum value of the C labels. This occurs at vertices $12k + 6$ and $12k + 5$, respectively. These are the last two vertices in the path. We re-label the last vertex in the path to $6k + 5$, giving an induced label of the last edge of 1. We now prove that $6k + 5$ has not been previously used as a vertex label. Note that $6k + 5$ is odd, so could not have been produced by the A or C formulas. Thus considering the B and D formulas, we see their respective maximum and minimum values are $6k + 3$ and $6k + 7$, thus $6k + 5$ has not been previously used.

We also have to show that 1 has not previously been used as an edge label. Observe that the smallest edge labels occur whenever the A, B vertices are maximized and the C, D vertices are minimized. We note that the maximum values of A and B are respectively $6k+4$ and $6k+3$, and that the minimum values of C and D labels are $6k+4$ and $6k+7$. The only possible scenario giving an edge label of 1 would be an edge whose endpoints are labeled with formulas B and C, but note that the vertices where their respective maximum and minimum values occur are $12k+2$ and $12k+5$, which are not the endpoints of an edge (it is worth pointing out that although the A and B maximum values differ by 1, the formulas only accept even inputs, so that no two adjacent vertices are labeled with formulas A and B). Thus, 1 does not previously appear as an edge label. And so our choice of $6k+5$ as the last vertex label is justified.

Note also the subcase (i) in the proof where there was a gap of four, and there we noted that this implies some odd label is repeated. One can easily verify that the repeated vertex is $6k+1$, and this repetition occurs on the last two vertices in the path. So we may change the last vertex label to $6k+2$, thus giving the last induced edge label of 1. Also, notice that the edge label of 1 has not previously occurred; this is seen by observing the respective maximum and minimum values of A, B, C, and D, as in the previous subcase in the last paragraph.

Case 2. Second, we postponed the subcase comparing the labels produced by the A formula and the D formula, where $n = 12k+9$. There, we discovered that the maximum value of formula A is $6k+6$, which is obtained by plugging in $12k+8$, and the minimum value of formula D is also $6k+6$, obtained by plugging in $12k+9$. Note that these are the last two vertices of P_{12k+9}. So, re-label the last vertex $6k+5$, giving the last induced edge label of 1. Notice that $6k+5$ was not used in the labeling previously. The reason is two-fold: (1) $6k+5$ is odd and formulas A and D yield even outputs and (2) so considering formulas B and C, we see that the maximum value of B is $6k+3$ and the minimum value of C is $6k+7$, thus leaving $6k+5$.

We claim that 1 is not an induced edge label. As noted above, the maximum value of the A labels is $6k+6$, the maximum value of the B labels is $6k+3$, the minimum value of the C labels is $6k+7$, and the minimum value of the D labels is $6k+6$. As reasoned in the last case (see the parenthetical note in that case as to how we eliminate the other possibilities) we only consider the maximum A label to the minimum C label. Note, however, that the maximum A label is not adjacent to the minimum C label because the maximum A label occurs at the $12k+8$ vertex and the minimum C label occurs at the $12k+5$ vertex, which are non-adjacent. Thus, the edge label 1 has not appeared thus far. Thus, our choice of the $12k+9$ vertex being re-labeled $6k+5$ is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

Case 3. Next, we consider the subcase comparing the labels produced by the B formula with the labels produced by the C formula when $n = 12k+3$. We discovered in this case that the maximum and minimum values are both $6k+3$, which occurs at the $12k + 2$ and $12k + 3$ vertices, respectively, for B and C, which are the last two vertices in the path. Now, re-label the last vertex in the path $6k + 2$, giving 1 as the last induced edge label. Again, note that $6k + 2$ was not previously used as a vertex label. This is because (1) $6k + 2$ is even and formulas B and C give odd labels and so (2) considering formulas A and D, we see their respective maximum and minimum values are $6k$ and $6k + 4$, thus leaving $6k + 2$.

We also must check that 1 has not previously been used as an induced edge label. To do this, we proceed as in the previous case. The maximum values of A and B are $6k$ and $6k + 3$ respectively. The minimum values of C and D are $6k + 3$ and $6k + 4$, respectively. Note that the maximum B value occurs at vertex $12k + 2$ and the minimum D value occurs at vertex $12k - 1$, so they are non-adjacent, and so will not produce a 0 induced edge label. Again, 1 was not previously an induced edge label, so our re-labeling is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

Case 4. Finally, we look at the subcase involving labels produced by the B formula and those produced by the D formula, where $n = 12k$. In that case, the respective maximum and minimum labels are $6k+1$, and they occur at vertices $12k$ and $12k-1$, respectively. So, we change the label of the last vertex to $6k + 2$, giving the last edge of the path the induced edge label 1. We again show that this is a permissible change.We note that (1) $6k + 2$ is even, and formulas B and D yield odd labels and so (2) considering formulas A and C, we see their respective maximum and minimum values are $6k$ and $6k+4$, respectively. Thus, $6k+2$ has not yet appeared as a vertex label.

We also have to show that 1 has not previously occurred as an induced edge label. The maximum values of A and B are, respectively, $6k$ and $6k + 1$. The minimum values of C and D are, respectively, $6k+4$ and $6k+1$. Note that the maximum value of A occurs at vertex $12k - 8$ and the minimum value of D occurs at vertex $12k - 1$, which are non-adjacent. Since this is the only case that might give an induced edge label of 1 and the vertices are non-adjacent, this induced edge label has yet to be used. Thus, our re-labeling technique is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

□

Label $C_6 + P_n$ in a similar fashion. To illustrate how to label the cycle, we will start by naming the six vertices consecutively a, b, c, d, e, and f. Now, let a, c, and e be labeled 0, 1, and 3 respectively. Then label b, d, and f with $n + 5$, $n + 2$, and $n+3$ respectively. To label the path part of the graph, label an endpoint $n+4$. Then label the next vertex 2. This will give the next highest edge label. We continue to label the vertices trying to get the next highest induced edge label at each step. If this is not possible, then try to get the next highest, and wait until the next vertex to get the edge label we skipped. This process will always work. Then $C_6 + P_n$, when n is not congruent to 0 modulo 3, will be seen to be graceful.

Theorem 2 $C_6 + P_n$, when n is not congruent to 0 modulo 3, is graceful.

Proof. We first begin by labeling the 6-cycle as in the paragraph preceding the statement of the theorem.

Next, we define i to be the i^{th} vertex starting at one endpoint of the path. Let

$$f(i) = \begin{cases} i - 6\lfloor \frac{i-2}{12} \rfloor & \text{if } i \equiv 2, 4, 6 \pmod{12} \\ (i-3) - 6\lfloor \frac{i-8}{12} \rfloor & \text{if } i \equiv 0, 8, 10 \pmod{12} \\ n - ((i-9) - 6\lfloor \frac{i-5}{12} \rfloor) & \text{if } i \equiv 1, 3, 11 \pmod{12} \\ n - ((i-12) - 6\lfloor \frac{i-13}{12} \rfloor) & \text{if } i \equiv 5, 7, 9 \pmod{12} \end{cases} \tag{2.2}$$

We first show that the function f defined above is one-to-one. Again, it is not always one-to-one, but we do point out the cases where it is not so, and we fix these cases in the lemma after the proof. We do this in a manner similar to the proof of Theorem 1. We first note that the A and B formulas are increasing, so that if $i \neq j$, and $f(i)$ and $f(j)$ are both computed by formula A or by formula B, then $f(i) \neq f(j)$. Similarly, the C and D formulas are decreasing, and by a similar argument, $f(i) \neq f(j)$

if they use the same formula.

Next, note that formula A yields even labels and formula B yields odd labels, thus formula A never equals formula B. Similarly, formula C and formula D are never equal.

Thus we check whether formula A ever equals formula C or D, and whether formula B ever equals formula C or D, and if so, we modify our labeling to make $C_6 + P_n$ graceful.

Case 1. We first compare A labels to C labels. Since A yields even labels, we only consider C labels that are even. Since $n \equiv 1, 2 \pmod 3$, we have $n \equiv 1, 2, 4, 5, 7, 8, 10, 11 \pmod{12}$. For C labels to be even, n must be even. Thus $n \equiv 2, 4, 8, 10 \pmod{12}$. We consider each case separately.

Subcase (i) $n = 12k + 2$. Proceeding as in the proof of Theorem 1, the maximum value of formula A is $6k + 2$ and the minimum value of formula C is $6k + 4$.

Subcase (ii) $n = 12k + 4$. The maximum value of formula A is $6k + 4$ and the minimum value of formula C is $6k+4$. We postpone this case to a separate discussion (see the lemma after the proof).

Subcase (iii) $n = 12k + 8$. The maximum value of A is $6k + 6$ and the minimum value of C is $6k + 8$.

Subcase (iv) $n = 12k + 10$. The maximum value of A is $6k + 6$ and the minimum value of C is $6k + 10$. Again, there is a gap of four, so we pass this along to the lemma.

Case 2. We next compare formula A labels to formula D labels. In order for D labels to be even, n must be odd. Thus, $n \equiv 1, 5, 7, 11 \pmod{12}$. Again, we consider each case separately.

Subcase (i) $n = 12k + 1$. The maximum value of A is $6k$ and the minimum value of D is $6k + 4$. Again, see the lemma.

Subcase (ii) $n = 12k + 5$. The maximum value of A is $6k + 4$ and the minimum value of D is $6k + 6$.

Subcase (iii) $n = 12k + 7$. The maximum value of A is $6k + 6$ and the minimum value of D is also $6k + 6$. See the lemma after the proof for more on this subcase.

Subcase (iv) $n = 12k + 11$. The maximum value of A is $6k + 6$ and the minimum value of D is $6k + 8$.

Case 3. Next, we compare formula B to formula C. Thus we require n to be odd, so $n \equiv 1, 5, 7, 11 \pmod{12}$.

Subcase (i) $n = 12k+1$. The maximum value of B is $6k+3$ and so is the minimum value of C. Again, see the lemma below.

Subcase (ii) $n = 12k + 5$. The maximum value of B is $6k + 3$ and the minimum value of C is $6k + 5$.

Subcase (iii) $n = 12k + 7$. The maximum value of B is $6k + 3$ and the minimum value of C is $6k + 7$. See the lemma.

Subcase (iv) $n = 12k + 11$. The maximum value of B is $6k + 7$ and the minimum value of C is $6k + 9$.

Case 4. Finally, we compare formula B to formula D. In this case, n must be even, so that $n \equiv 2, 4, 8, 10 \pmod{12}$.

Subcase (i) $n = 12k+2$. Here, the maximum value of B is $6k+3$ and the minimum value of D is $6k+5$.

Subcase (ii) $n = 12k+4$. The maximum value of B is $6k+3$ and the minimum value of D is $6k+7$. Once more, see the lemma.

Subcase (iii) $n = 12k+8$. The maximum value of B is $6k+5$ and the minimum value of D is $6k+7$.

Subcase (iv) $n = 12k+10$. The maximum value of B is $6k+7$ and so is the minimum value of D. Once again, see the lemma below.

Now, we show that all the edge labels are distinct.

Case I: We compare the $C-A$ edge and the $D-A$ edge. We show that $|f(i) - f(i+1)| \neq |f(j) - f(j+1)|$, where $i \neq j$. The C-A edge is $|n - ((i-9) - 6\lfloor \frac{i-5}{12} \rfloor) - (i+1-6\lfloor \frac{i+1-2}{12} \rfloor)|$. The D-A edge is $|n - ((j-12) - 6\lfloor \frac{j-13}{12} \rfloor) - (j+1-6\lfloor \frac{j+1-2}{12} \rfloor)|$.

Subcase (i): Both i-floors and the j-floors are the same. We add the two floor functions in both edges using the first specific floor function. Since the floor functions are assumed to be the same, it does not matter which one is used in the simplification. After simplifying the $C-A$ edge we get $|n - 2i + 8 + 12\lfloor \frac{i-5}{12} \rfloor|$. After simplifying the $D-A$ edge we get $|n - 2j + 11 + 12\lfloor \frac{j-13}{12} \rfloor|$.

Now, $12\lfloor \frac{i-5}{12} \rfloor$ and $12\lfloor \frac{j-13}{12} \rfloor$ must differ by a multiple of 12. Thus, $-2i = -2j + 3 + 12m$ for some integer m. But this is impossible since the right hand side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $C-A$ edge and the $D-A$ edge which does not change the parity. We get the same result as in Subcase (i).

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $C-A$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $D-A$ edge only. This does not change the parity since 6 is even.

Hence, the $C-A$ edge and the $D-A$ edge are never equal.

Case II: We compare the $C-B$ edge and the $D-B$ edge. The $C-B$ edge is $|n-((i-9)-6\lfloor\frac{i-5}{12}\rfloor)-((i+1-3)-6\lfloor\frac{i+1-8}{12}\rfloor)|$. The $D-B$ edge is $|n-((j-12)-6\lfloor\frac{j-13}{12}\rfloor)-((j+1-3)-6\lfloor\frac{j+1-8}{12}\rfloor)|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $C-B$ edge we get $|n-2i+11+12\lfloor\frac{i-5}{12}\rfloor|$. After simplifying the $D-B$ edge we get $|n-2j+14+12\lfloor\frac{j-13}{12}\rfloor|$.

Now, $12\lfloor\frac{i-5}{12}\rfloor$ and $12\lfloor\frac{j-13}{12}\rfloor$ must differ by a multiple of 12. Thus, $-2i=-2j+3+12m$ for some integer m. But this is impossible since the right hand side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $C - B$ edge and the $D - B$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $C - B$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $D - B$ edge only. This does not change the parity since 6 is even.

Hence, the $C - B$ edge and the $D - A$ edge are never equal.

Case III: We compare the $A - C$ edge and the $B - C$ edge. The $A - C$ edge is $|(i - 6\lfloor\frac{i-2}{12}\rfloor)) - ((n - ((i + 1 - 9) - 6\lfloor\frac{i+1-5}{12}\rfloor))|$. The $B - C$ edge is $|((j - 3) - 6\lfloor\frac{j-8}{12}\rfloor)) - (n - ((j + 1 - 9) - 6\lfloor\frac{j+1-5}{12}\rfloor))|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $A - C$ edge we get $|2i - n - 8 - 12\lfloor\frac{i-2}{12}\rfloor|$. After simplifying the $B - C$ edge we get $|2j - n - 11 - 12\lfloor\frac{j-8}{12}\rfloor|$.

Now, $12\lfloor\frac{i-2}{12}\rfloor$ and $12\lfloor\frac{j-8}{12}\rfloor$ must differ by a multiple of 12. Thus, $2i = 2j - 3 + 12m$ for some integer m. But this is impossible since the right hand side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $A - C$ edge and the $B - C$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $A - C$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $B - C$ edge only. This does not change the parity since 6 is even.

Hence, the $A - C$ edge and the $B - C$ edge are never equal.

Case IV: We compare the $A - D$ edge and the $B - D$ edge. The $A - D$ edge is $|(i - 6\lfloor \frac{i-2}{12} \rfloor) - (n - ((i + 1 - 12) - 6\lfloor \frac{i+1-13}{12} \rfloor))|$. The $B - D$ edge is $|(j - 3) - 6\lfloor \frac{j-8}{12} \rfloor - (n - ((j + 1 - 12) - 6\lfloor \frac{j+1-13}{12} \rfloor))|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $A - D$ edge we get $|2i - n - 11 - 12\lfloor \frac{i-2}{12} \rfloor|$. After simplifying the $B - D$ edge we get $|2j - n - 14 - 12\lfloor \frac{j-8}{12} \rfloor|$.

Now, $12\lfloor \frac{i-2}{12} \rfloor$ and $12\lfloor \frac{j-8}{12} \rfloor$ must differ by a multiple of 12. Thus, $2i = 2j - 3 + 12m$ for some integer m. But this is impossible since the right hand side is even and the left hand side is odd.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $A - D$ edge and the $B - D$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. In this subcase, we subtract 6 from the $A - D$ edge only. This does not change the parity since 6 is even.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, we subtract 6 from the $B-D$ edge only. This does not change the parity since 6 is even.

Hence, the $A-D$ edge and the $B-D$ edge are never equal.

Case V: We compare the $D-A$ edge and the $C-B$ edge. The $D-A$ edge is $|n-((i-12)-6\lfloor\frac{i-13}{12}\rfloor))-(i+1-6\lfloor\frac{i+1-2}{12}\rfloor)|$. The $C-B$ edge is $|(n-((j-9)-6\lfloor\frac{j-5}{12}\rfloor))-((j+1-3)-6\lfloor\frac{j+1-8}{12}\rfloor)|$. In this case, all possible values for i make the floor functions in the $D-A$ edge round down to different values. Also, all possible values for j make the floor functions round down to the same values. So the only case to consider is when the i-floors are different and the j-floors are the same. Here we subtract 6 from the $D-A$ edge only. After simplifying the $D-A$ edge we get $|n-2i+5+12\lfloor\frac{i-13}{12}\rfloor|$. After simplifying the $C-B$ edge we get $|n-2j+11+12\lfloor\frac{j-5}{12}\rfloor|$.

Now, $12\lfloor\frac{i-13}{12}\rfloor$ and $12\lfloor\frac{j-5}{12}\rfloor$ must differ by a multiple of 12. Thus, $-2i=-2j+6+12m$ for some integer m.

But, the only values for i are $i\equiv 5,7,9 \mod 12$ and the only values for j are $j\equiv 1,3,11 \mod 12$. So, $-2i\equiv -2j \mod 12$. Since any multiple of 12 is zero modulo 12, then we have that $-2i$ and $-2j+6$ are the same modulo 12. This is impossible since $2i\equiv 2,6,10 \mod 12$ and $2j\equiv 2,6,10 \mod 12$ and the only possible differences are $4,8 \mod 12$.

Hence, the $D-A$ edge and the $C-B$ edge are never equal.

Case VI: We compare the $A-C$ edge and the $B-D$ edge. The $A-C$ edge is $|(i-6\lfloor\frac{i-2}{12}\rfloor)) - (n-((i+1-9)-6\lfloor\frac{i+1-5}{12}\rfloor))|$. The $B-D$ edge is $|(j-3)-6\lfloor\frac{j-8}{12}\rfloor) - (n-((j+1-12)-6\lfloor\frac{j+1-13}{12}\rfloor))|$.

Subcase (i): Both i-floors and the j-floors are the same. After simplifying the $A-C$ edge, we get $|2i-n-8-12\lfloor\frac{i-2}{12}\rfloor|$. After simplifying the $B-C$ edge we get $|2j-n-11-12\lfloor\frac{j-8}{12}\rfloor|$.

Now, $12\lfloor\frac{i-2}{12}\rfloor$ and $12\lfloor\frac{j-8}{12}\rfloor$ must differ by a multiple of 12. Thus, $2i = 2j-6+12m$ for some integer m.

But, the only values for i are $i \equiv 2,4,6 \mod 12$ and the only values for j are $j \equiv 0,8,10 \mod 12$. So, $2i \equiv 2j \mod 12$. Since any multiple of 12 is zero modulo 12, then we have that $2i$ and $2j-6$ are the same modulo 12. This is impossible since $2i \equiv 0,4,8 \mod 12$ and $2j \equiv 0,4,8 \mod 12$ and the only possible differences are $4,8 \mod 12$.

Subcase (ii): Both the i-floors and the j-floors are different. In this subcase, we are subtracting 6 from both the $A-C$ edge and the $B-D$ edge which does not change the parity. We get the same result as in Subcase I.

Subcase (iii): The i-floors are different and the j-floors are the same. This subcase occurs when $i \equiv 2 \mod 12$ and $j \equiv 0 \mod 12$. This implies that $2i \equiv 4 \mod 12$ and $2j \equiv 0 \mod 12$. Since $2i \equiv 4 \mod 12$ then $2i \equiv 4 \mod 24$, and since $2j \equiv 0 \mod 12$ then $2i \equiv 0 \mod 24$. But we know that $4 \mod 24$ is not the same as $0 \mod 24 + 12m$ for some integer m.

Subcase (iv): The j-floors are different and the i-floors are the same. In this subcase, $i \equiv 4, 6 \mod 12$ and $j \equiv 8, 10 \mod 12$. This implies that $2i \equiv 0, 8 \mod 12$ and $2j \equiv 4, 8 \mod 12$. Since $2i \equiv 0, 8 \mod 12$ then $2i \equiv 8, 12 \mod 24$, and since $2j \equiv 4, 8 \mod 12$ then $2i \equiv 16, 20 \mod 24$. Given this information, we need to know if $8, 12 \mod 24$ is the same as $16, 20 \mod 24 + 12m$ for some integer m. If both the i-floors and the j-floors are the same, then their parity is the same and then they will differ by an even number. So, we have that both floor functions are a multiple of 12. If we take the difference of these floor functions, then we have the product of 12 and an even number. Then we have 24 times an integer k. Thus, $8, 12 \mod 24$ is not the same as $16, 20 \mod 24$.

If the i-floors and the j-floors have a different parity, then there are two cases to consider, $i < j$ and $j < i$.

If $i < j$, let $i = 4$ and $j = 10$. After simplifying, we get $8 = 20$. This is impossible. So if $i < j$, then $2i + 12\lfloor \frac{i-2}{12} \rfloor$ is not the same as $2j + 12\lfloor \frac{j-8}{12} \rfloor$ modulo 12.

If $j < i$, let $i = 16$ and $j = 10$. After simplifying, we get $42 = 20$. This is impossible. So if $j < i$, then $2i + 12\lfloor \frac{i-2}{12} \rfloor$ is not the same as $2j + 12\lfloor \frac{j-8}{12} \rfloor$ modulo 12.

Hence, the $A - C$ edge and the $B - D$ edge are never equal, and so $C_6 + P_n$, when n is not congruent to 0 modulo 3, is graceful.

$\square$

We now consider those cases we postponed in proving that f was one-to-one.

Lemma *For the subcases in the one-to-one proof that were postponed, one can make* *an appropriate change to our labeling function to ensure that $C_6 + P_n$ is graceful.*

Proof. Again, we break this into four cases, each one actually being a subcase of its respective case above.

Case 1. We first consider the subcase $12k+4$ for the A label and the C label. We found that the maximum value of formula A is $6k+4$, and it is also the minimum value of formula C. This occurs on vertex $12k+4$ and vertex $12k+3$, respectively. Now, change the label of last vertex in the path to $6k+5$, thus giving a label of 1 to the last edge in the path. To see that $6k+5$ has not yet been used, we consider those vertices labeled by formulas B and D, since only those produce odd outputs. Note that the maximum value of B is $6k+3$ and the minimum value of D is $6k+7$, thus showing that $6k+5$ had not yet been used in the path.

Now we check that 1 has not previously been an edge label. Recall that the maximum values of A and B are respectively $6k+4$ and $6k+3$, and that the minimum values of C and D are respectively $6k+6$ and $6k+5$. The only way one could conceivably get an induced edge label of 1 using these would be to use A and D, since they differ by 1 (note that one cannot have a vertex labeled by A and one labeled by B adjacent to one another, because the functions both only accept even inputs; similarly, for formulas C and D). Note, however, that the vertices where these maximum and minimum values occur are $12k+4$ and $12k-3$, respectively, which are non-adjacent. Thus the edge label of 1 had not been previously used, and so our choice of our new label to the last vertex in the path is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

Case 2. We next consider the case where $n = 12 + 7$ for the A and D vertex labels that we postponed in the proof of the theorem. There, we found that the maximum A label is $6k + 6$ and so is the minimum D label. These occur at vertices $12k + 6$ and $12k + 7$, respectively, in the path. Now, change the label of the last vertex in the path to $6k + 5$. To see this has not yet been used, we check B labels and C labels. We note that the maximum B label is $6k + 3$ and the minimum C label is $6k + 7$, so that $6k + 5$ has not been previously used, and we only have to check that the edge label 1 has not previously been used on the path.

We note that the maximum values of A and B are $6k+6$ and $6k+3$, respectively, and the minimum values of C and D are respectively $6k + 7$ and $6k + 6$. The only potential edge label of 1 is seen to be using the A label and the C label. But notice that their maximum and minimum values occur at vertices $12k+6$ and $12k+3$, which are not adjacent. Thus, our new label for the last vertex in the path is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

Case 3. Next, we go back to the case $n = 12k + 1$ for the comparison between the B labels and the C labels that was postponed earlier. We found that the maximum value of the B labels is $6k + 3$, which is the same as the minimum C label. These occur at vertices $12k$ and $12k + 1$, respectively (the last two vertices in the path). Now, change the label of the last vertex in the path to $6k + 2$, giving an edge label of the last edge equal of 1. We must check that the A and D formulas never give this vertex label. So note that the maximum value of the A labels is $6k$ and the minimum of the D labels is $6k + 4$, thus allowing $6k + 2$ to be used.

To see that 1 does not occur on any other edge, note that the maximum values of A and B are respectively $6k$ and $6k + 3$, and the minimum values of C and D are $6k + 3$ and $6k + 4$, respectively. Comparing B to D, the maximum value of B occurs at vertex $12k$ and the minimum of D occurs at $12k - 3$, so they are not adjacent, and we therefore have that 1 is not used on any prior edge to the last one in the path.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

Case 4. Finally, we consider $n = 12k + 10$, where we compared B labels to D labels. In this case, we found earlier that the maximum value of the B labels is $6k + 7$ which is the same as the minimum value of the D labels. These occur at vertices $12k + 10$ and $12k + 9$, respectively, the last two vertices in the path. We change the label of the last vertex in the path to $6k+8$, giving the last edge an induced label of 1. To see that this new label has not been previously used on the path, note that the maximum value of A is $6k+6$ and the minimum value of C is $6k+10$, so label $6k+8$ has not been previously used.

To see that edge label 1 has not occurred on the edges, note that the maximum values of A and B are respectively $6k+6$ and $6k+7$, and that the minimum values of C and D are $6k+10$ and $6k+7$, respectively. Thus we consider the the A and D labels. The maximum value of A occurs at vertex $12k+6$ and the minimum value of D occurs at $12k+9$. Thus, the edge label of 1 did not occur on any other edge, and so our new choice of the last last vertex label is justified.

Again, there was also another subcase where there was a gap of four between the maximum and minimum values, thus there was a vertex label repeated. And, again, it occurs on the last two vertices in the path. We modify the last vertex as earlier, but leave the details to the reader, since the argument is the same.

□

Chapter 3

Harmonious Labeling of the Odd Cycle Union an Edge

Here, we ask the question: Is the disjoint union of an edge P_2 and the odd cycle C_s, denoted $C_s + P_2$, harmonious? A function f is called a *harmonious* labeling of a graph G with m edges if there is an injection from the vertices of G to the group of integers modulo m such that when each edge xy is assigned the label $(f(x) + f(y))$ mod m, the resulting edge labels are distinct. In this chapter we present a harmonious labeling of $C_s + P_2$, s-odd.

Theorem 3 Let s be odd. Then $C_s + P_2$ is harmonious.

Proof. To show that C_s+P_2 is harmonious, we must present a vertex labeling function, f, of $C_s + P_2$ with m edges that is an injection from the vertices of $C_s + P_2$ to the group of integers modulo m such that when each edge xy is assigned to the label

$(f(x) + f(y)) \mod m$, the resulting edge labels are distinct. Note that $m = s + 1$.

Let

$$f(i) = \begin{cases} s - \frac{i-1}{2} & \text{if } i \text{ is odd} \\ \\ \frac{s-i+1}{2} & \text{if } i \text{ is even} \end{cases} \tag{3.1}$$

where i= i^{th} vertex starting at a certain vertex in the cycle, $1 \leq i \leq m$.

We must first show that no edge label is repeated. We do this in cases depending on the endpoints of the edge.

Case I: *The endpoints of one edge are* $s - \frac{i-1}{2}$ *and* $\frac{s-(i+1)+1}{2}$ *and the endpoints of the other edge are* $s - \frac{j-1}{2}$ *and* $\frac{s-(j+1)+1}{2}$.

We prove that $(s - \frac{i-1}{2} + \frac{s-(i+1)+1}{2}) \mod m \neq (s - \frac{j-1}{2} + \frac{s-(j+1)+1}{2}) \mod m$, if $i < j$. After simplifying, the left hand side is $\frac{1}{2}s - i - \frac{1}{2}$ and the right hand side is $\frac{1}{2}s - j - \frac{1}{2}$. These do not equal since $i < j$.

Case II: *The endpoints of one edge are* $\frac{s-i+1}{2}$ *and* $s - \frac{(i+1)-1}{2}$ *and the endpoints of the other edge are* $\frac{s-i+1}{2}$ *and* $s - \frac{(j+1)-1}{2}$.

We prove that $(\frac{s-i+1}{2} + s - \frac{(i+1)-1}{2}) \mod m \neq (\frac{s-j+1}{2} + s - \frac{(j+1)-1}{2}) \mod m$, if $i < j$. After simplifying, the left hand side is $\frac{1}{2}s - i - \frac{1}{2}$ and the right hand side is $\frac{1}{2}s - j - \frac{1}{2}$. These do not not equal since $i < j$.

Case III: *The endpoints of one edge are* $s - \frac{u-1}{2}$ *and* $\frac{s-(u+1)+1}{2}$ *and the endpoints of the other edge are* $s - \frac{v-1}{2}$ *and* $s - \frac{(v+1)-1}{2}$.

We prove that $(s - \frac{u-1}{2} + \frac{s-(u+1)+1}{2}) \mod m \neq (\frac{s-v+1}{2} + m - \frac{(v+1)-1}{2}) \mod m$, if $u < v$. After simplifying, the left hand side is $\frac{1}{2}s - u - \frac{1}{2}$ and the right hand side is $\frac{1}{2}s - v - \frac{1}{2}$. These do not not equal since $u < v$.

Now, we must show that the label of the edge with endpoints $i = 1$ and $i = s$ is not equal to any other edge label. Using our labeling function, f, we get the two labels of the endpoints of this edge; s and $\frac{s+1}{2}$. So, this join-edge label is $\frac{1}{2}s - \frac{1}{2}$. Previously, we found that all the other edge labels are of the form $\frac{1}{2}s - i - \frac{1}{2}$ for $1 \leq i \leq s$. Thus, the label of the join-edge is always different from the other edge labels. The only edge label thus far not used is $\frac{s+1}{2} \mod m$. To get this edge label we use 0 and $\frac{s+1}{2}$ to label the two vertices in the path, which gives $\frac{s+1}{2}$ as the induced edge label. Thus, all edge labels are distinct. Also note that we only used the vertex label $\frac{s+1}{2}$ twice, which is the requirement for a harmonious labeling for cyclic graphs.

We must also check that no vertex label has been repeated. First, note that f is one-to-one. Observe that no two vertices using the first formula are the same since it is decreasing. Similarly, no two vertices using the second formula are the same. Next, if we set $s = \frac{i-1}{2} = \frac{s-j+1}{2}$, where i is odd and j is even, the we have $s + i - j + 2$, which is a contradiction since s is odd and $s + i - j$ is even.

Also, 0 was not used in the cycle, so that only $\frac{s+1}{2}$ was repeated. Thus, $C_s + P_2$, where s is odd, is harmonious.

$\square$

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