PLATE BUCKLING DUE TO COMBINED BENDING AND COMPRESSION USING LINEAR FINITE ELEMENT METHOD.

by

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ABSTRACT

The objective of this study was to develop new solutions for linear elastic buckling coefficients of rectangular flat plates with various support conditions. Critical buckling coefficients were found for rectangular flat plates subjected to compression and bending on two opposite edges with one free unloaded edge using the finite element method. Plates with different aspect ratios (0.5, 0.75, 1, 1.5, 2, 2.4, 2.8, 3.2, 3.6, 4) were analyzed. Currently, no solutions are available in the literature for plates subjected to compression and bending with a free unloaded edge.

Thin-walled structures have the characteristic of susceptibility of failure by instability or buckling. It is important to the design engineer that accurate methods are available to determine the critical buckling strength. The method developed in this work was verified on problems where closed form mathematical solutions exist.

An engineer will be able to use the solutions developed in this work in the design of components that are susceptible to instability failures. Another benefit of this work is to demonstrate to practicing engineers that reliable instability results can be obtained by using standard finite element analysis (FEA) methods. This work considers a small subset of instability problems but the FEA method utilized herein can be effectively used to model a large class of practical instability problems.

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NOMENCLATURE

a	length of the plate (in)
a'	supported length of an unloaded, longitudinal edge (in)
b	width of plate (in)
b_e	"effective-width"
D	flexural rigidity of the plate (lb*in): $\frac{Et^3}{12(1-\nu^2)}$
{d}	Global nodal displacement vector
E	Young's modulus (psi)
E_T	Tangent modulus (strain-hardening modulus) (psi)
ΔF	Unbalanced nodal force vector
{F}	global nodal force vector
g	gap between intermittent supports (in)
I_s	moment of inertia of a stiffener (in^4)
[K]	global stiffness matrix
K	buckling coefficient (--)
K'	modified buckling coefficient, $K\pi^2/12$ (--)
K_s	buckling coefficient for shear buckling stress (--)
K_L	linear stiffness matrix
K_G	geometric stiffness matrix
M_y	edge moment along the x-axis (in*lb)
M_{xy}	twisting moment (in*lb)
N_x	edge force per unit length in the x-direction (lb/in)

N_y	edge force per unit length in the y-direction (lb/in)
N_{xy}	edge shearing force per unit length (lb/in)
P_{ult}	ultimate load (lb)
Q_x	shearing force (lb)
q(x,y)	Intensity of a distributed lateral load (lb)
ΔT_1	Work done by external forces (lb*in)
t	thickness of plate (in)
ΔU	strain energy of bending (lb*in)
U_n	global displacement vector (non-linear finite element method)
V_e	total potential energy of a plate element (lb*in)
w	displacement function (in)
z	distance from neutral surface (in)
α	the ratio a/b (--)
δ	nodal displacement vector
ε_D	displacement tolerance (--)
ε_x	Strain in the x-direction (--)
ε_y	Strain in the y-direction (--)
ϕ	a/b
γ_{xy}	shear strain (--)
γ	$\frac{EI_s}{bD}$ (--)
λ_{cr}	buckling load factor
ν	Poisson's ratio (--)

$1/\rho_x$	curvature of the neutral surface in a section parallel to the xz-plane (in)
$1/\rho_y$	curvature of the neutral surface in a section parallel to the yz-plane (in)
σ_C	critical buckling stress (psi)
σ^*_C	critical stress for the case with compression only (psi)
σ_{Cb}	compressive stress due to bending (psi)
σ^*_{Cb}	critical buckling stress for bending (psi)
σ_x	normal stress in the x-direction (psi)
σ_y	normal stress in the y-direction (psi)
s_y	yield point of the material under consideration (psi)
σ_{VM}	Von Mises equivalent 1-D stress (psi)
$\sigma_{1,2,3}$	principal stresses (psi)
σ_e	edge stress (psi)
σ_{av}	average stress at ultimate load
τ_c	critical shear stress (psi)
τ^*_c	critical stress for the case with shear only (psi)
τ_{xy}	shear stress (psi)

1. INTRODUCTION

Linear elastic buckling of plates that are subjected to in-plane forces is a problem of great practical importance that has been extensively researched over the past 60 years. Elastic instability of flat rectangular plates became an important research area when the design of the lightweight airframes was introduced. Later, the theory of thin plates has been applied to engineering structures (Fok, 1984). Some advantages of thin-walled structures are the high strength coupled with the ease of manufacturing and the relative low weight. However, thin-walled structures have the characteristic of susceptibility of failure by instability or buckling. It is, therefore, important to the design engineer that accurate methods are available to determine the critical buckling strength.

Most research on instability of flat plates has been done on rectangular shapes of various proportions. Usually the plates are supported continuously along all edges with loading occurring along two opposite sides. A limited amount of work has been done on plates with an unsupported or partially supported unloaded edge.

Norris, et. al., (1951) studied the buckling behavior of intermittently supported rectangular plates with both analytical methods and laboratory experiments. According to Wang et. al. (1993), engineers tend to use design charts and formulas rather than using accurate but more complex solution methods such as finite element analysis in everyday design work. Approximate formulas and solutions will continue to be used until inexpensive and much more user-friendly computer software is available to all engineers.

The objective of this study was to develop new solutions for the linear elastic critical buckling stress of rectangular isotropic flat plates with one unsupported unloaded edge using the finite element method.

As mentioned earlier, only a few published papers investigate the problem of flat plates subjected to uniform compression with non-continuous boundary conditions. An example of such a problem could be a welded plate structure with openings. As an example, consider the web of an I-section under bending.

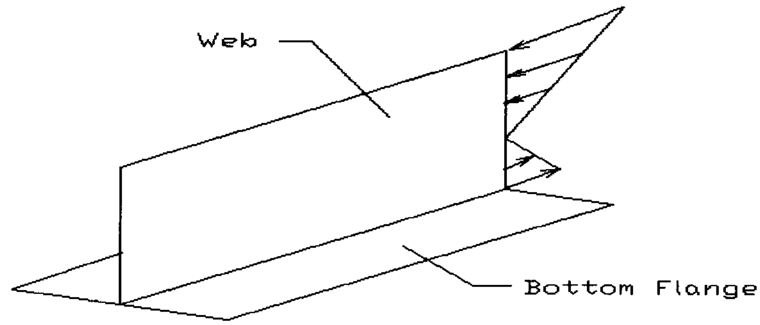


Figure 1-1: I-beam under bending.

In this case, the web is not supported at the top section. Currently, there are no solutions that include buckling coefficients for the above case. This paper presents buckling coefficients for cases with an unsupported unloaded edge with combinations on the other edges i.e., simply supported and fixed.

In this work, buckling coefficients of flat plates are obtained by using the finite element method. The linear elastic buckling stress was obtained using a linear eigenvalue buckling analysis solver. Verification of the finite element method was done by comparing to known closed-form solutions. Charts are drawn for the buckling coefficients of flat plates with an unsupported unloaded edge for various ratios of bending stress to compression stress with other edges either simply supported or fixed.

A literature review of previous research of buckling of flat plates is presented in the coming section. Then in chapters 3, 4 and 5, the theories of thin plates and elastic stability are reviewed. Chapter 6 presents the finite element analysis and the results that were obtained. A conclusion and a discussion of the results are included in chapter 7.

2. LITERATURE REVIEW

Numerous research studies have been done over the years for rectangular plates with different boundary and loading conditions (Timoshenko, 1936 and Bulson, 1970). Bryan (1891) presented what seems to be the first published paper on elastic critical stress. Bryan analyzed a rectangular flat plate under uniform compression with simply supported edges.

Over the years different combinations of simply supported and clamped edges have been studied. Different loading conditions have been studied, for example pure bending, combined bending and compression, uniform compression, and shear. The elastic critical stress, a function of the material (E, ν), with width (b), thickness (t) and the boundary conditions, is given by (Timoshenko, 1936):

$$\sigma_c = K \frac{\pi^2 E}{12(1-\nu^2)\left(\frac{b}{t}\right)^2} \qquad \textbf{Eq.2-1}$$

where: σ_c = critical buckling stress (psi)

K = buckling coefficient (--)

E = Young's modulus (psi)

ν = Poisson's ratio (--)

b = width of the plate (in)

t = thickness of the plate (in)

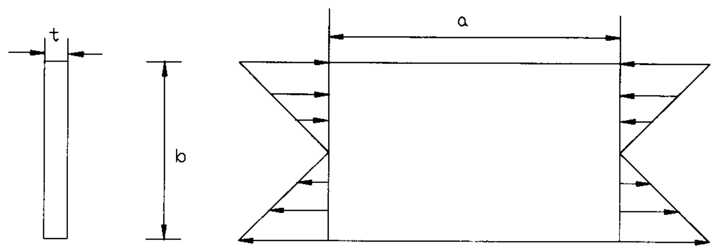


Figure 2-1: Sketch of flat plate under pure bending.

Some excellent references for plate buckling are Galambos (1988), Bulson (1970), Young (1989), Timoshenko (1936, 1961)

2.1 Flat plates under combined bending and compressive stresses:

In this case the plate is subjected to a combination of compression and bending stresses and the compressive stress along the edges will vary from a maximum to a minimum, as shown below:

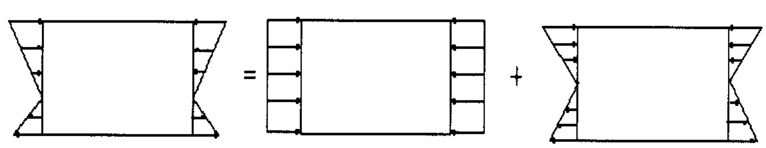


Figure 2-2: Combined bending and compression of a flat plate.

Buckling stresses for flat plates in bending and compression have been investigated by Timoshenko (1936), Heck and Ebner (1936), Bijlaard (1957) and Brockenbrough and Johnston (1974).

For the following ratios between bending and compressive stress, for plates with all simply supported edges minimum buckling coefficients were published by Galambos (1988):

$$\frac{\sigma_{cb}}{\sigma_c} = \infty, 5.00, 2.00, 1.00, 0.50, 0.0$$

where: σ_{cb} = compressive stress due to bending

σ_c = compressive stress due to uniform compression

The first case corresponds to pure bending and the last case to pure compression. For plates with loaded edges simply supported and unloaded edges fixed, minimum buckling coefficients were given for $\frac{\sigma_{cb}}{\sigma_b} = \infty$, 1.00 and 0.0.

2.2 Flat plates under uniform compression:

Flat plates under uniform compression with different types of boundary conditions are shown below:

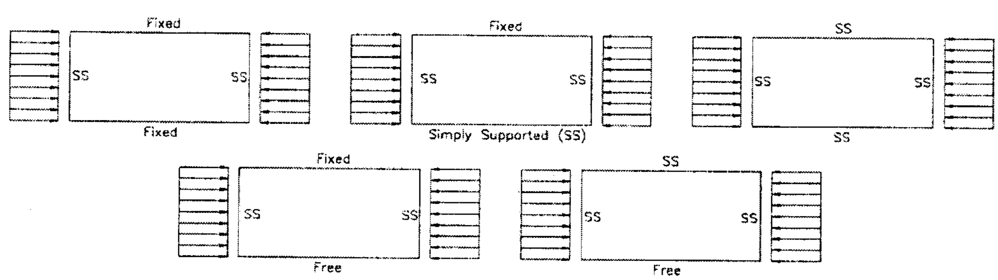


Figure 2-3: Flat plates under uniform compression.

All the cases above were studied by Timoshenko (1936) and the cases when edges were fixed were investigated by Heck and Ebner (1936) and Maulbetsch (1937).

2.3 Flat plates under uniform tension / compression in two perpendicular directions:

Research has been done by Timoshenko (1936) and Heck and Ebner (1936) for the case of flat plates under uniform tension or compression in two perpendicular directions with simply supported edges as shown below. Timoshenko (1936) also studied the cases with clamped edges.

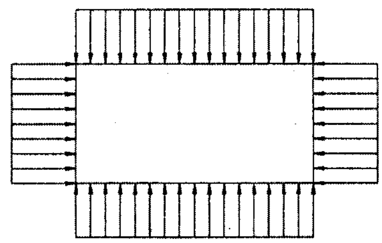


Figure 2-4: Rectangular plate under uniform compression in two perpendicular directions.

2.4 Combined bending and compressive stresses in two perpendicular directions:

Yoshizuka and Narmoka (1971) published the buckling coefficients for the case of combined bending and compressive stresses in two perpendicular directions for a rectangular plate.

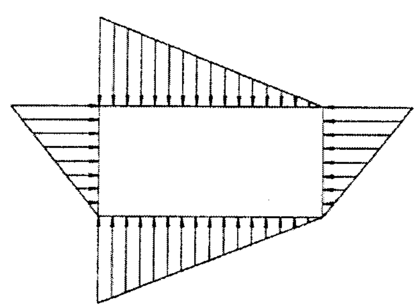


Figure 2-5: Flat plate subjected to compression and bending in two perpendicular directions.

2.5 Rectangular plates subjected to edge shear stresses on all edges:

The plate shown below is subjected to edge shear stresses on all edges, this stress state is called "pure shear".

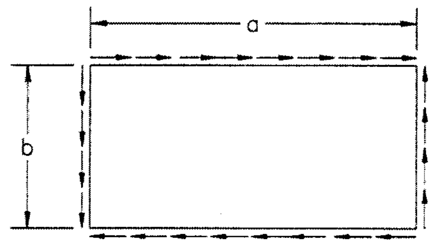


Figure 2-6: Plate subjected to pure shear.

Timoshenko (1910), Bergmann and Reissner (1932) and Seydel (1933) developed critical buckling coefficients for the case of plate with all edges simply supported. The critical shear stress can be found by substituting τ_c and K_s for σ_c and K in Eq. 2-1. Approximate expressions for the critical buckling coefficient, K_s are given below:

$$K_s = 4.00 + \frac{5.34}{\alpha^2} \quad \text{valid for } \alpha \leq 1$$

$$K_s = 5.34 + \frac{4.00}{\alpha^2} \quad \text{valid for } \alpha \geq 1$$

where: $\alpha = a/b$

Moheit (1939) developed the following expressions for the critical buckling coefficients for the plates with all edges clamped:

$$K_s = 5.6 + \frac{8.98}{\alpha^2} \quad \text{valid for } \alpha \leq 1$$

$$k_s = 8.98 + \frac{5.6}{\alpha^2} \quad \text{valid for } \alpha \geq 1$$

The case for a plate with two opposite edges clamped and the other two edges simply supported, Iguchi (1938) developed a solution for a general rectangular plate and Leggett (1941) for a square plate.

2.6 Shear combined with direct stress:

Shown below is a plate subjected to shear and uniform compression. Iguchi (1938) investigated the case with a plate subjected to shear and uniform compression with all edges simply supported.

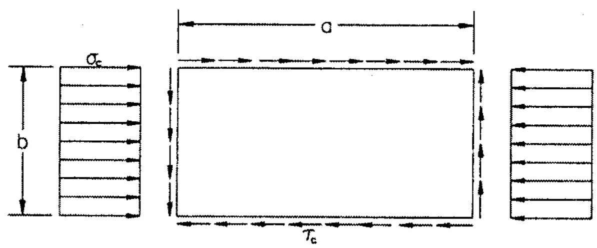


Figure 2-7: Plate subjected to shear and uniform compression.

Iguchi developed an approximate interaction equation between the critical buckling stress for uniform compression and for shear alone. This interaction equation given below predicts instability when satisfied.

$$\frac{\sigma_c}{\sigma_c^*} + \left(\frac{\tau_c}{\tau_c^*}\right)^2 = 1 \text{ valid for a/b} > 1$$

where: σ_c^* = critical stress for the case with compression only

τ_c^* = critical stress for the case with shear only

σ_c = actual compression stress present

τ_c = actual shearing stress present

2.7 Shear combined with bending:

A plate subjected to a combination of pure shear and pure bending is shown below:

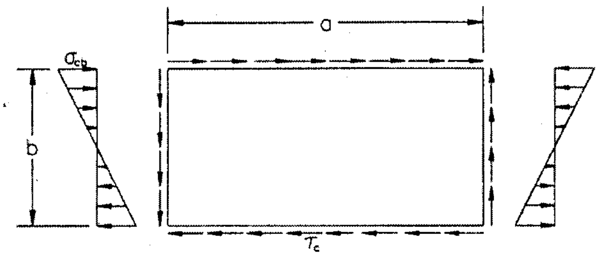


Figure 2-8: Plate subjected to shear and bending.

This problem was investigated by Timoshenko (1934) and a value for reduced critical buckling coefficient K_c, which is a function of τ_c / τ_c^* was presented. This value for K_c is valid for $\alpha = 0.5$, 0.8 and 1.0.

Where: $\alpha = a/b$

τ_c = the actual shearing stress present

τ_c^* = the buckling stress for pure shear

This case was also investigated by Stein (1936) and Way (1936). Chwalla (1936) presented the following approximate interaction formula predicting failure, which corresponds well with the results from Stein and Way.

$$\left(\frac{\sigma_{cb}}{\sigma_{cb}^*}\right)^2 + \left(\frac{\tau_c}{\tau_c^*}\right)^2 = 1$$

where: σ_{cb}^* = the critical buckling stress for bending

σ_{cb} = the actual bending stress present

τ_c = the actual shearing stress present

τ_c^* = the critical buckling stress for pure shear

2.8 Shear combined with bending and uniform compression:

An interaction formula was presented by Gerard and Becker (1957/1958) for a plate which is subjected to a combination of shear, bending and compression and having all four edges simply supported as shown below:

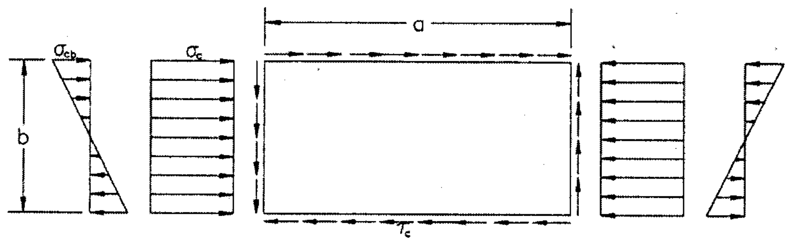


Figure 2-9: Plate subjected to shear, compression and bending.

The three-part interaction formula is given by:

$$\left(\frac{\sigma_c}{\sigma_c^*}\right)+\left(\frac{\sigma_{cb}}{\sigma_{cb}^*}\right)^2+\left(\frac{\tau_c}{\tau_c^*}\right)^2=1$$

where: σ_c^* = the critical buckling stress for compression

σ_c = the actual compression stress present

McKenzie (1964) presented interaction graphs that took into account, in addition to the above loading condition, a uniform compressive load applied on the horizontal edge, as shown below.

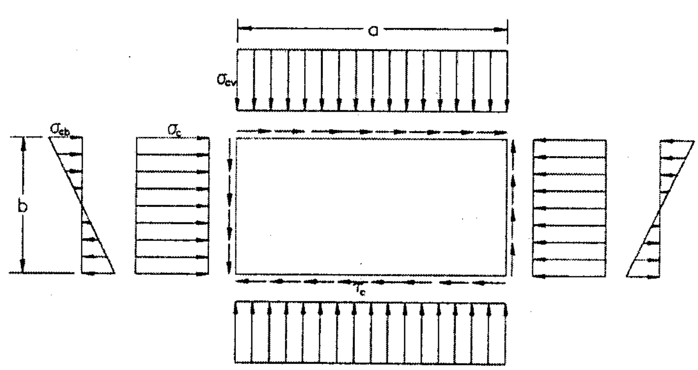


Figure 2-10: Plate subjected to compression, bending, shear and a vertical applied compression loading.

2.9 Buckling strength of stiffened plates: Uniaxial compression, combined compression and shear:

To increase the stiffness of a plate, stiffeners are added, as shown below. Both longitudinal and transverse stiffeners are used, either separately or in a combination.

Timoshenko and Gere (1961), Bleich and Ramsay (1951) and Seide and Stein (1949) have presented solutions to plate with one, two or three longitudinal stiffeners equally spaced parallel to the applied loading. Tables and graphs are given by them to determine the critical stress for plates simply supported on all edges.

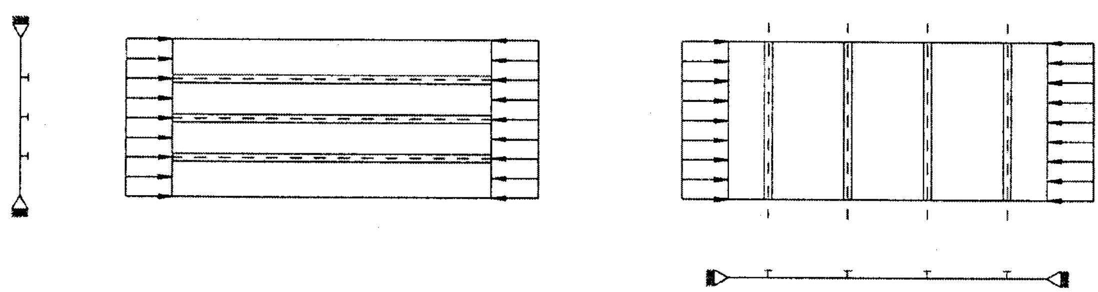


Figure 2-11: Plates with longitudinal and transverse stiffeners.

Timoshenko and Gere (1961) investigated plates in uniaxial compression with transverse stiffeners. For one, two, or three equally spaced stiffeners, Timoshenko and Gere defined the required size of the stiffeners. Klitchieff (1949) defined required size of stiffeners when any numbers of stiffeners are used. Gerard and Becker (1957/1958) investigated the case with both longitudinal and transverse stiffeners and supplied the minimum value of γ as a function of α for different combinations of longitudinal and transverse stiffeners in graphs.

where: $\gamma = \dfrac{EI_s}{bD}$

EI_s = flexural rigidity of one transverse stiffener

$$D = \frac{Et^3}{12(1-\nu^2)}$$; flexural rigidity of the plate

$\alpha = a/b$

2.10 Buckling strength of flat plates with partial boundary conditions:

There are just a few papers published on the buckling of rectangular plates with partially edge boundary conditions (Persson, T. S Thesis 1996, Hamada et. Al. 1967 and Norris et. Al. 1951).

2.10.1 Flat plate under uniform compression with simply supported but partially clamped edges:

Hamada et. Al. (1967) investigated flat plates with simply supported loaded edges in uniform compression and simply supported but partially clamped unloaded edges.

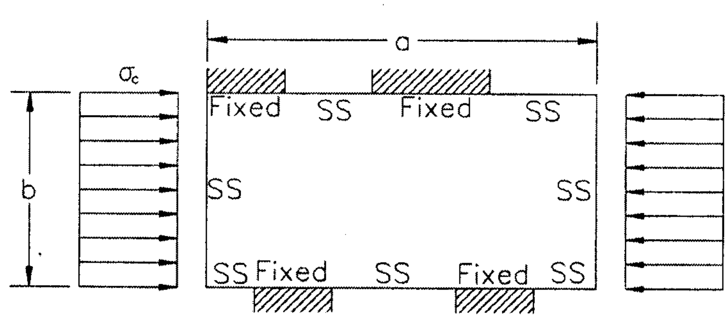


Figure 2-12: Flat plate under uniform compression with simply supported but partially clamped unloaded edges and simply supported loaded edges.

Hamada and Ota (1958, 9159) proposed an energy method which was used to investigate this problem. Ten cases were presented with diagrams for plates with aspect ratio 1, 2/3 and 1/2. Laboratory experiments were done to verify the results.

2.10.2 Flat plate under compression with intermittently simply supported edges:

Shown below is the sketch of the intermittently supported plate. Norris et. Al. (1951) studied the buckling behavior of long rectangular plates intermittently supported along the unloaded edges.

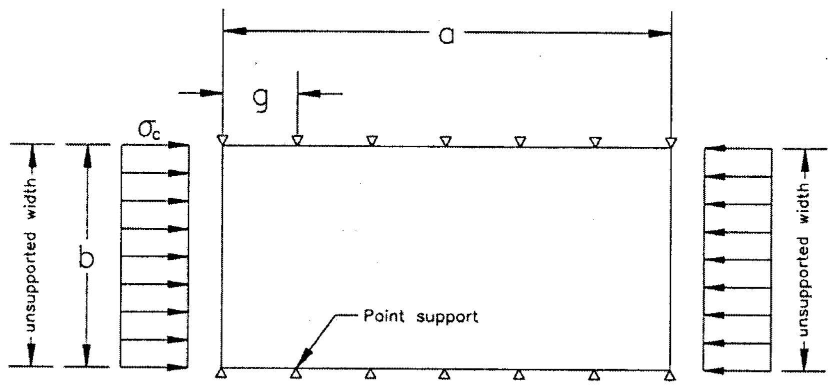


Figure 2-13: Long flat plate with intermittently simply supported edges.

The objective of the research was to develop specifications that could be used to determine the required length of intermittent fillet welds to connect component parts of structural members. The critical buckling coefficient, K was given as follows:

$$K = 4.0 \qquad \text{valid for } 0 < g/b < 0.5$$

$$K = (b/g)^2 \qquad \text{valid for } g/b > 0.5$$

Where: b = unsupported width of the plate

g = gap between intermittent supports

3. PLATE THEORY BACKGROUND

A summary of the elastic bending and buckling theory of thin plates is discussed next. This summary gives the needed theoretical background to elastic bending and buckling of flat plates and to the mathematical definitions of the boundary conditions.

3.1 Basic assumptions used in the theory of elasticity:

Described below are the basic assumptions that are used in the theory of elasticity. These assumptions are common to all elastic problems.

1. Perfectly elastic material, i.e., if a body is deformed by external forces, the body will return completely to its initial shape when the external forces are removed.
2. Homogeneous, i.e., the physical properties of the entire body are the same as any small element cut out from the body.
3. Isotropic, i.e., the elastic properties of the body are the same in all directions.

3.2 Theory of bending of thin plates:

The theory for thin plates is similar to the theory for beams. In pure bending of beams, "the stress distribution is obtained by assuming that cross-sections of the bar remain plane during bending and rotate only with respect to their neutral axes so as to be always normal to the deflection curve." (Timoshenko, 1936, p. 319) as shown in figure 2-1 below.

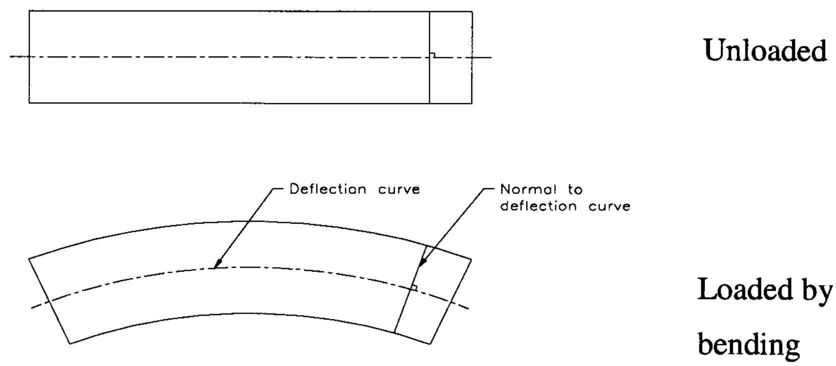


Figure 2-1. Beam under pure bending.

A rectangular plate element is shown below. For a thin plate, bending in two perpendicular directions occur.

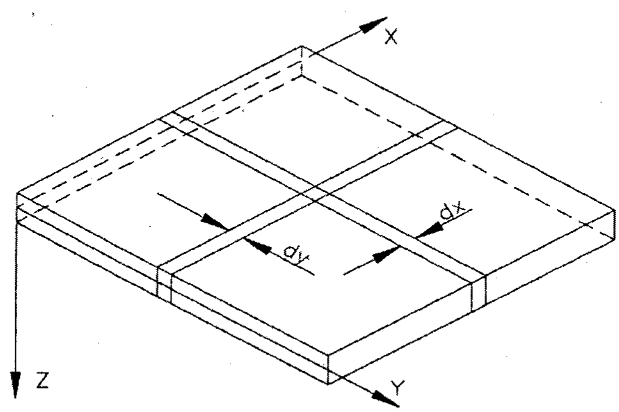


Figure 3-2: Thin plate notation.

Consider a small element cut out by two pairs of planes, parallel to the xz-plane and the yz-plane. This small differential element is shown below:

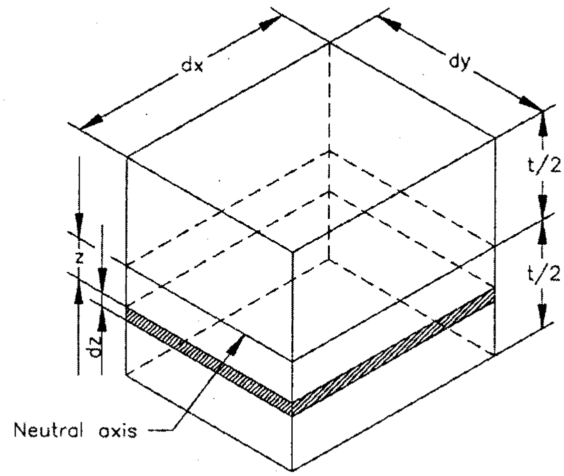


Figure 3-3: Differential element notation.

The basic assumptions of elastic plate bending are:

1. Perfectly flat plate and of uniform thickness.
2. The thickness of the plate is small compared with other dimensions. For plate bending, the thickness, *t*, is less than or equal to 1/4 of the smallest width of the plate. For plate buckling equations, the thickness, *t*, should be 1/10 of the smallest width of the plate, (Young, 1989).
3. Deflections are small, i.e., smaller or equal to 1/2 of the thickness, (Young, 1989).
4. The middle plane of the plate does not elongate during bending and remains a neutral surface.
5. The lateral sides of a differential element remain plane during bending and rotate only to be normal to the deflection surface. Therefore, the stresses and strains are proportional to their distance from the neutral surface.
6. The bending and twisting of the plate element resist the applied loads. The effect of shearing forces is neglected.

From the above assumptions the strains in a plane element, in the x- and y-directions are given by:

$$\varepsilon_x = \frac{z}{\rho_x} \qquad \varepsilon_y = \frac{z}{\rho_y} \qquad \text{..................................Eq.3-1}$$

where: z = distance from neutral surface

$1/\rho_x$ = curvatures of the neutral surface in a section parallel to the xz-plane

$1/\rho_y$ = curvatures of the neutral surface in a section parallel to the yz-plane

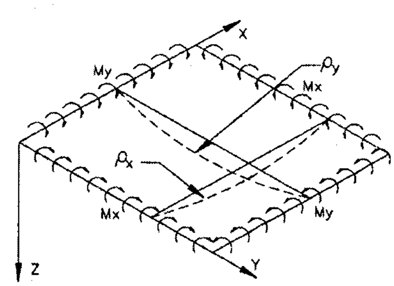


Figure 3-4: Curvatures of the neutral surface in a plate section.

and from Hooke's law, the strains ε_x and ε_y are related to the stresses σ_x and σ_y as follows:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \text{ ....Eq.3-2}$$

where: E = Young's modulus

σ_x = normal stresses in the x-direction

σ_y = normal stresses in the y-direction

ν = Poisson's ratio

By combining Eq.3-1 and Eq.3-2, and solving for the normal stresses σ_x and σ_y, yields

$$\sigma_x = \frac{Ez}{(1-\nu^2)}\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right) \qquad \sigma_y = \frac{Ez}{(1-\nu^2)}\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right) \quad \text{.............................}\textbf{Eq.3-3}$$

The normal stresses above act on the lateral sides of the element in Figure 3-3 and can be reduced to couples, which must equal the externally applied moments. It can then be shown that, the edge moments M_x and M_y are given by:

$$M_x = D\left(\frac{1}{\rho_x}+\nu\frac{1}{\rho_y}\right) \qquad M_y = D\left(\frac{1}{\rho_y}+\nu\frac{1}{\rho_x}\right) \quad \text{..................................}\textbf{Eq.3-4}$$

where: $D = \dfrac{Et^3}{12(1-\nu^2)}$; flexural rigidity of the plate....................................**Eq.3-5**

This quantity corresponds to the EI – value of a beam unit width. The term $(1-\nu^2)$ increases the rigidity of the plate compared to a beam of the same width. The reason for this is that moments in one direction create a curvature in a perpendicular direction, to form a so called *anticlastic* surface. The plate's resistance to this second curvature, has the effect of an increase in the rigidity of the plate.

Let the deflection, in the z-direction, of the plane be w, then by using the approximate formulas for the curvatures of a plate, the curvatures are given by:

$$\frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2} \qquad \frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2} \quad \text{..}\textbf{Eq.3-6}$$

Substituting Eq.3-6 into Eq.3-4 yields the following expressions for the moments:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad \text{.........................}\textbf{Eq.3-7}$$

From basic theory of elasticity, the relationship between shear strain and shear stress is given by:

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \quad \text{...}\textbf{Eq.3-8}$$

It can then be shown that the twisting moment is given by:

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \quad \text{..}\textbf{Eq.3-9}$$

Consider a plate that is subjected to a distributed lateral load as shown in figure below, acting perpendicular to the middle plane of the plate, where q(x,y) is the intensity of the load. In the general case, the intensity q is a function of x and y.

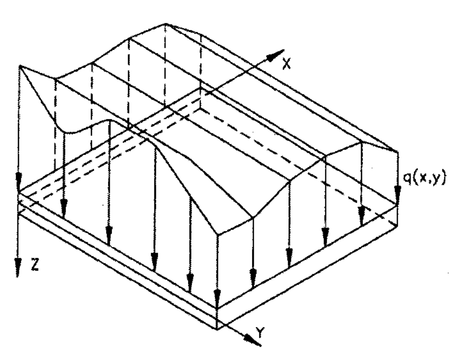


Figure 3-5: Distributed lateral load on a plate.

By integrating the following fourth-order partial differential equation, the deflection surface can be found:

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D} \quad \textbf{Eq.3-10}$$

If the deflection surface is known, all the stresses can be calculated. Next, the definition of the boundary conditions will be discussed.

3.3 Mathematical definitions of the boundary conditions:

Discussed below are the mathematical definitions for a simply supported edge, fixed (clamped or built-in) edge and for a free edge. In order to integrate the above differential equation, the distributed load and the boundary conditions must be known.

3.3.1 Simply Supported Edge:

A simply supported edge has no deflections along the supported edge, however the rotation with respect to the x-axis is not restricted. A sketch of a simply supported edge is shown below:

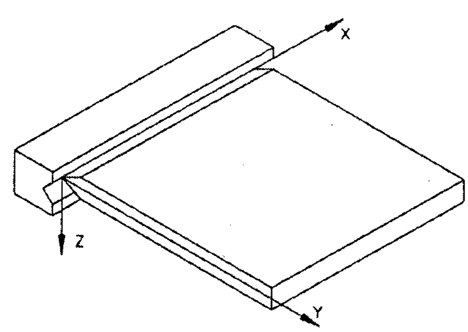


Figure 3-6: Simply supported edge.

Because there are no restrictions against rotation along the simply supported edge, this means that there is no bending moments along this edge. The mathematical definition of a simply supported edge is given below:

$$(w)_{y=0} = 0 \qquad \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_{y=0} = 0 \quad \text{Eq.3-11}$$

3.3.2 Fixed Edge:

The deflection is zero along the fixed edge and the slope of the middle plane of the plate is zero for a fixed edge. Shown below is a sketch of a fixed edge:

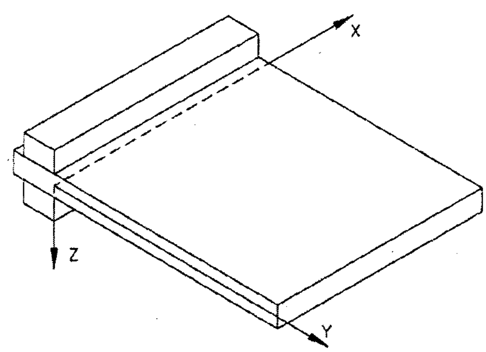


Figure3-7: Fixed edge

Using the above coordinate system, the boundary conditions are:

$$(w)_{y=0} = 0 \qquad \left(\frac{\partial w}{\partial y}\right)_{y=0} = 0 \quad \text{Eq.3-12}$$

3.3.3 Free Edge:

Along an edge that has no supports, the bending and twisting moments and the vertical shearing forces will all be zero. Let the edge $x = a$ be free in the sketch below:

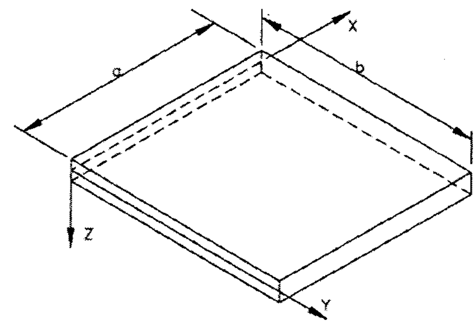


Figure 3-8: Free edge.

$(M_x)_{x=a} = 0$ $(M_{xy})_{x=a} = 0$ $(Q_x)_{x=a} = 0$ **Eq.3-13**

The two boundary conditions above for the twisting moment and the shearing forces can be combined to one boundary condition, as was proved by Kirchhoff (1850), i.e.,

$$\left(Q_x - \frac{\partial M_{xy}}{\partial y}\right)_{x=a} = 0 \quad \textbf{Eq.3-14}$$

The analytical expression for the above boundary condition can be shown to be:

$$\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a} = 0 \quad \textbf{Eq.3-15}$$

The expression for the requirement that the bending moment is zero can be expressed as follows:

$$\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a}=0 \quad \textbf{Eq.3-16}$$

3.4 Plate subjected to in-plane loads:

If a plate has, in addition to the distributed lateral load, forces that are applied in the middle plane of the plate, the effect on plate bending can be considerable. The differential equation for the deflection surface can then be shown to be (Saint Venant derived this differential equation in 1883):

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{1}{D}\left(q+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}\right) \quad \textbf{Eq.3-17}$$

where: q = distributed lateral load intensity

N_x = edge force per unit length in the x-direction

N_y = edge force per unit length in the y-direction

N_{xy} = edge shearing force per unit length

The above differential equation is not valid for large deflections, i.e., when the middle plane of the plate stretches. This would be the case when inelastic buckling occurs and/or when the plate carries ultimate load.

4. LOCAL BUCKLING OF PLATES

4.1 Introduction:

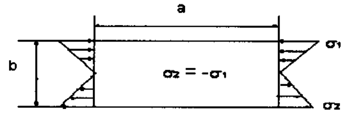
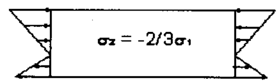
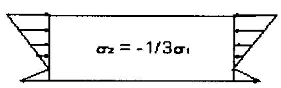
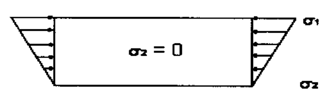
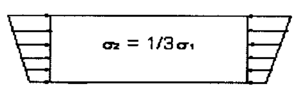
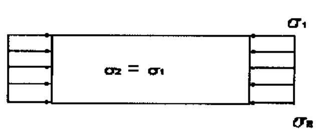
Most structural members and fabrications are composed of connected elements that, for purposes of analysis and design, may be treated as plates. When a plate is subjected to direct compression, bending, or shear, the plate may buckle locally before the member as a whole becomes unstable. This section considers local buckling of flat plates.

4.1.1 Behavior of Flat Plates Under Various Edge Loadings:

A. Compression and Bending: When in-plane bending stresses act simultaneously with uniform compression stresses, the sum of the applied edge stress varies along the loaded edges of the plate from a maximum compressive stress, σ_1, to a minimum stress σ_2, as shown in Figure. The behavior of plates under these loading conditions is generally similar to that discussed in succeeding sub section. For the elastic range, the value of σ_1 at buckling, f_{crl}, is

$$f_{crl} = \frac{k_1 \pi^2 E}{12(1-\nu^2)(b/t)^2} \quad \textbf{Eq.4-1}$$

where k_1 is a non dimensional plate buckling coefficient that depends primarily on the type of edge support and on the ratio of bending stress, to uniform compressive stress, f_b/f_c.

Loading	Ratio of Bending stress to uniform Compression stress, f_b/f_c	Minimum Buckling Coefficient,*k_1	
		Unloaded Edges Simply Supported	Unloaded Edges Fixed
 $\sigma_2 = -\sigma_1$	∞ (Pure Bending)	23.9	39.6
 $\sigma_2 = -2/3\sigma_1$	5.00	15.7	
 $\sigma_2 = -1/3\sigma_1$	2.00	11.0	
 $\sigma_2 = 0$	1.00	7.8	13.6
 $\sigma_2 = 1/3\sigma_1$	0.50	5.8	
 $\sigma_2 = \sigma_1$	0.00 (Pure Compression)	4.0	6.97

*Values given are based on plates having loaded edges simply supported and are conservative for plates having loaded edges fixed.

Figure 4-1: Buckling coefficients for flat plates under compression and bending

Minimum values of k_1 for two edge conditions are given in Figure 4-2. The plate buckling coefficient k_1 is independent of a/b for values of a/b>1.0, and are conservative for a/b<1.0. For stress ratios not shown in Figure values of k_1 can be obtained by linear interpolation.

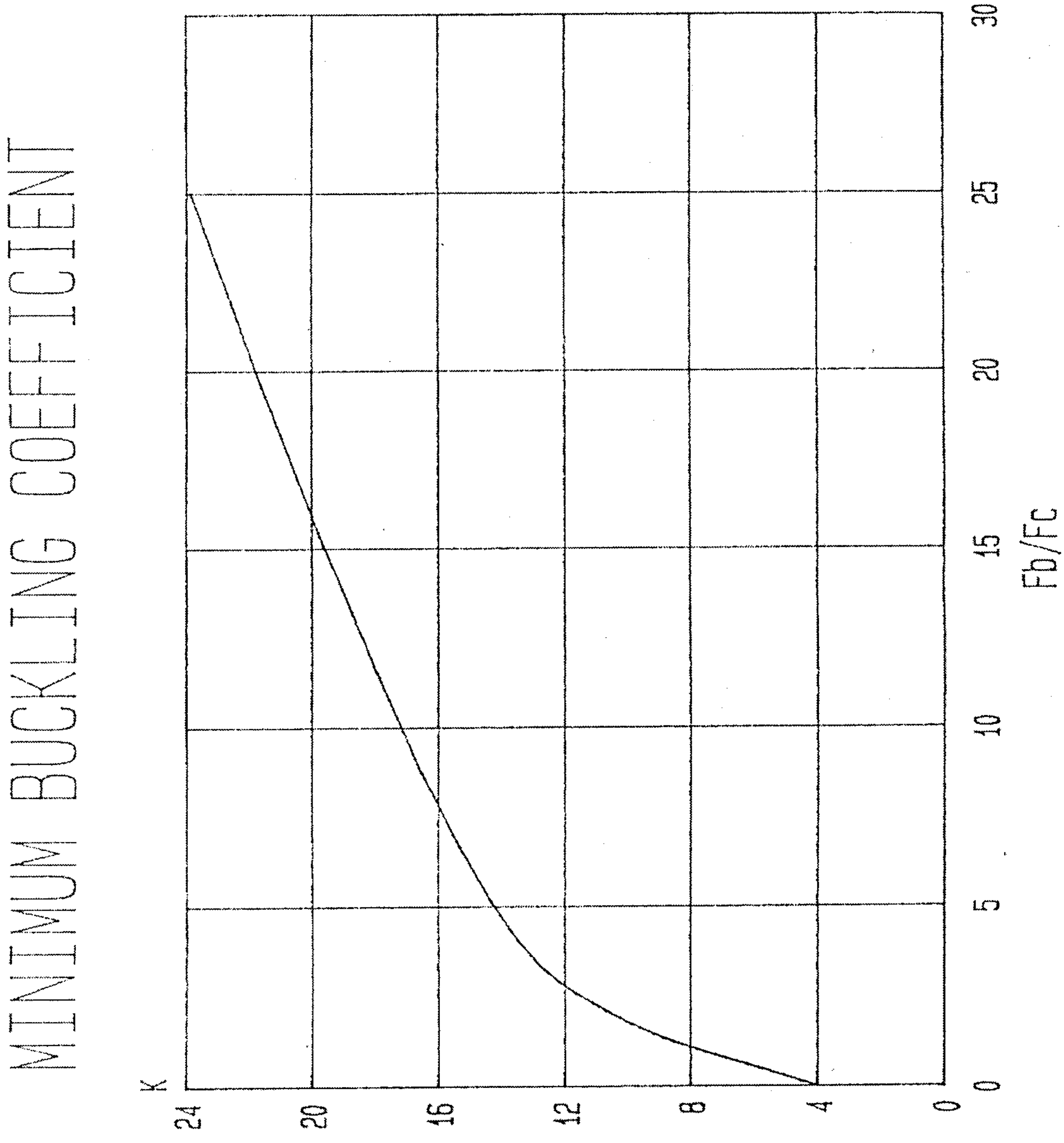


Figure 4-2: Chart for f_b/ f_c versus minimum buckling coefficient, k for all sides simply supported

B. Uniform Compression: A plate that is (1) made of isotropic material, (2) free of residual stresses, (3) perfectly flat, and (4) subjected to loads in its plane is referred to as a "perfect plate". When an increasing uniform compressive load is applied along opposite edges of a perfect rectangular plate, the plate shortens uniformly in its plane, and compressive stresses are uniformly distributed overall transverse cross sections. However, when the buckling stress is reached, the plate deflects from its initial plane in a series of waves, and the compressive stresses are redistributed over transverse cross sections. The buckling stress of a plate is not usually the maximum stress of a plate can withstand.

Within the elastic range, the buckling stress, f_{cr} is

$$f_{cr} = \frac{k_c \pi^2 E}{12(1-\nu^2)(b/t)^2} \quad \text{.................................} \textbf{Eq.4-2}$$

where b/t is the plate width-to-thickness ratio and k_c is a non-dimensional plate buckling coefficient that depends primarily on the type of edge supports and the length-to-width ratio, a/b, of the plate. The width is measured perpendicular to the direction of the applied load. The above equation is similar to the Euler equation for columns.

Figure 4-3 gives k_c for various loading conditions

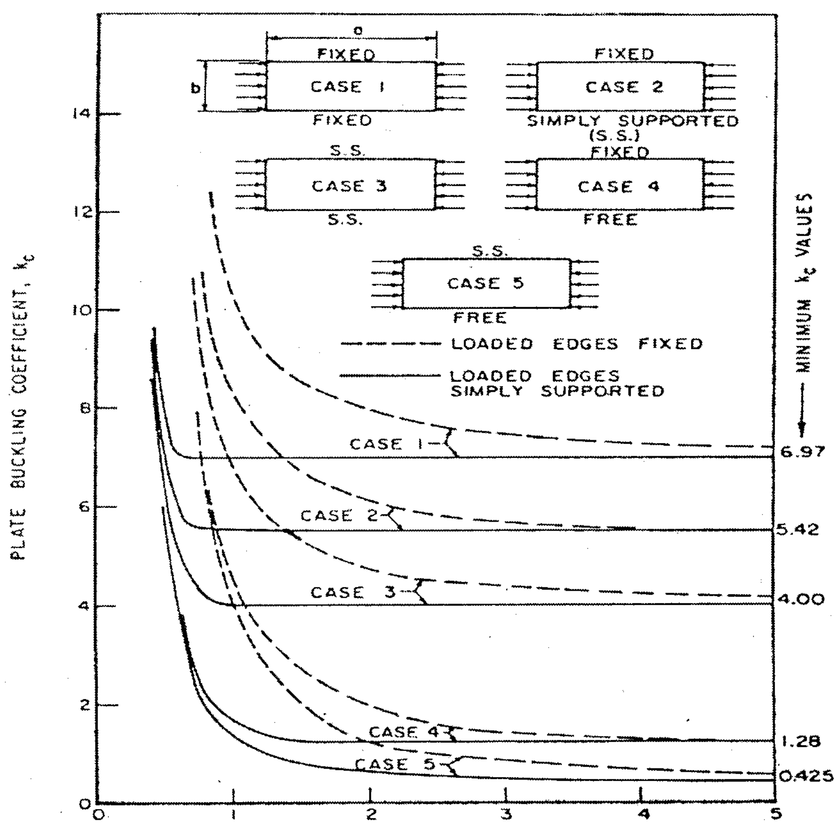
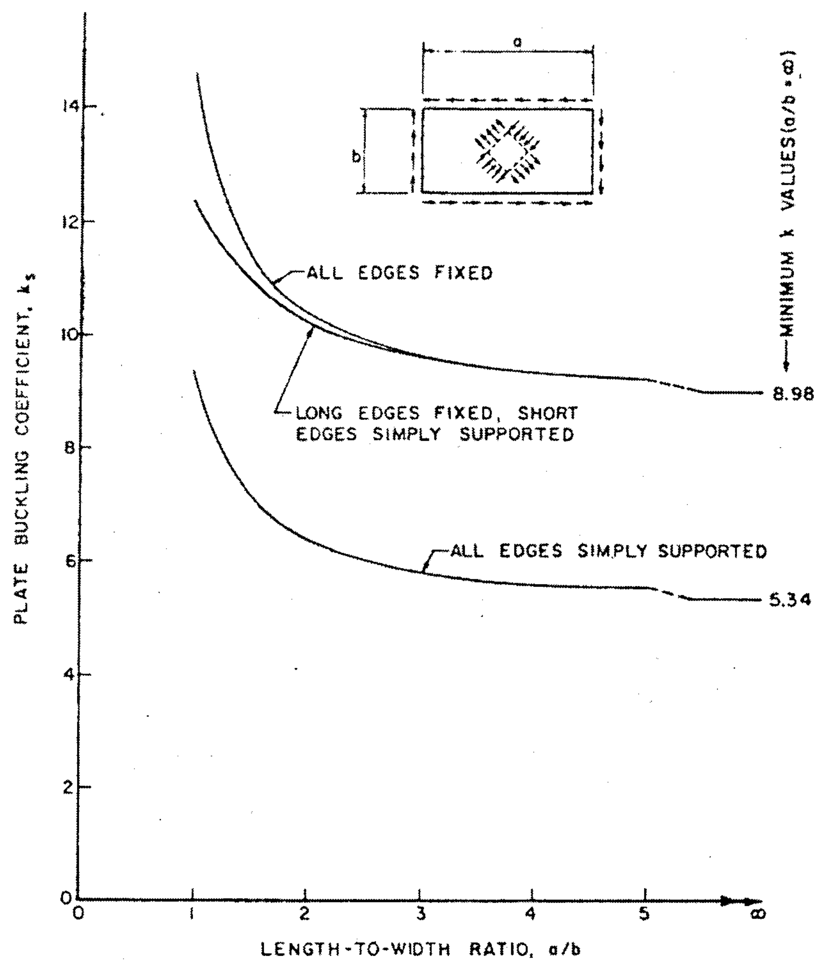


Figure 4-3: Buckling Coefficients for flat plates under uniform compression.

C. **Shear:** A plate subjected to uniformly distributed shear stress along all four edges develops internal tension and compression stresses that are a maximum on planes at 45 degrees to the edges and equal to the edge shear stress. Thus, when increasing shear stresses are applied, the internal compression stress increases and the plate eventually buckles.

Within the elastic range the value of the shear stress at buckling, f_{crs} , is

$$f_{crs} = \frac{k_s \pi^2 E}{12(1-\nu^2)(b/t)^2} \qquad \textbf{Eq.4-3}$$



where k_s is a non dimensional plate buckling coefficient that depends primarily on the type of edge supports and the a/b ratio. Except for small a/b ratios, plates with long edges fixed and short edges simply supported have the same buckling strength as plates with all edges fixed. If long edges are simply supported and short edges fixed, the buckling strength of long plates will be the same as if all edges were simply supported.

D. **Shear and Other Loadings:** When shear is simultaneously applied with bending and/or uniform compression, a plate will buckle before the applied stress reaches the lowest critical value calculated for independent loadings. Curves giving stress combinations that will cause buckling for three different combinations of loading and the interaction equations that define them are shown in figures 4-5, 4-6, and 4-7. The curves and equations are given in terms of stress ratios, that is, ratios of applied stress to buckling stress for independent loadings. For simultaneous loadings of shear, compression, and bending, a series of curves is given in 4-7 for various shear-stress ratios. The curve for zero shear stress ratio is an interaction curve for compression and bending.

Each curve has been derived for plates having a/b 2/1, and are conservative for smaller a/b ratios. The curves are based on elastic behavior, but will give approximate results in the inelastic range if the inelastic buckling stresses are calculated by the equations given in the preceding sections.

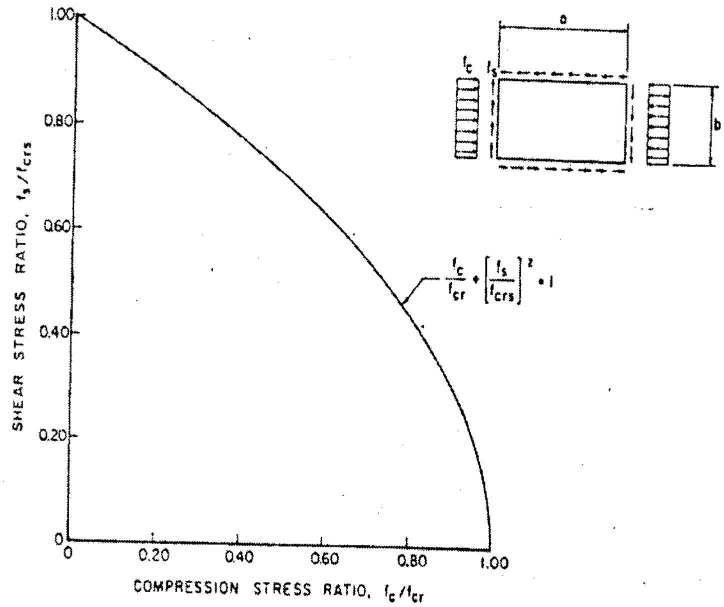


Figure 4-5: Interaction curve for buckling of flat plates under shear and uniform compression.

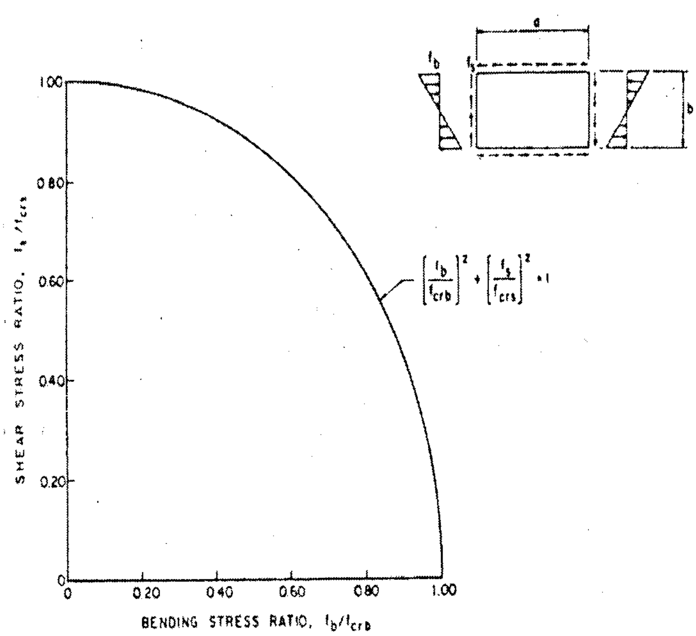


Figure 4-6: Interaction curve for buckling of flat plates under shear and bending.

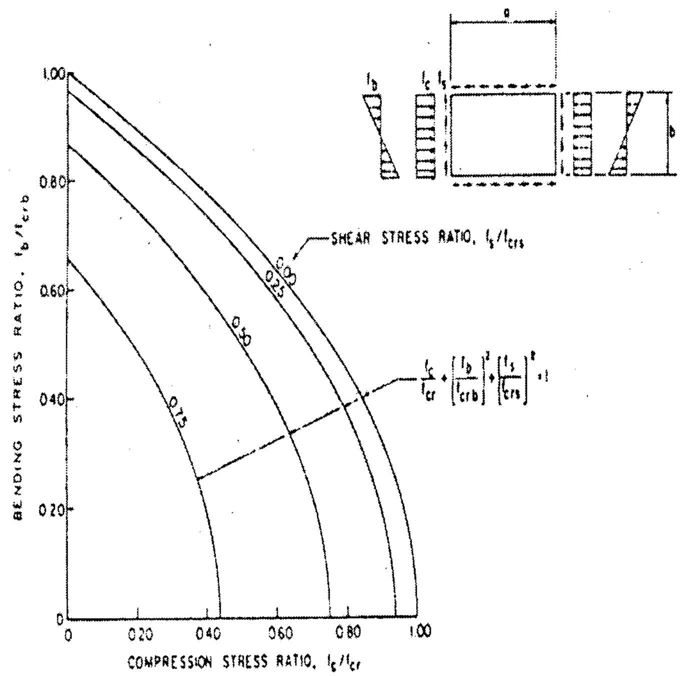


Figure 4-7: Interaction curve for buckling of flat plates under shear, compression, and bending.

5. ELASTIC STABILITY OF THIN PLATES

5.1 Deflections of Rectangular Plates with Simply Supported Edges:

In the case of a rectangular plate with simply supported edges (Figure 5-1), the deflection surface can be represented by the double trigonometric series

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \textbf{Eq.5-1}$$

Each term of this series vanishes for $x = 0$, $x =$ a and also for $y = 0$, $y =$ b. Hence the deflection w is zero along the boundary as required.

Calculating the derivatives $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$, we find again that each term of the calculated series becomes zero at the boundary. From this it can be concluded that the bending moments are zero along the boundary as they should be in the case of simply supported edges. The expression for the potential energy of bending for this case is

$$U=\frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dxdy$$

Substituting for the w expression, it can be shown that the integral of the term in the brackets vanishes and we obtain

$$U=\frac{1}{2}D\int_0^a\int_0^b\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right\}^2 dxdy$$

Only the squares of the terms of the infinite series give integrals different from zero.

Only the squares of the terms of the infinite series give integrals different from zero.

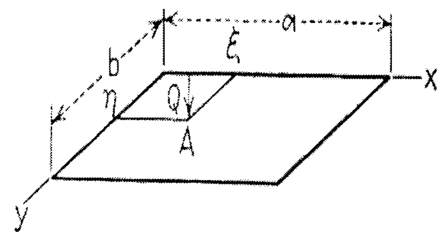


Figure 5-1

Then observing that

$$\int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} dxdy = \frac{ab}{4}$$

we obtain

$$U = \frac{ab}{8} D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}{}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 \qquad \text{.......................................} \textbf{Eq.5-2}$$

Having this expression for U, we can obtain the deflection of the plate for any kind of loading by using the principle of virtual displacements. Assume, for instance, that a concentrated load force Q is acting at a point A with coordinates ξ and η. To determine any coefficient a_{mn} of the series in this case, we give to this coefficient a small increment δa_{mn}. The corresponding virtual deflection of the plate is

$$\delta a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$Q\delta a_{mn} \sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

and from the principle of virtual displacements we obtain the following equation:

$$Q\delta a_{mn} \sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} = \frac{\partial U}{\partial a_{mn}}\delta a_{mn} = \frac{ab}{4}Da_{mn}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2 \delta a_{mn} \quad \text{..............(a)}$$

from which

$$a_{mn} = \frac{4Q\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{abD\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \quad \text{................................(b)}$$

When this is substituted in Eq. 5-1, the deflection of the plate produced by a concentrated load Q is obtained. Having this deflection and using the principle of superposition, we can determine the deflections for any kind of loading.

5.2 Buckling of a Simply Supported Rectangular Plate under Combined Bending and Compression:

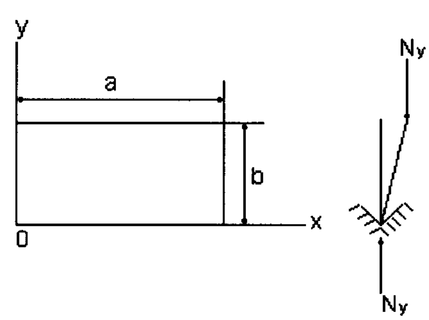


Figure 5-2

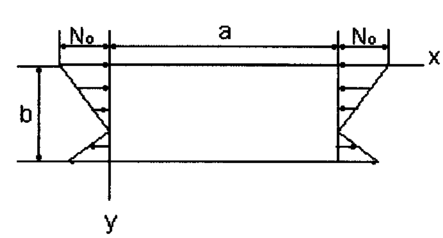


Figure 5-3

Consider a simply supported rectangular plate (Fig. 5-3) along whose sides x = 0 and x = a distributed forces, acting in the middle plane of the plate, are applied, their intensity being given by the equations

$$N_x = N_0\left(1-\alpha\frac{y}{b}\right) \qquad \textbf{Eq.5-3}$$

where N_0 is the intensity of compressive force at the edge y =0 and α is a numerical factor. By changing α, we can obtain various particular cases. For example, by taking $\alpha = 2$ we obtain the case of pure bending. If α is less than 2, we have a combination of bending and compression as indicated in Figure 5-1. If $\alpha > 2$, there will be a similar combination of bending and tension.

The deflection of buckled plate simply supported on all sides can be taken in the form of the double trigonometric series

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

For calculating the critical value of the compressive force N_0 we use the energy method. For the strain energy of bending due to the deflections w use

$$\Delta U = \frac{D}{2}\frac{ab\pi^4}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}{}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \qquad \textbf{Eq 5-4}$$

The work done by the external forces during buckling of the plate is

$$\Delta T = \frac{1}{2}\int_0^a\int_0^b N_0\left(1-\alpha\frac{y}{b}\right)\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad \textbf{Eq.5-5}$$

substituting the value of w, we obtain, for the work done by external forces during buckling, the following expression:

$$\Delta T = \frac{N_0}{2}\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}{}^2 \frac{m^2\pi^2}{a^2} - \frac{N_0}{2}\frac{\alpha a}{2b}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[\frac{b^2}{4}\sum_{n=1}^{\infty} a_{mn}{}^2 - \frac{8b^2}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}^{\infty}\frac{a_{mn}a_{mi}ni}{\left(n^2-i^2\right)}\right]$$

where for i only such numbers are taken that n ± i is always odd.

Equating this work to the strain energy of bending ΔU, we obtain for the critical value of N_0 an equation which states that $(N_0)_{cr}$ is equal to

$$\frac{\pi^4 D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}{}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}{}^2\frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[\sum_{n=1}^{\infty} a_{mn}{}^2 - \frac{32}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}^{\infty}\frac{a_{mn}a_{mi}ni}{(n^2-i^2)^2}\right]}$$

The coefficients a_{mn} must be adjusted now so as to make the obtained expression for $(N_0)_{cr}$ a minimum. By taking the derivatives of this expression with respect to each coefficient a_{mn} and equating these derivatives to zero, we finally obtain a system of linear equations of the following form:

$$Da_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 = (N_0)_{cr}\left\{a_{mn}\frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\frac{m^2\pi^2}{a^2}\left[a_{mn} - \frac{16}{\pi^2}\sum_{i}^{\infty}\frac{a_{mi}ni}{\left(n^2-i^2\right)^2}\right]\right\} \quad \textbf{....Eq5-6}$$

Collect all equations with a certain value of the number m. These equations will contain coefficients a_{m1}, a_{m2}, a_{m3}, ... All other coefficients equal to zero, so for the deflection of plate, take the expression

$$w = \sin\frac{m\pi x}{a}\sum_{n=1}^{\infty} a_{mn}\sin\frac{n\pi y}{b}$$

which is equivalent to the assumption that the buckled plate is subdivided along the x axis into m half-waves. Consider one half-wave between two nodal lines as a simply supported plate which is buckled into one half-wave. Substituting $m = 1$ in Eq. 5-6 and using the notation

$$\sigma_{cr} = \frac{(N_0)_{cr}}{t}$$

we obtain a system of equations of the following kind:

$$a_{1n}\left[\left(1+n^2\frac{a^2}{b^2}\right)^2 - \sigma_{cr}\frac{a^2 t}{\pi^2 D}\left(1-\frac{\alpha}{2}\right)\right] - 8\alpha\sigma_{cr}\frac{a^2 t}{\pi^4 D}\sum_i^{\infty}\frac{a_{1i}n_i}{\left(n^2-i^2\right)^2} = 0 \quad \textbf{Eq 5-7}$$

where the summation is taken over all numbers i such that $n \pm i$ is an odd number.

These are homogeneous linear equations in a_{11}, a_{12}, ... which will be satisfied by putting a_{11}, a_{12}, ... equal to zero, which corresponds to the flat form of equilibrium of the plate. To get for the coefficients a_{11}, a_{12}, ... solutions different from zero, which indicates the possibility of buckling of the plate, the determinant of the Eq. 5-7 must be zero. In this way an equation for calculating the critical values of compressive stresses is obtained. The calculation can be made by successive approximations. Begin by taking only the first of the Eq. 5-7 and assuming that all coefficients except a_{11} are zero. In this way we obtain

$$\left(1+\frac{a^2}{b^2}\right)^2 - \sigma_{cr}\frac{a^2 t}{\pi^2 D}\left(1-\frac{\alpha}{2}\right) = 0$$

from which

$$\sigma_{cr} = \frac{\pi^2 D}{a^2 t}\left(1+\frac{a^2}{b^2}\right)^2\frac{1}{1-\alpha/2} = \frac{\pi^2 D}{b^2 t}\left(\frac{b}{a}+\frac{a}{b}\right)^2\frac{1}{1-\alpha/2} \quad \textbf{Eq.5-8}$$

This first approximation gives a satisfactory result only for small values of α. To obtain a second approximation, two equations of the system 5-7 with coefficients a_{11} and a_{12} should be taken, and we obtain

$$a_{11}\left[\left(1+\frac{a^2}{b^2}\right)^2-\sigma_{cr}\frac{a^2t}{\pi^2D}\left(1-\frac{\alpha}{2}\right)\right]-8\alpha\sigma_{cr}\frac{a^2t}{\pi^4D}\frac{2}{9}a_{12}=0$$

$$-8\alpha\sigma_{cr}\frac{a^2t}{\pi^4D}\frac{2}{9}a_{11}+\left[\left(1+4\frac{a^2}{b^2}\right)^2-\sigma_{cr}\frac{a^2t}{\pi^2D}\left(1-\frac{\alpha}{2}\right)\right]a_{12}=0$$

Equating to zero the determinant of these equations, we obtain

$$\left(\frac{\sigma_{cr}a^2t}{\pi^2D}\right)^2\left[\left(1-\frac{\alpha}{2}\right)^2-\left(\frac{8\alpha}{\pi^2}\frac{2}{9}\right)^2\right]-\frac{\sigma_{cr}a^2t}{\pi^2D}\left(1-\frac{\alpha}{2}\right)\left[\left(1+\frac{a^2}{b^2}\right)^2+\left(1+4\frac{a^2}{b^2}\right)^2\right]+\left(1+\frac{a^2}{b^2}\right)^2\left(1+4\frac{a^2}{b^2}\right)^2=0$$

..........................Eq.5-9

From this equation the second approximation for σ_{cr} can be calculated. The accuracy of this approximation decreases as α increases; for pure bending, when $\alpha = 2$, and for a square plate the error is about 8 per cent, so that the calculation of a further approximation is necessary to obtain a more satisfactory accuracy. By taking three equations of the system Eq. 5-7 and assuming $\alpha = 2$, we obtain

$$\left(1+\frac{a^2}{b^2}\right)^2a_{11}-16\sigma_{cr}\frac{a^2t}{\pi^4D}\frac{2}{9}a_{12}=0$$

$$-16\sigma_{cr}\frac{a^2t}{\pi^4D}\frac{2}{9}a_{11}+\left(1+4\frac{a^2}{b^2}\right)^2a_{12}-16\sigma_{cr}\frac{a^2t}{\pi^4D}\frac{6}{25}a_{13}=0$$

$$-16\sigma_{cr}\frac{a^2 t}{\pi^4 D}\frac{6}{25}a_{12}+\left(1+9\frac{a^2}{b^2}\right)^2 a_{13}=0$$

Equating to zero the determinant of these equations, we obtain an equation for calculating the third approximation which is sufficiently accurate for the case of pure bending.

The final expression for σ_{cr} can be represented by the equation

$$\sigma_{cr}=k\frac{\pi^2 D}{b^2 t}$$

6. FINITE ELEMENT ANALYSIS

The linear elastic buckling coefficient was obtained by using a linear critical buckling analysis program using FEA. The results obtained from the FEA were verified as follows: A flat plate with known linear elastic buckling stress and buckling coefficient was modeled using the FEA program. The linear elastic buckling coefficient obtained from the FEA program was then compared to a closed-form solution.

6.1 Linear elastic buckling strength

The linear elastic buckling load was obtained by using a commercially available FEA-program, ALGOR –software. This program determines the load which brings the structure to the bifurcation point. The predicted buckling load is the Euler buckling load. The bifurcation point is defined as: The compressed member can be in equilibrium in two different configurations at the bifurcation point. (Galambos,1988). The member can either be straight or in a slightly deflected shape. If a plot of axial load versus lateral deflection is made, a branch point occurs at the bifurcation point. After the bifurcation point two different load/deflection curves are mathematically valid. Idealized load-displacement paths are shown in Figure 6-1 below:

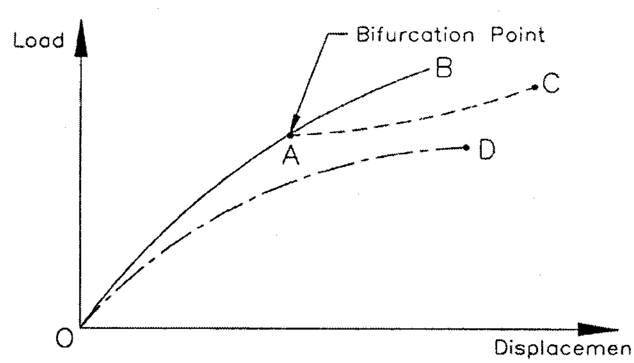


Figure 6-1: Idealized load-displacement paths.

The solid line represents a perfect structure and displaces along a basic path (OAB) and the bifurcation occurs at point A. The dashed line AC represents the post-buckling path, and this path can either rise or descend depending on what type of structure and loading is considered. The third line (OD), applies to structures with initial imperfections and as can be seen a bifurcation point does not exist.

The basic equation to be solved in finite element analysis is

$$\{F\}=[K]\{d\} \qquad \textbf{Eq.6-1}$$

where: F = global nodal force vector
K = global stiffness matrix
d = global nodal displacement vector

By using the minimum potential energy principle the total potential energy of a plate element can be written as (Allen, 1980):

$$V_e=\frac{1}{2}\{\delta\}^T[K_L+K_G]\{\delta\} \qquad \textbf{Eq.6-2}$$

Where: V_e = total potential energy of the plate element
δ = nodal displacement vector
K_L = linear stiffness matrix
K_G = geometric stiffness matrix

At the critical load, the total potential energy is a minimum and the homogeneous equation given below has a non-trivial solution.

$$[K_L+K_G]\{\delta\}=0 \qquad \textbf{Eq.6-3}$$

The critical stress is then the smallest root of the determinant of the following equation:

$$\det(K_L + \lambda_{cr} K_G) = 0 \quad \textbf{Eq.6-4}$$

where: λ_{cr} = the buckling load factor.

As was the case in the theoretical development of the critical buckling load, the FEA-model of the flat plate is assumed to be perfectly flat and entirely elastic, i.e., the buckling takes place in the elastic region. The linear elastic buckling load was found for the flat plate shown below with non-continuous boundary conditions. The plate is subjected to combined bending and compression and has one unloaded edge free.

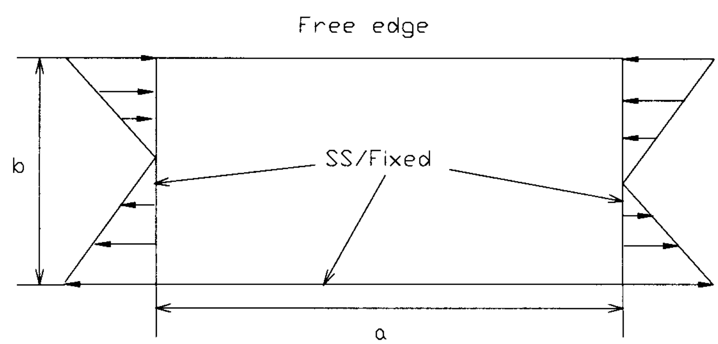


Figure 6-2: Flat plate under pure bending with top edge free.

where: a = length of the plate

b = width of the plate

SS/Fixed = simply supported or fixed edge

Ten aspect ratios, *a/b* were investigated for the above case: *a/b* = 0.5, 0.75, 1, 1.5, 2, 2.4, 2.8, 3.2, 3.6, 4. Materials properties for these ten different models are:

Thickness, t (in): 0.05

Young's modulus (psi): 30E6

Poisson's ratio: 0.30

Dimensions for the ten different plate models are given below:

a/b	Length, a (in)	Width, b (in)
0.50	1.00	2.00
0.75	1.50	2.00
1.00	2.00	2.00
1.50	3.00	2.00
2.00	4.00	2.00
2.40	4.80	2.00
2.80	5.60	2.00
3.20	6.40	2.00
3.60	7.20	2.00
4.00	8.00	2.00

The same above ten aspect ratios were also investigated for the plate with three side's totally fixed and top unloaded edge free. The closed-form solutions are known for plates when the top edge is supported along the entire edge. The finite element solution should, therefore, correspond to the closed-form solution at this point. One verification of the FEA – model and the corresponding buckling load can therefore be done by considering the above case.

The linear elastic buckling strength was obtained by using a linear eigenvalue solver (ALGOR-software). In the next section, the finite element models will be described showing elements used, boundary conditions and applied loads.

6.2 Grid of cases to consider: The following six grid of cases were investigated with the following two boundary conditions:

(1) Flat plate with 3 sides simply supported and the top unloaded edge free

(2) Flat plate with 3 sides totally fixed and the top unloaded edge free

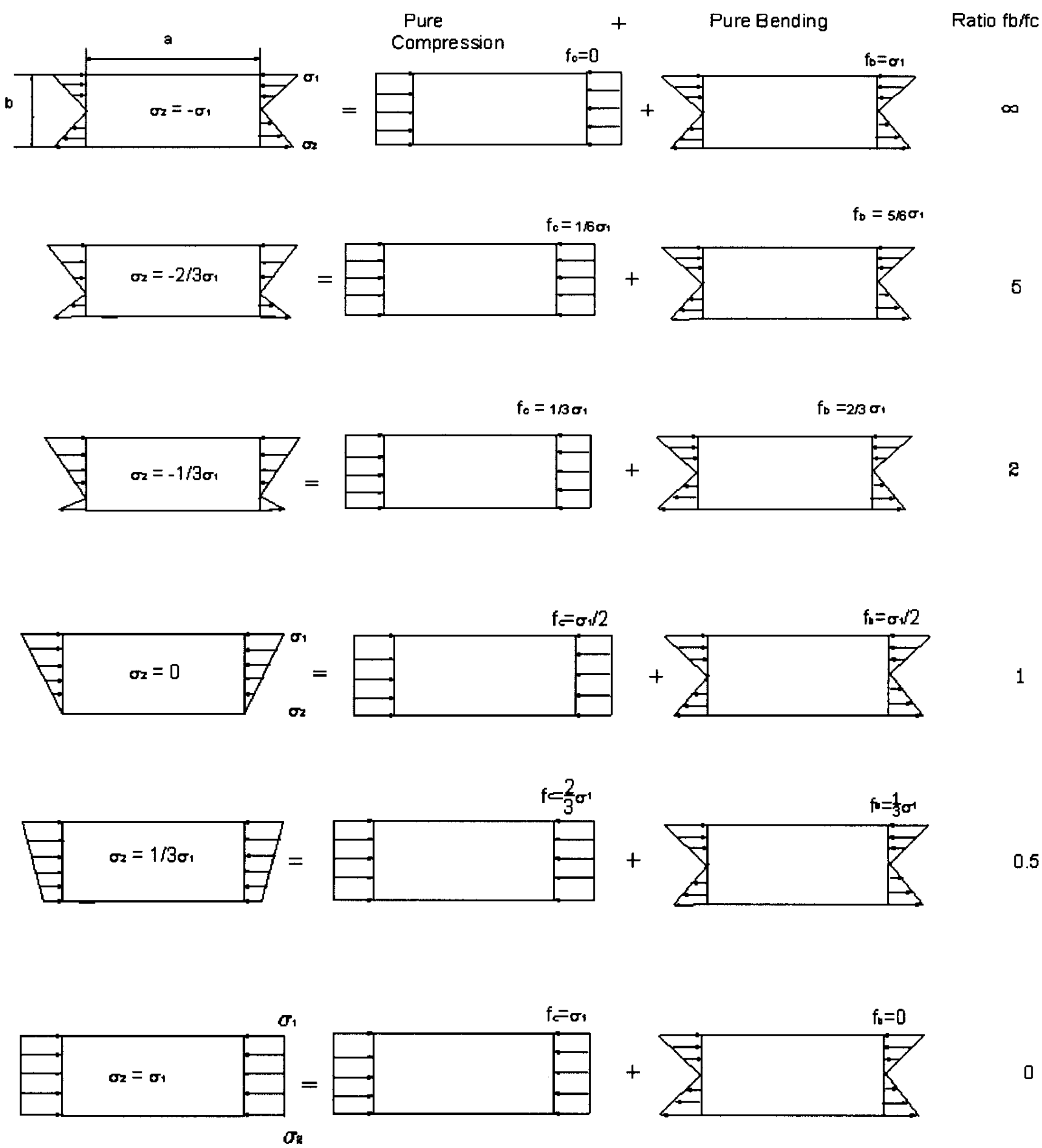
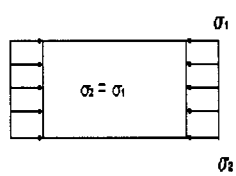
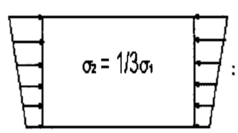
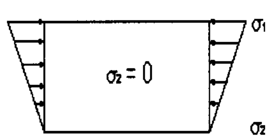


Figure 6-3: Flat plates under compression and bending

6.3 Verification of model results:

There are no closed form solutions available with one side of the plate unsupported. To verify the FEA method used a known solution was modeled. This verification solution has all four sides simply supported. The results of this verification are given below and show good correlation.

Minimum Buckling Coefficient, K			
Loading	Ratio of Bending Stress to Uniform Compresion Stress σ_{cb}/σ_c	All Edges Simply Supported	
		Closed-form Solution*	FEA
	0.0	4.0	4.0032
	0.50	5.8	5.9
	1.00	7.8	7.84

***Values given are from "*Guide to Stability Design Criteria for Metal Structures*" edited by Theodore V. Galambos, 4th Edition, pg. 103.**

6.4 Description of the finite element models:

6.4.1 FEA-model of the plate with 3 sides Simply Supported and the unloaded top edge Free:

edge Free: Shown in Figure 6-4 is a sketch of the finite element model used in the linear elastic buckling. Boundary conditions are as follows: Translational boundary conditions are denoted by T_x, T_y, T_z and rotational boundary conditions are denoted by R_x, R_y, R_z

where the subscript denotes the axis along which the translational and rotational boundary constraint is applied.

In Figure 6-4 below, the boundary conditions are given as simply supported along the edges. From Eq.3-6 in the theory section, the mathematical definition of a simple support was given as:

$$(w)_{y=0} = 0 \quad \text{and} \quad \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_{y=0} = 0$$

The first equation states that there are no deflections along the supported edges, and the second equation indicates that there are no bending moments along the supported edges. Translational boundary conditions must therefore be used in the FEA-model. Referring to Figure 6-4, on the left edge, boundary conditions T_x and T_y are applied along the edge. Translations are therefore restricted in the x-and y-directions, where the x-direction is the out of plane direction and the y-direction is the longitudinal direction. The bottom and the right edge only have the out-of-plane boundary condition, T_x. With these boundary conditions, there are no deflections in the x-direction along the edges and the plate can still contract in the y-and z-directions; also, no rotational restrictions are imposed on the plate model. The applied load was distributed along one edge of the plate. At the two corner nodes one half the nodal load was applied, as shown below:

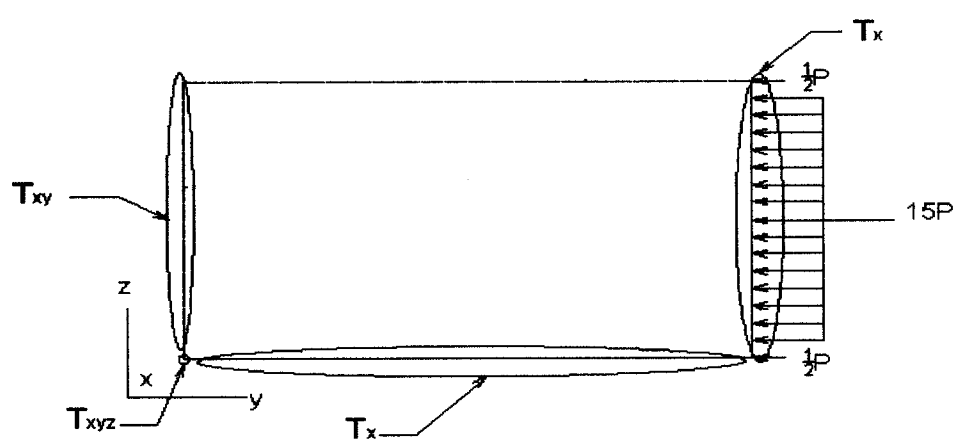


Figure 6-4: FEA-model of plate with 3 sides simply supported and top unloaded edge free.

6.4.2 FEA-model of the plate with 3 sides totally Fixed and the unloaded top edge Free:

Shown in Figure 6-5 is the sketch of FEA-model used. From Eq.3-7 the mathematical definition of a fixed support was given as:

$$(w)_{y=0} = 0 \quad \text{and} \quad \left(\frac{\partial w}{\partial y}\right)_{y=0} = 0$$

The equations state that deflection is zero along the fixed edge and the slope of the middle plane of the plate is zero for a fixed edge. Translational and rotational boundary conditions must therefore be used in the FEA-model. Referring to the Figure 6-5, on the left edge, boundary conditions T_x and T_y and R_{yz} are applied along the edge. Translations are therefore restricted in *x*-and *y*-directions and rotations about *y*- and *z*-directions. On the bottom edge the boundary conditions T_x and R_y are applied which constraints the translation in the *x*-direction and the rotation about the *y*-direction. The right edge has the out-of-plane translational boundary condition, T_x and transverse rotational boundary condition, R_z. The applied load was distributed along one edge of the plate. At the two corner nodes one half the nodal load was applied, as shown below:

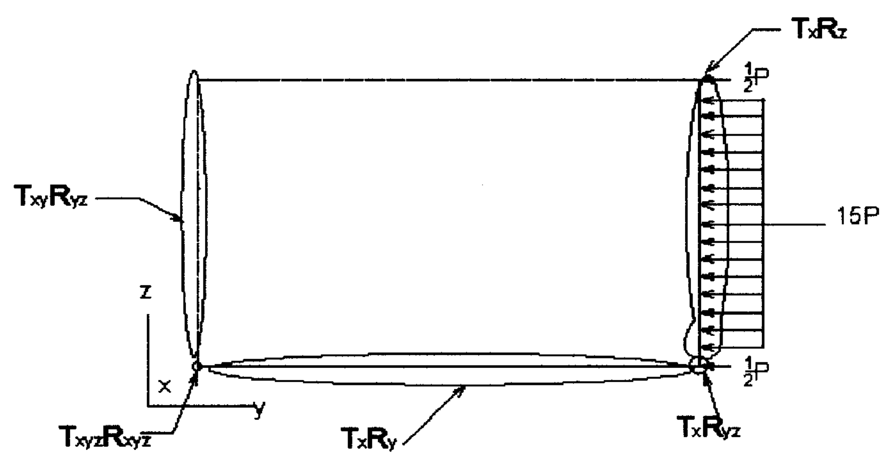


Figure 6-5: FEA-model of plate with 3 sides fixed and top unloaded edge free

In the next sections, results are presented for different grid of cases for different types of plates that were investigated in this work.

6.4.3 Pure Bending:

In the case where the plate is under pure bending only bending stress f_b exists and the compressive stress f_c is zero. Shown below is a chart for the critical buckling coefficient, K, for the pure bending case of a flat plate with different aspect ratios. The different aspect ratios are plotted against the *x*-axis and the respective critical buckling coefficients are plotted against the *y*-axis.

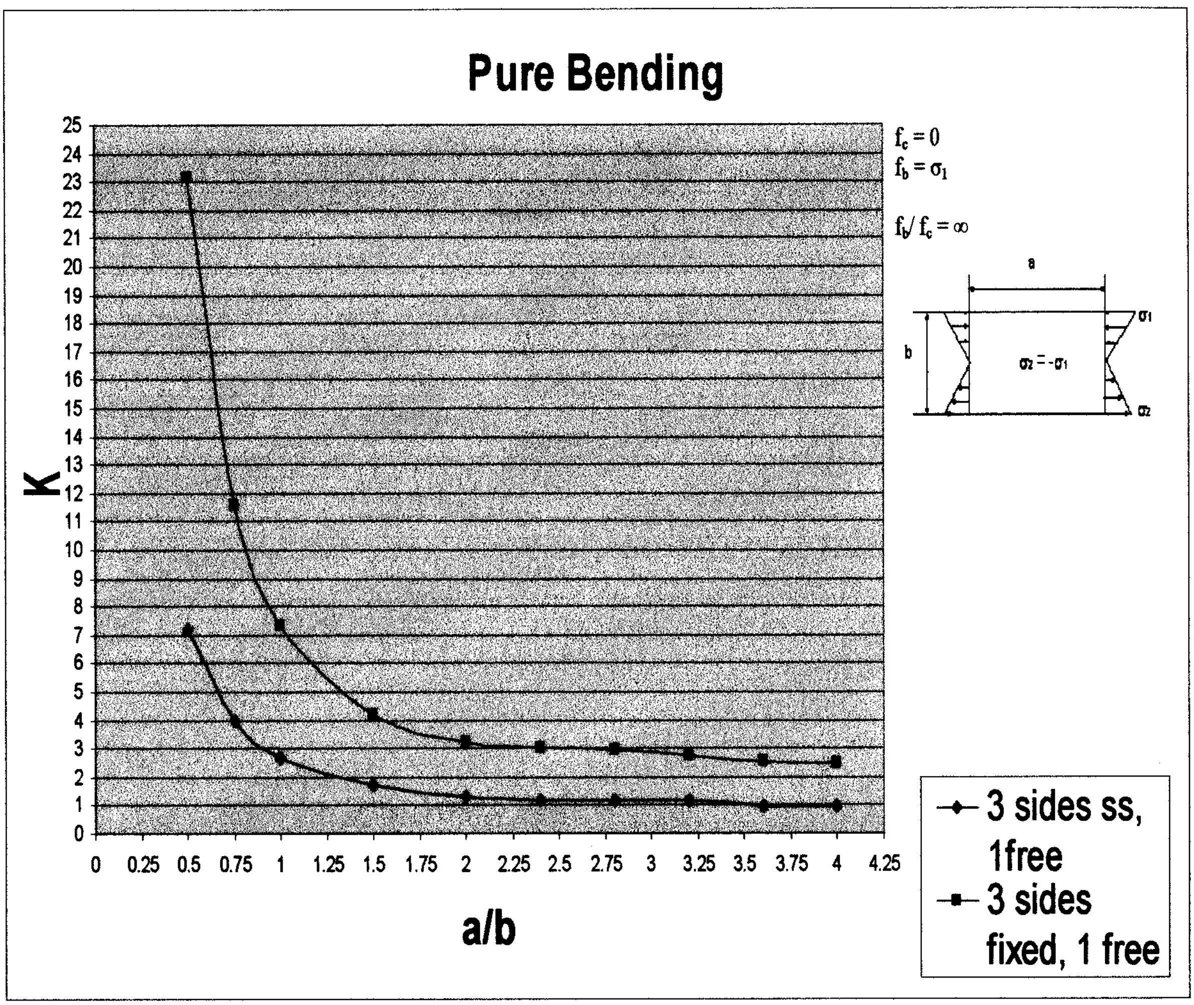


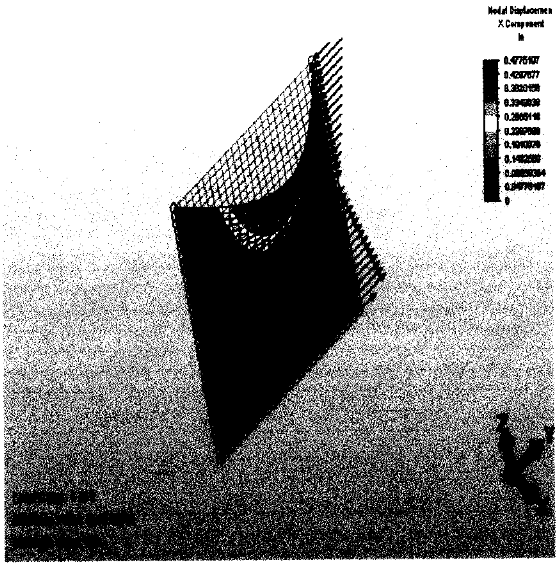
Figure 6-6: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = -\sigma_1$

From the above chart it is interesting to note that the buckling coefficient increases as the aspect ratio decreases. The buckling coefficients for different aspect ratios of the plates with 3 sides fixed and 1 side free are higher when compared to the plates with 3 sides simply supported and 1 side free as seen from the chart.

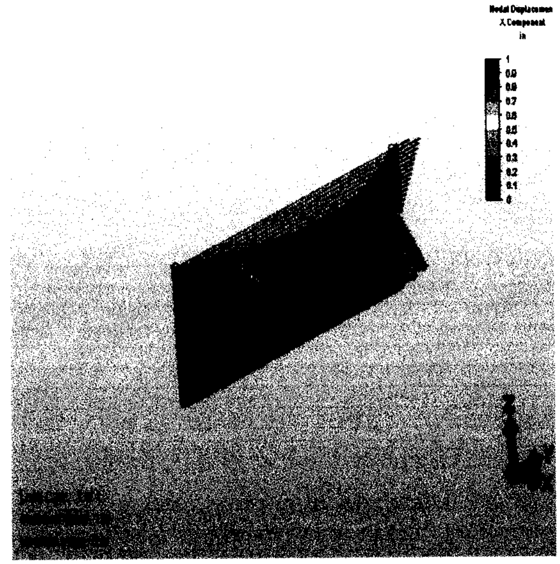
Shown below are pictures of the buckled mode shapes for the plates with two different boundary conditions under pure bending.

<u>Case 1</u>: Pure Bending: $\sigma_2 = -\sigma_1$

Figure 6-7: Buckled mode shapes of plates with 3 sides simply supported and top edge free:



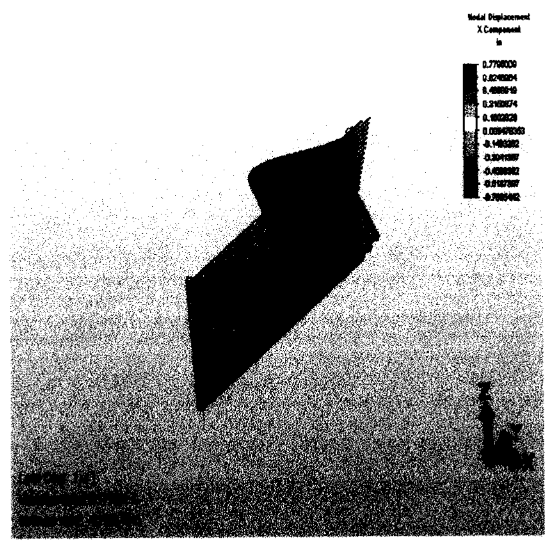
Buckled mode shape with a/b = 0.75



Buckled mode shape with a/b = 3.2

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:

Buckled mode shape with a/b = 0.75



Buckled mode shape with a/b = 3.2

6.4.4 Grid of case with $\sigma_2 = -2/3\ \sigma_1$:

In this case the plate is under combined bending and compression where both the f_b and f_c exist. Chart for the critical buckling coefficient, K, of a flat plate with different aspect ratios is shown below with aspect ratios plotted against the horizontal-axis and the respective critical buckling coefficients against vertical-axis.

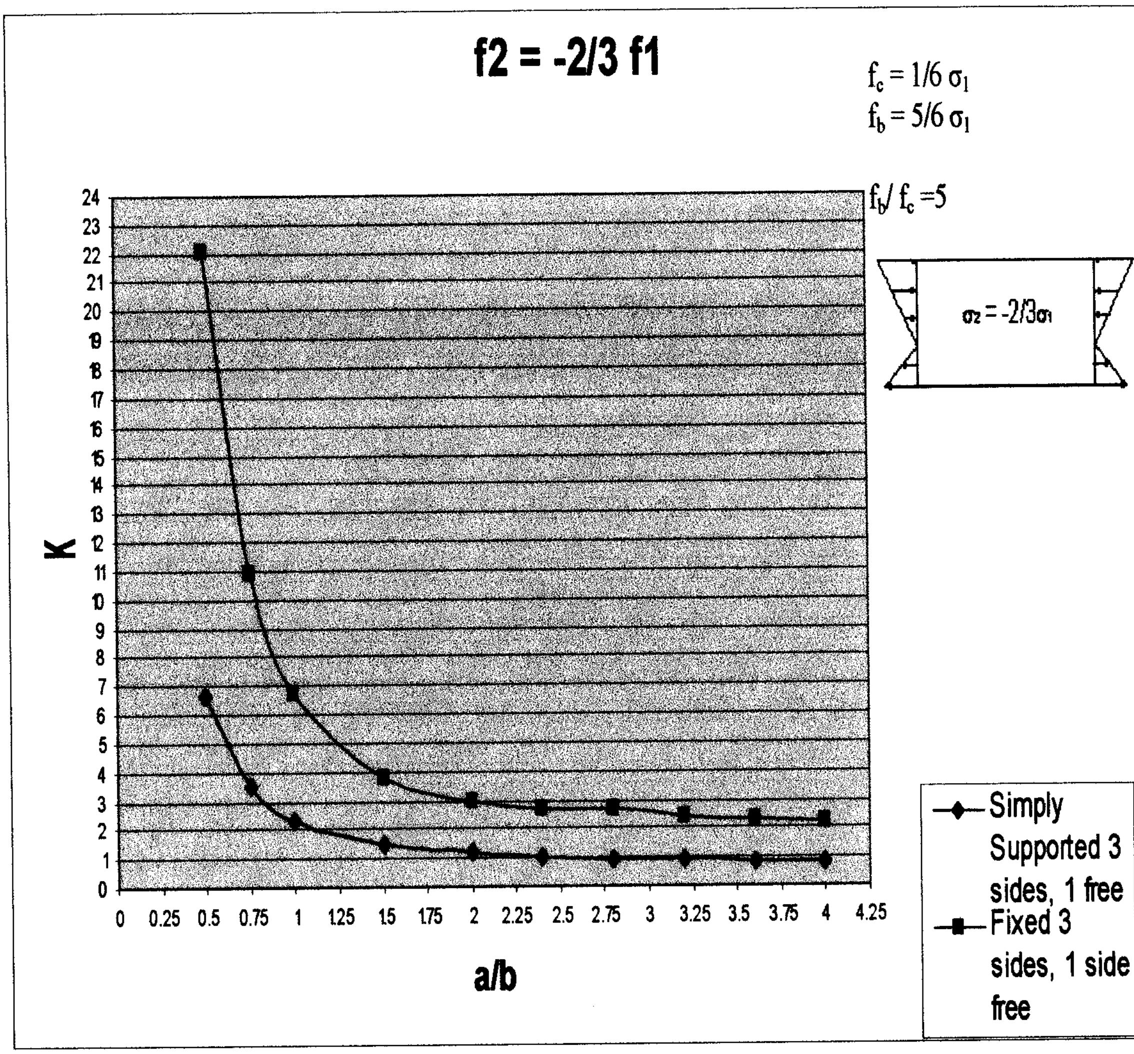
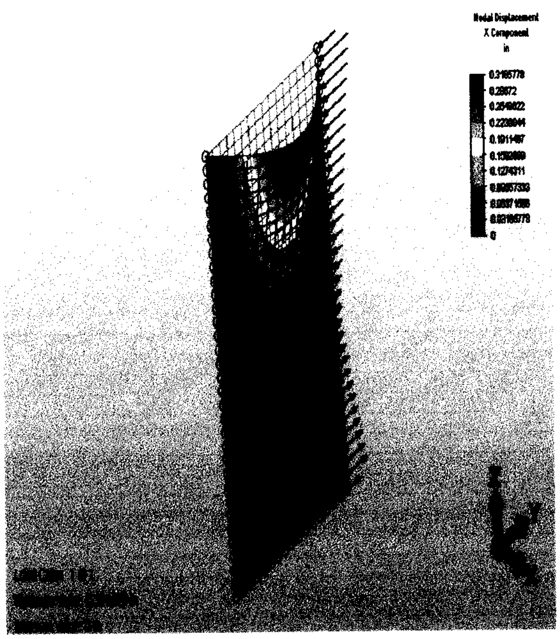


Figure 6-8: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = -2/3\ \sigma_1$

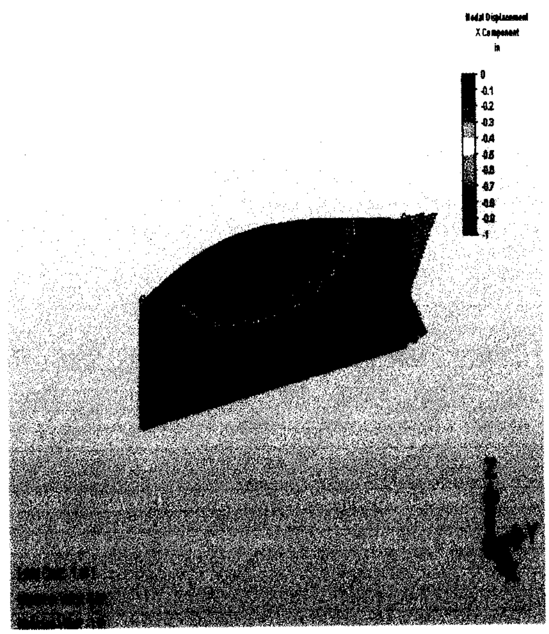
As seen from the chart the buckling coefficients for different aspect ratios of the plates with 3 sides fixed and 1 side free are higher when compared to the plates with 3 sides simply supported and 1 side free. The K increases with decreasing a/b ratio. Buckled mode shapes are shown below.

Case 2: $\sigma_2 = -2/3\ \sigma_1$

Figure 6-9: Buckled mode shapes of plates with 3 sides simply supported and top edge free:

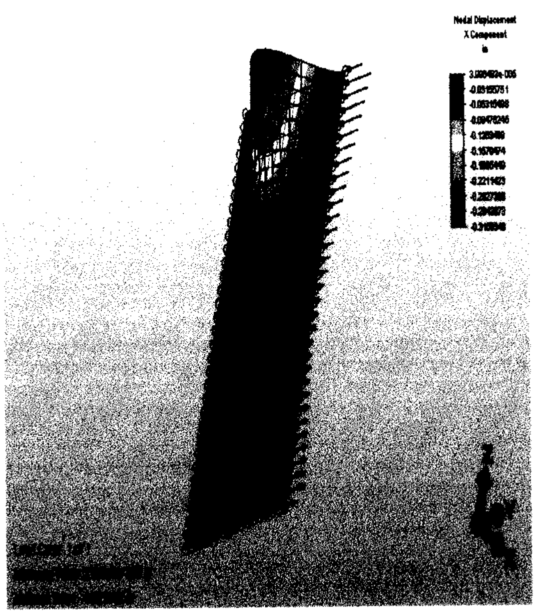


Buckled mode shape with a/b = 0.5

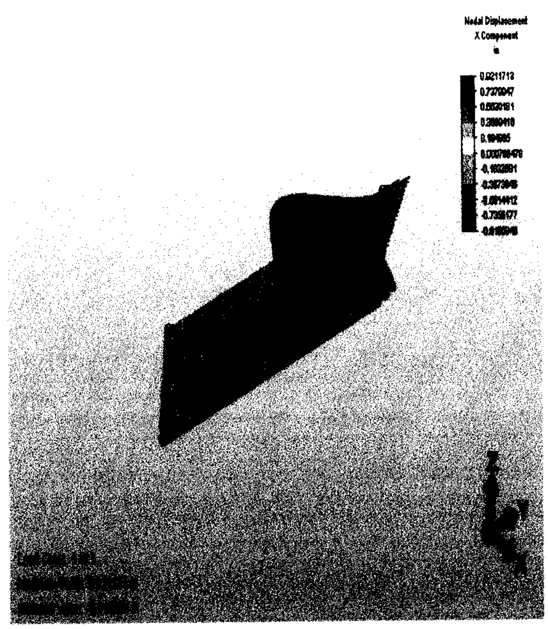


Buckled mode shape with a/b = 4

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:



Buckled mode shape with a/b =0.5



Buckled mode shape with a/b = 4

6.4.5 Grid of case with $\sigma_2 = -1/3\ \sigma_1$:

The flat plate in this case is under combined bending and compression with different values of f_b and f_c when compared to the last case. Shown below is a chart for the critical buckling coefficient, K, for the case $\sigma 2 = -1/3\ \sigma_1$ with different aspect ratios.

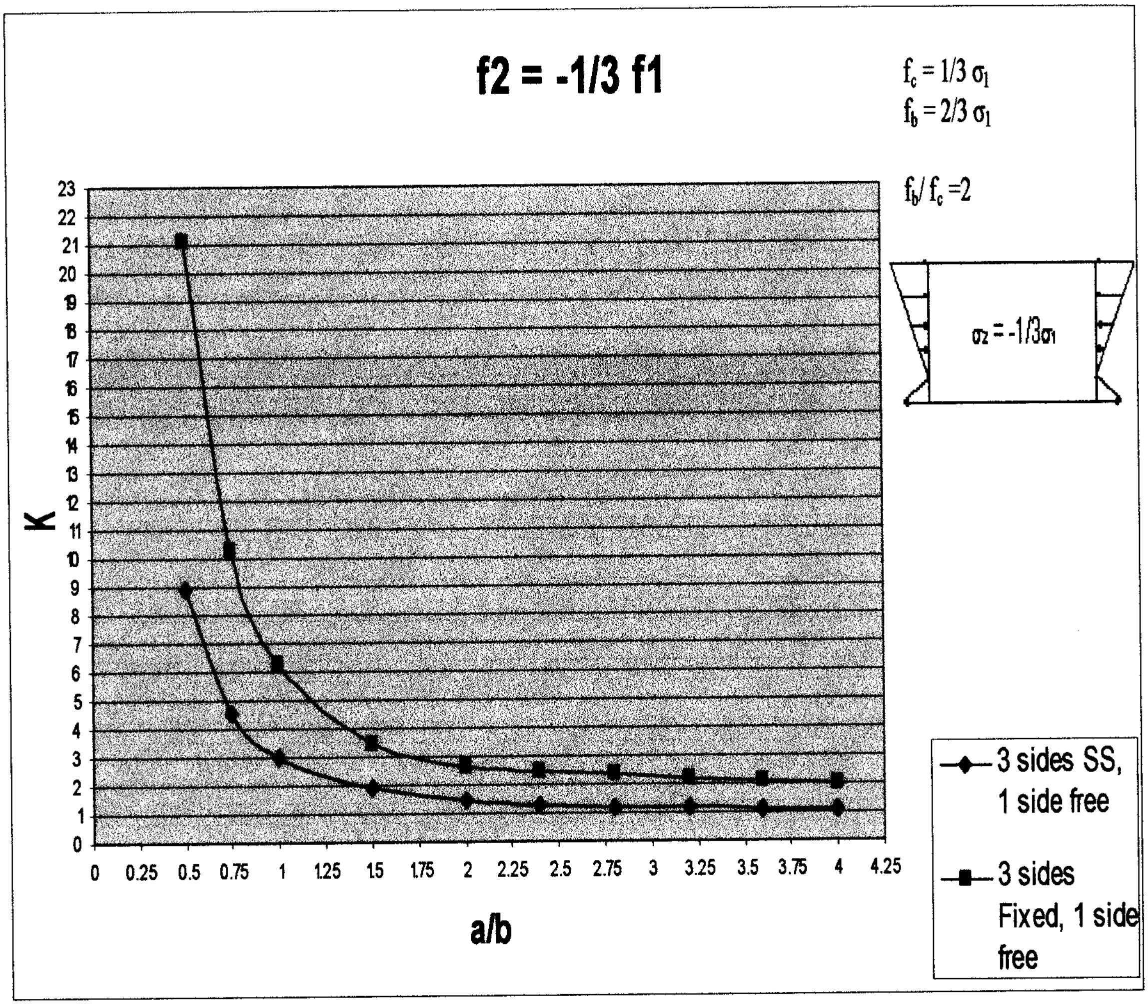


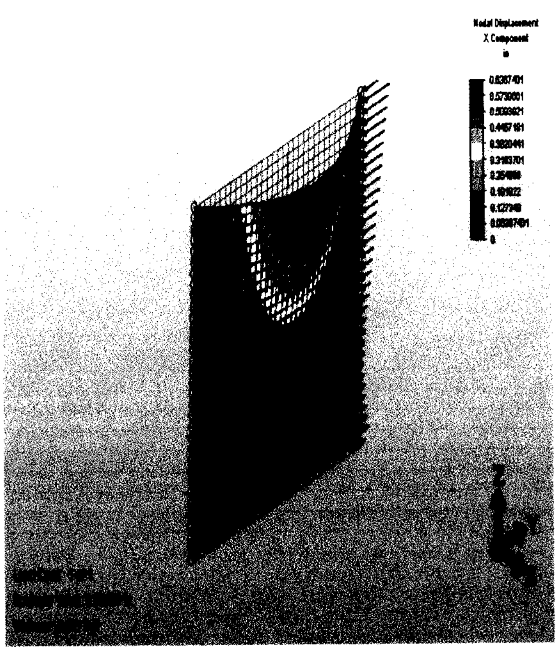
Figure 6-10: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = -1/3\ \sigma_1$

It can be noted from the chart above that the buckling coefficients for different aspect ratios of the plates with 3 sides fixed and 1 side free are higher when compared to the plates with 3 sides simply supported and 1 side free and also that with decreasing a/b ratio the value of K increases.

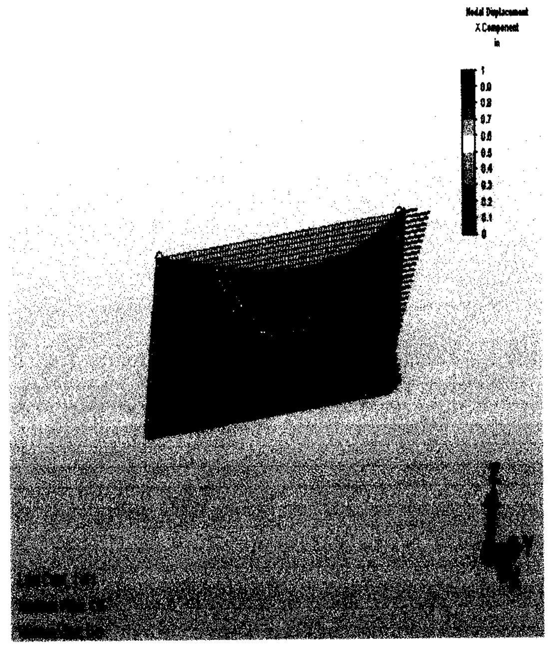
Pictures of the buckled mode shapes for the plates with two different boundary conditions are shown below.

Case 3: $\sigma_2 = -1/3\,\sigma_1$

Figure 6-11: Buckled mode shapes of plates with 3 sides simply supported and top edge free:

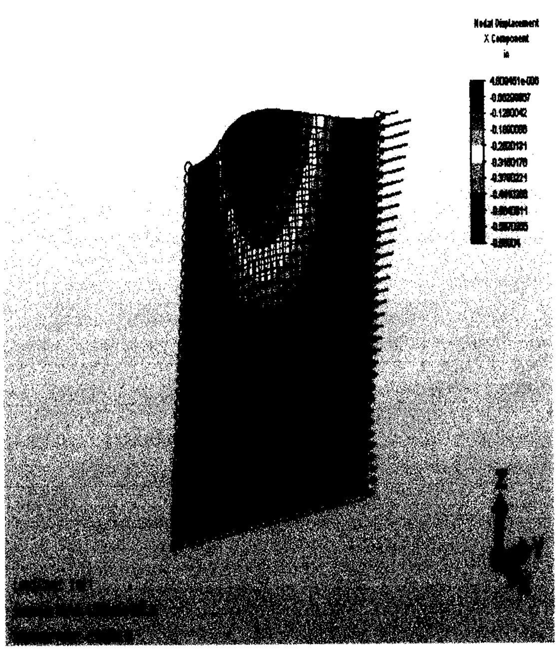


Buckled mode shape with a/b = 1

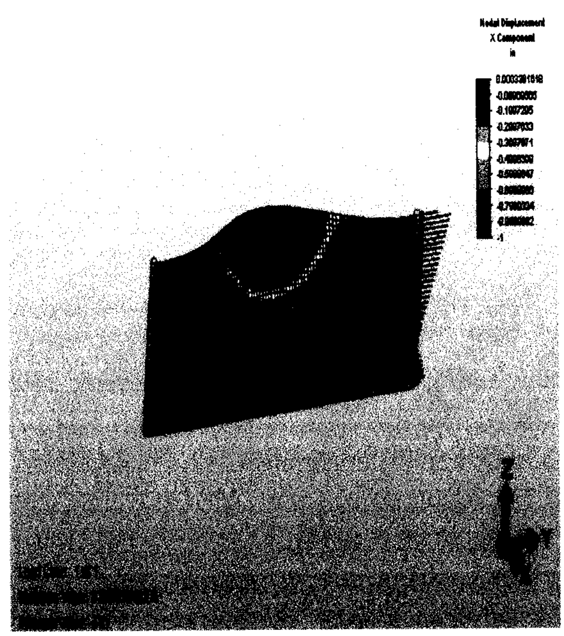


Buckled mode shape with a/b = 2.8

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:



Buckled mode shape with a/b = 1



Buckled mode shape with a/b = 2.8

6.4.6 Grid of case with $\sigma_2 = 0$:

In this case where the flat plate is under combined bending and compression both the bending and compressive stresses are equal. Chart for the critical buckling coefficient, K, of a flat plate with different aspect ratios is shown below. The aspect ratios are plotted against the x-axis and K along y-axis.

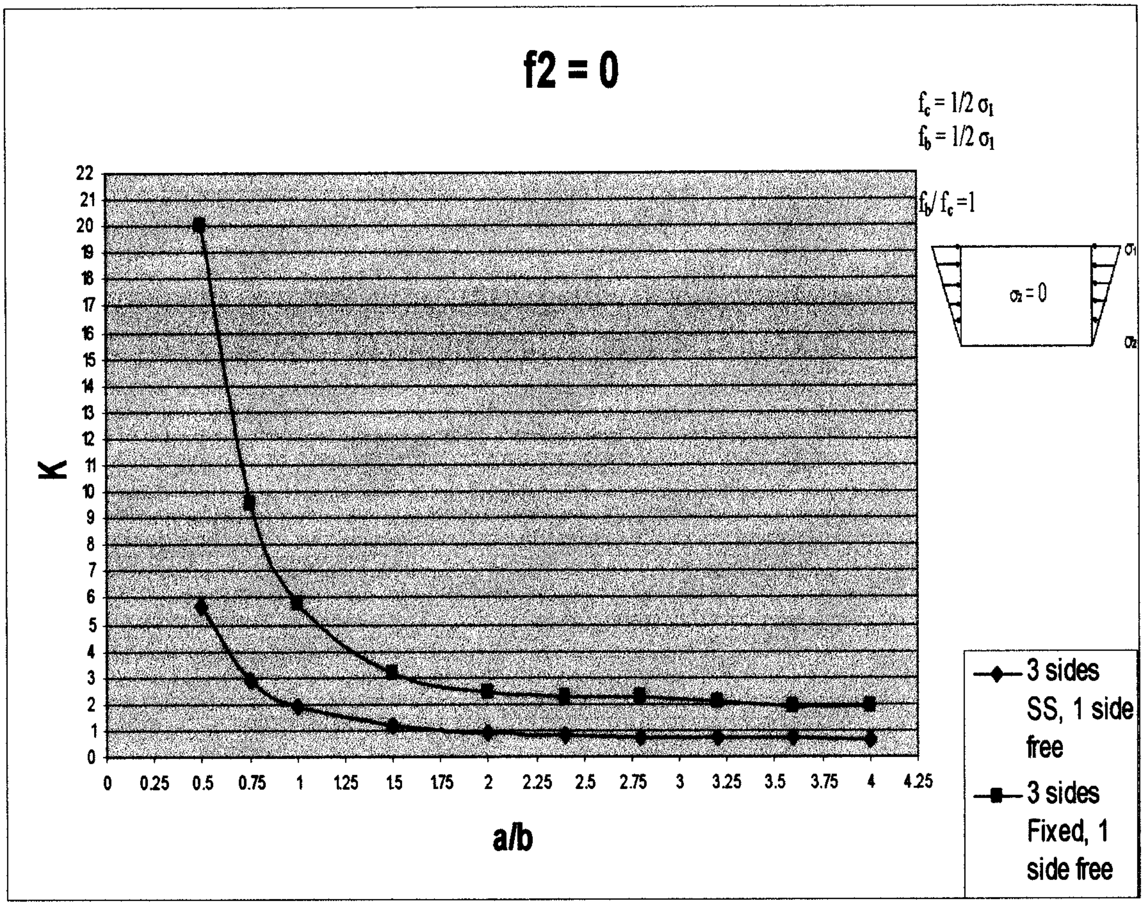


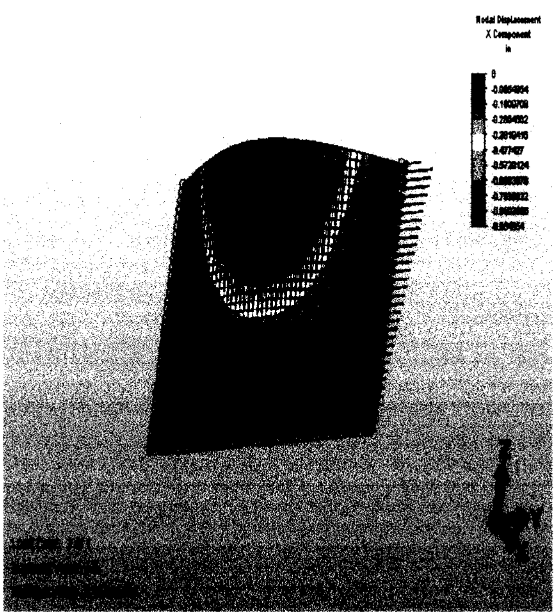
Figure 6-12: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = 0$

The values of critical buckling coefficients for plates with 3 sides simply supported and 1 side free are lower than that for plates with 3 sides totally fixed and 1 side free and K increases with decreasing a/b ratio.

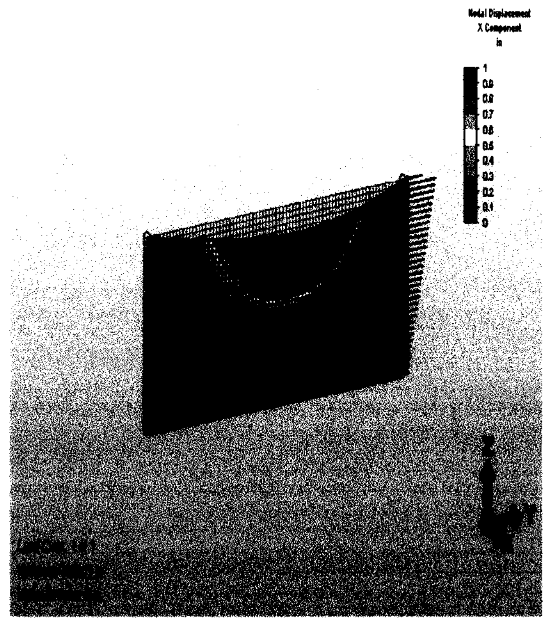
Buckled mode shapes for the above two different boundary conditions for this grid of case are shown below.

<u>Case 4</u>: $\sigma_2 = 0$

Figure 6-13: Buckled mode shapes of plates with 3 sides simply supported and top edge free:

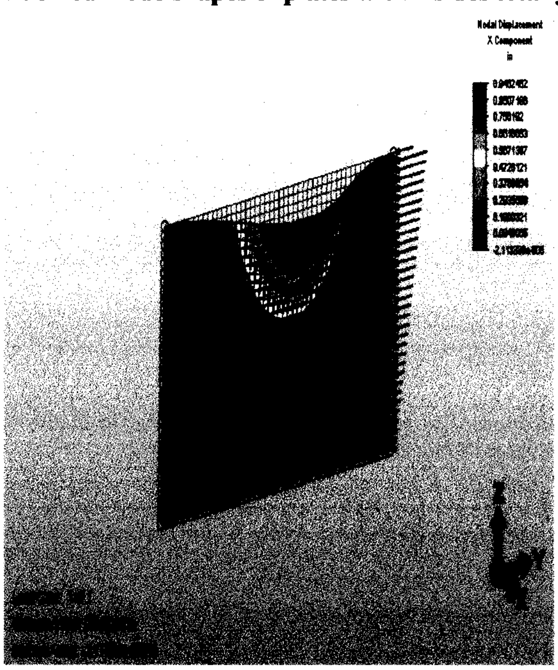


Buckled mode shape with a/b = 1.5

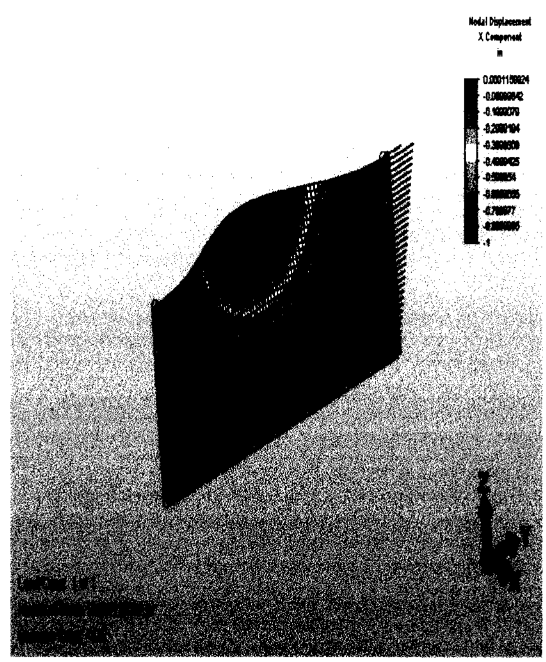


Buckled mode shape with a/b =2.4

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:



Buckled mode shape with a/b =1.5



Buckled mode shape with a/b = 2.4

6.4.7 Grid of case with $\sigma_2 = 1/3\ \sigma_1$:

The plate is under combined bending and compression in this case where compressive stress is greater than the bending stress. The ratio of bending to compressive stress is 0.5. Shown below is a chart for the critical buckling coefficient, K.

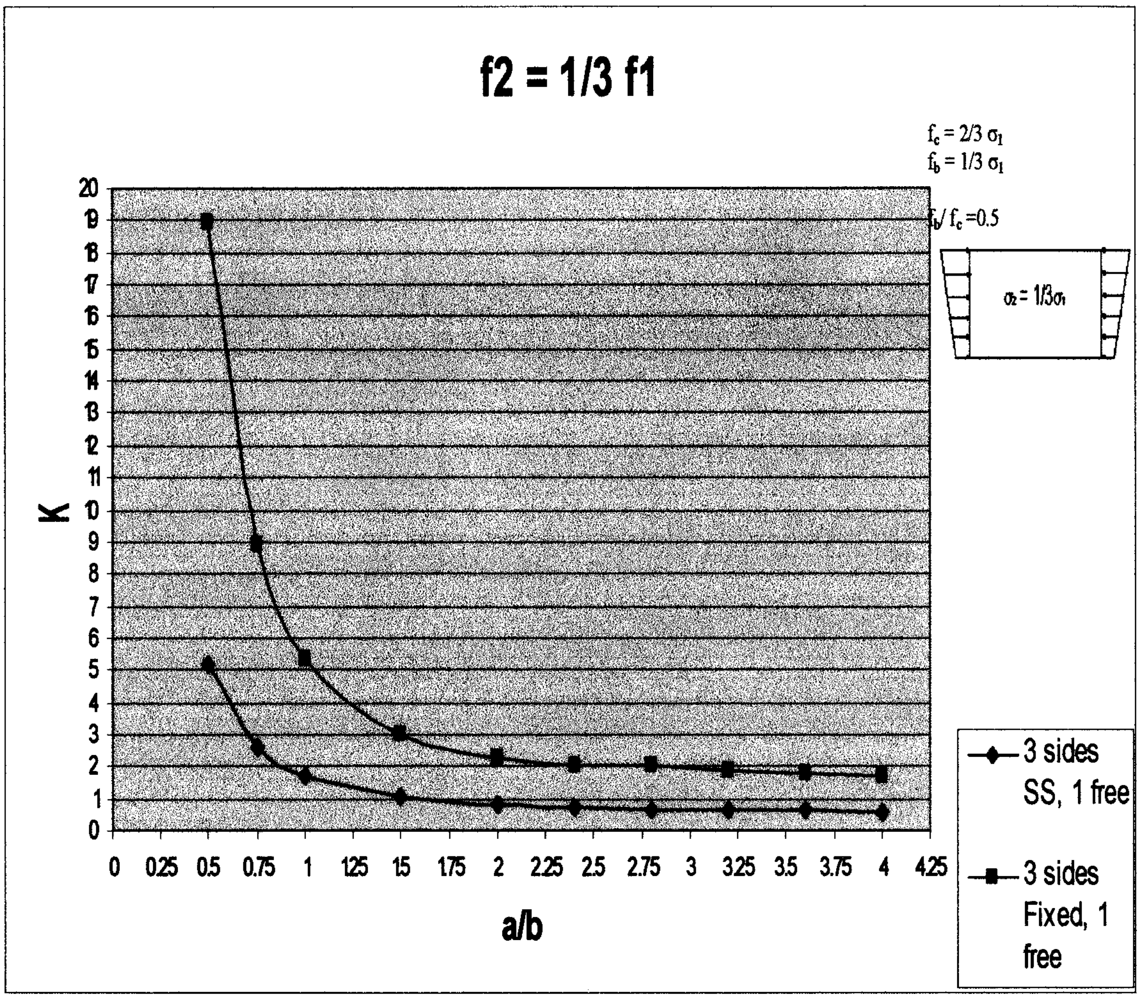


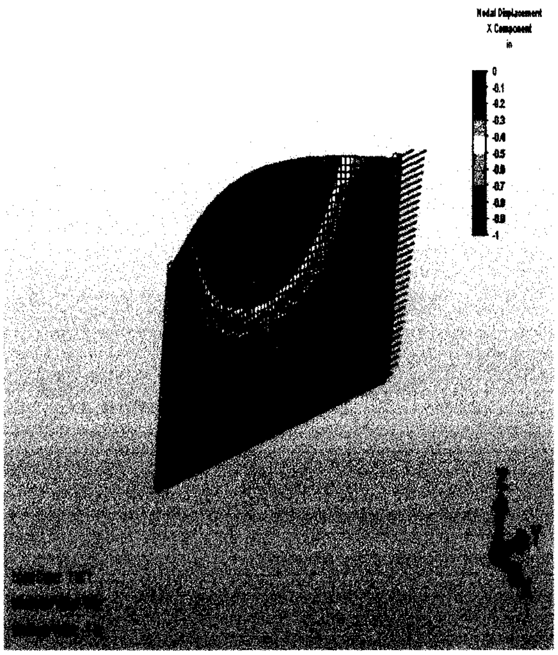
Figure 6-14: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = 1/3\ \sigma_1$

It can be noted from the plot that the values of buckling coefficients when compared are greater for plates with 3 sides simply supported and 1 side free than plates with 3 sides fixed and 1 free. As in all the above cases the value of K increases with decreasing a/b ratios.

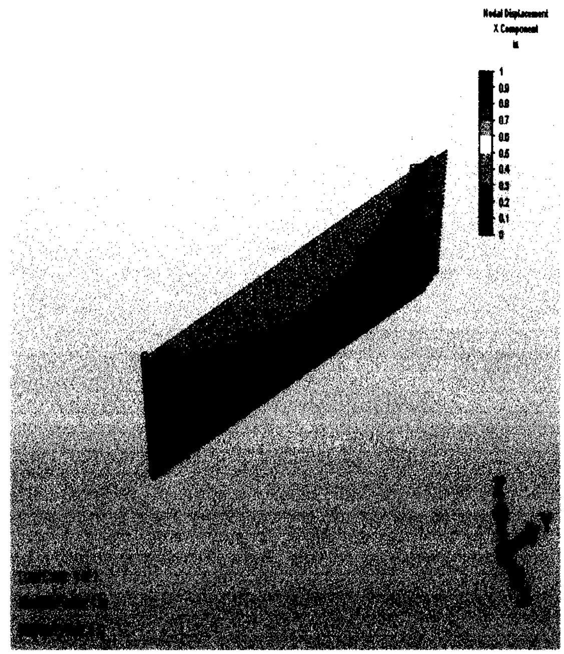
Some of the buckled mode shapes are shown below.

Case 5: $\sigma_2 = 1/3\ \sigma_1$

Figure 6-15: Buckled mode shapes of plates with 3 sides simply supported and top edge free:

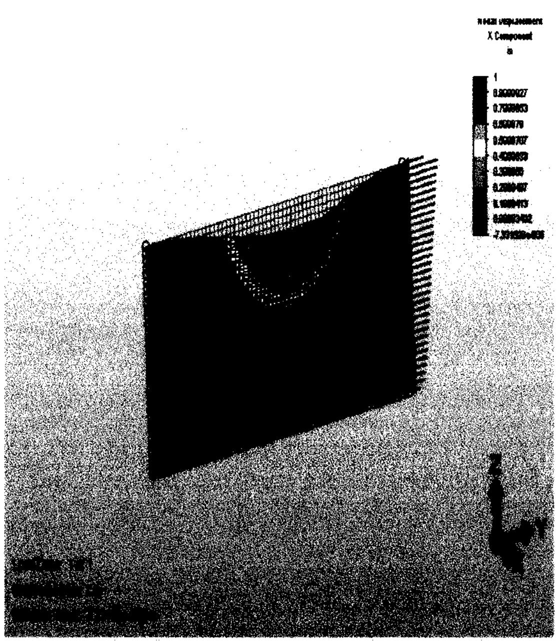


Buckled mode shape with a/b = 2

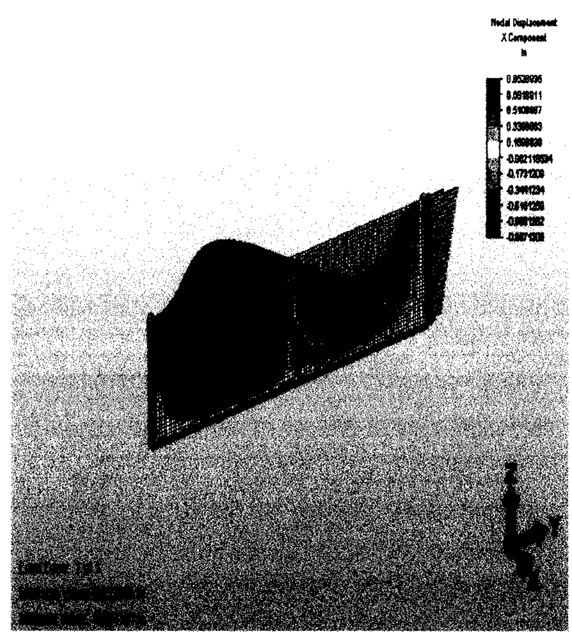


Buckled mode shape with a/b = 3.6

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:



Buckled mode shape with a/b =2



Buckled mode shape with a/b = 3.6

6.4.8 Grid of case with $\sigma_2 = \sigma_1$:

In the case where the plate is under pure compression only compressive stress f_c exists and the bending stress f_b is zero. The ratio of bending to compressive stress is hence zero. Chart shown below gives the different values of K for plates under pure compression with different a/b ratios.

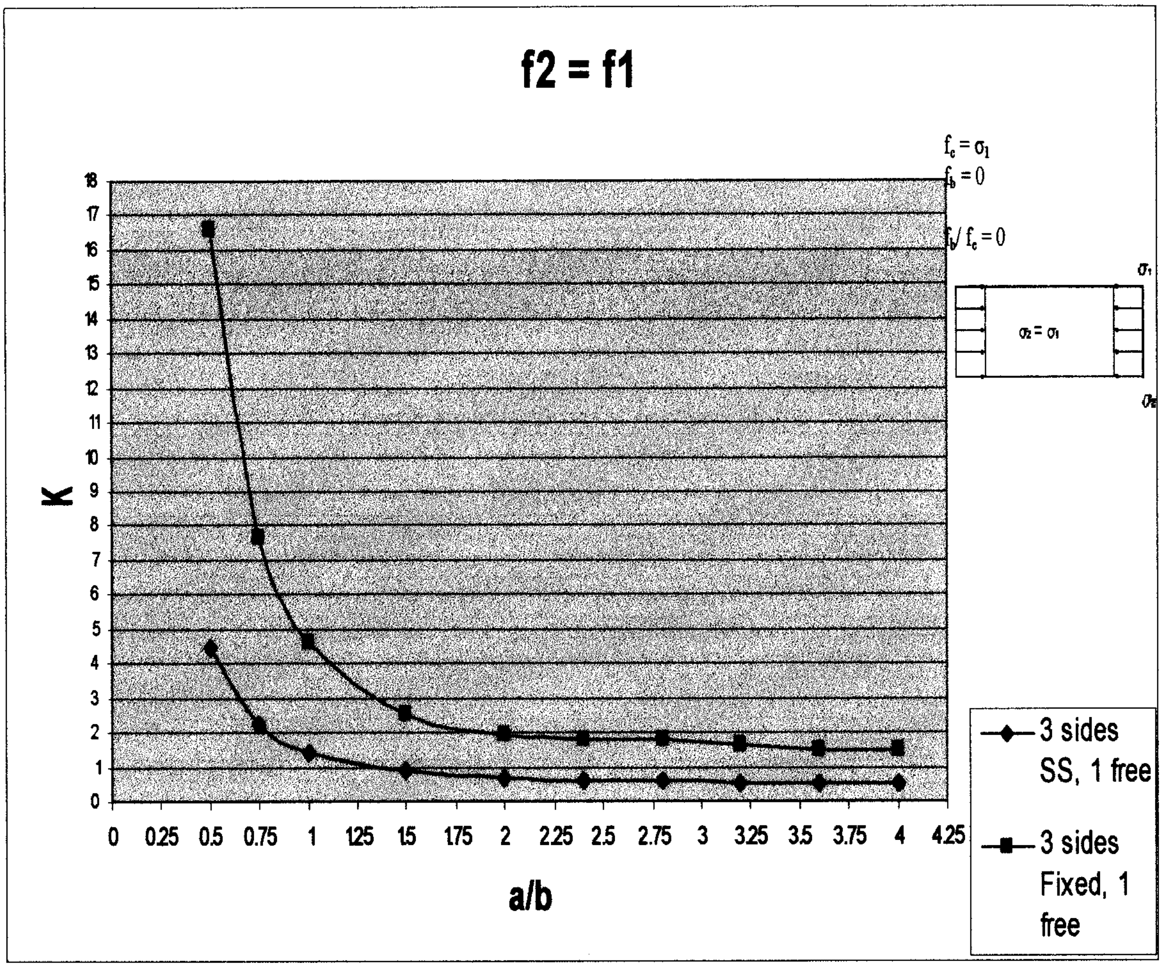


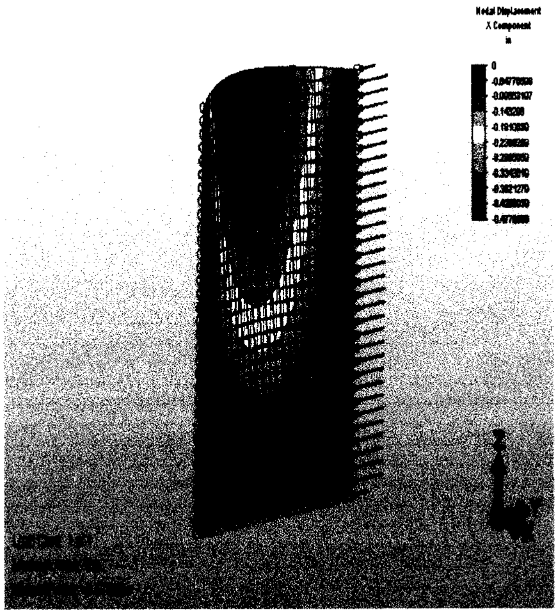
Figure 6-16: Buckling coefficient K for plates with different ratios of a/b with $\sigma_2 = \sigma_1$

As seen from the chart the values of K for the plates with 3 sides simply supported and 1 free are lower than the plates with 3 sides fixed and 1 side free.

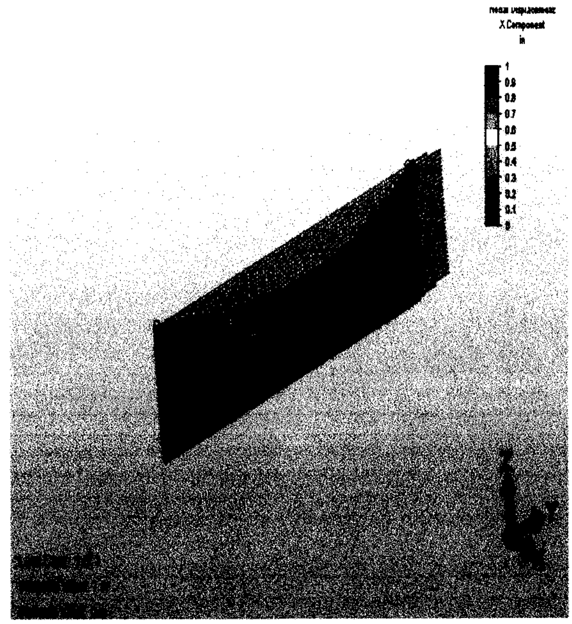
Shown below are pictures of the buckled mode shapes of plates for different aspect ratios under pure compression.

<u>**Case 6**</u>**: $\sigma_2 = \sigma_1$**

Figure 6-17: Buckled mode shapes of plates with 3 sides simply supported and top edge free:

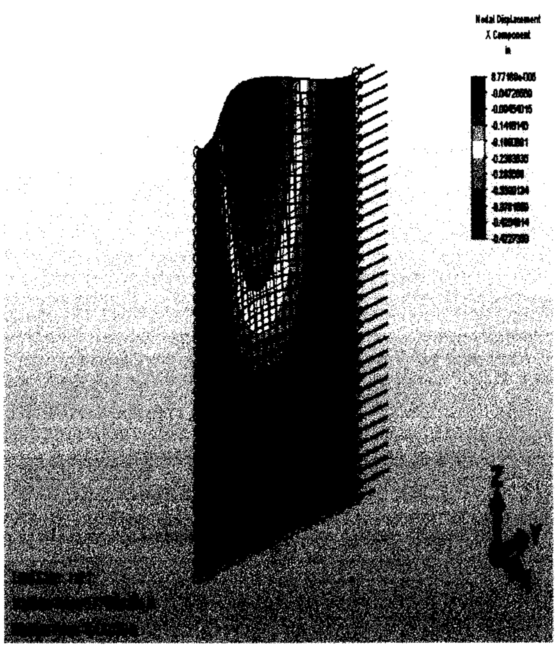


Buckled mode shape with a/b = 0.75

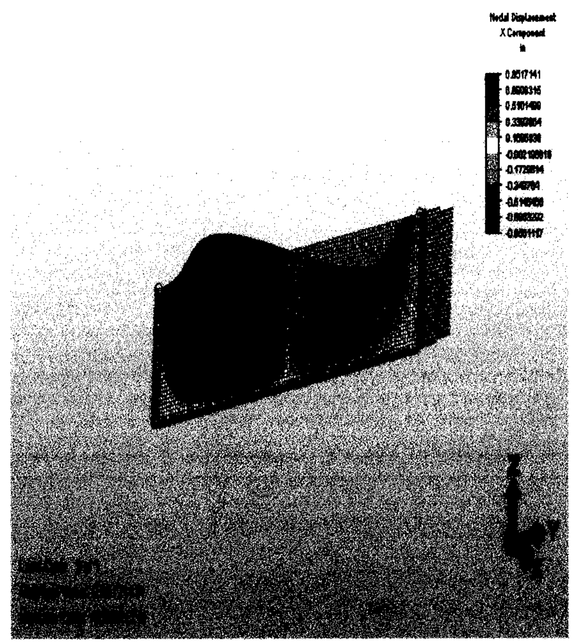


Buckled mode shape with a/b = 3.6

Buckled mode shapes of plates with 3 sides totally fixed and top edge free:



Buckled mode shape with a/b =0.75



Buckled mode shape with a/b = 3.6

Shown below is a chart for the minimum critical buckling coefficient, K, for different grid of cases considered where a/b is large enough to have a minimum K value.

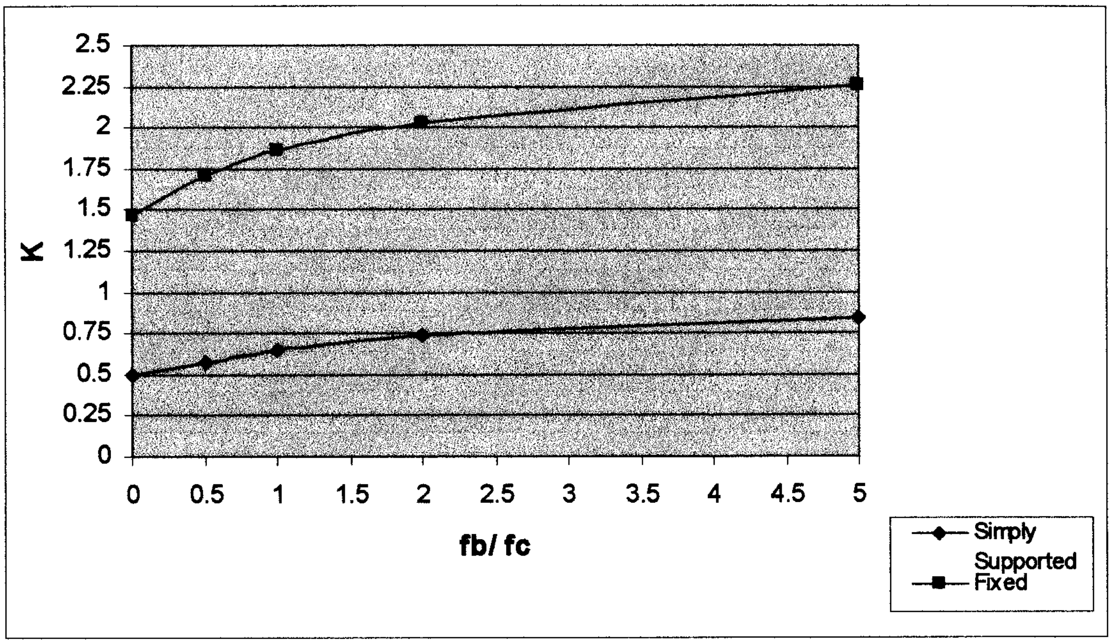


Figure 6-18: Minimum buckling coefficient K for different grid of cases.

7. CONCLUSION

New solutions for linear elastic buckling coefficients of rectangular plates with various support conditions were developed in this work. Design charts for the critical buckling coefficients for flat plates under combined bending and compression with two different boundary conditions with unloaded top edge free were presented in this work.

The finite element method is proved to be an accurate tool to determine the critical buckling load for flat plates with complex boundary conditions. When faced with plates under combined bending and compression with one unloaded edge free design engineers will be able to use the design charts developed in this work without having to make additional approximations.

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